Prioritized Sequent-Based Argumentation

Ofer Arieli
School of Computer Science
Tel-Aviv Academic College, Israel
oarieli@mta.ac.il

AnneMarie Borg
Institute of Philosophy II
Ruhr-University Bochum, Germany
annemarie.borg@rub.de

Christian Straßer
Institute of Philosophy II
Ruhr-University Bochum, Germany
christian.strasser@rub.de

ABSTRACT

In this paper we integrate priorities in sequent-based argumentation. The former is a useful and extensively investigated tool in the context of non-monotonic reasoning, and the latter is a modular and general way of handling logical argumentation. Their combination offers a platform for representing and reasoning with maximally consistent subsets of prioritized knowledge bases. Moreover, many frameworks of the resulting formalisms satisfy common rationality postulates and other desirable properties, like conflict preservation.

ACM Reference Format:

1 INTRODUCTION

Logical (or structural) argumentation is a branch of argumentation theory in which arguments have a specific structure. Among others, it has been shown useful for reasoning about knowledge, beliefs, goals and norms in agent and multi-agent systems (see, e.g., [18] for a survey and further references). In logical argumentation, arguments are expressed in terms of formal languages and acceptance of arguments is determined by logical entailments. A wealth of research has been conducted on formalizing this kind of argumentation. This includes methods that are based on Tarskian logics, like Besnard and Hunter’s approach [10], in which classical logic is the deductive base (the so-called core logic). This approach was generalized to sequent-based argumentation [7], in which Gentzen’s sequents [15], extensively used in proof theory, are incorporated for representing arguments, and attacks are formulated by special inference rules called sequent elimination rules. The result is a generic and modular approach to logical argumentation, in which any logic with a corresponding sound and complete sequent calculus can be used as the underlying core logic.

An important feature of reasoning in many contexts, including of course multi-agent systems, is the use of priorities, e.g. to model the agents’ preferences. For many existing argumentation frameworks prioritized settings are already available, see, e.g. [2, 13, 19]. The main contribution of this paper is that we extend some of those settings to arbitrary propositional languages and logics, where arguments and the attacks among them are captured in a more moderated way. For this, we extend sequent-based argumentation frameworks with a priority function on the well-formed formulas of the core logic. By keeping the exact definition of the priority function unspecified, we are able to create a general sequent-based framework that can handle different types of preferences, specified in different languages and for various logics and purposes.

The adequacy of this prioritized version is shown by the validity, for particular attack rules, of common rationality postulates [1, 12] and by the fact that in the obtained framework conflicts are tolerated: any extension in the prioritized setting is conflict-free in the flat (i.e., the non-prioritized) case. Moreover, the use of priorities allows us to extend to the preferential case some recent results (see [6, 8]) that sequent-based argumentation frameworks provide a useful platform for representing and reasoning with maximally consistent subsets of the premises [22].

The usefulness of our approach will be demonstrated (among others) on the following toy example (to which we shall return in the conclusion of the paper), involving agents and preferences.

Example 1.1. [5, 17] An agent, representing a flat owner, negotiates the construction of a swimming pool (s), a tennis-courts (t) and a private car-park (p) with other agents, representing potential tenants. It is known that any investment in two or more of these facilities will increase the rent (r), otherwise the rent will not be changed. The tenants’ representatives do not have a particular preference among these options, but if they have to make a choice, they prefer not to have two sport facilities (s and t) and definitely do not want to increase the rent. Based on these inputs, that flat owner’s representative needs to reach a recommendation about the facility (or facilities) to be constructed.

The remainder of the paper is organized as follows: the next section is a survey of the most important notions of sequent-based argumentation, followed by a section in which the general setting for the preferences is introduced. In Section 4 we consider some basic properties of the prioritized frameworks and show their adequacy for defeasible reasoning. Then, in Section 5, we give some representation results in terms of maximally consistent subsets of the premises. In Section 6 we consider some related approaches and conclude.

2 SEQUENT-BASED ARGUMENTATION

Throughout the paper we will consider propositional languages, denoted by L. Atomic formulas are denoted by p, q. Formulas are denoted by γ, δ, φ, ψ, sets of formulas are denoted by S, T, and finite sets of formulas are denoted by Γ, Δ, all of which can be primed or indexed.

Definition 2.1. A logic for a language L is a pair L = ⟨L, ⊢⟩, where ⊢ is a (Tarskian) consequence relation for L, having the following properties: reflexivity: if φ ∈ S, then S ⊢ φ; transitivity: if S ⊢ ψ and S′, φ ⊢ ψ, then S, S′ ⊢ ψ; and monotonicity: if S′ ⊢ φ and S′ ⊆ S, then S ⊢ φ.
We assume that the underlying language $L$ contains the following connectives:
- $\neg p \text{ if and only if } \neg \neg p$, for every atom $p$,
- a $\land$-conjunction $\Delta : \phi \land \psi$ iff $\phi \land \psi$.

Other connectives $L$ may contain are the following:
- a $\lor$-disjunction $\star : \phi \lor \psi$ iff $\phi$ or $\psi$,
- a $\Rightarrow$-implication $\star : \phi \Rightarrow \psi$ iff $\phi \Rightarrow \psi$.

We shall abbreviate $(\phi \lor \psi) \land (\psi \lor \phi)$ by $\phi \lor \psi$, denote by $\land \Gamma$ (respectively, by $\lor \Gamma$), the conjunction (respectively, the disjunction) of all the formulas in $\Gamma$, and let $\neg S = \{ \neg \phi \mid \phi \in S \}$.

As usual in logical argumentation (see, e.g., [10, 20, 21, 23]), arguments have a specific structure based on the underlying formal language, the so-called core logic. In the current setting arguments are represented by the well-known proof theoretical notion of a sequent.

**Definition 2.2.** Let $L = \langle \mathcal{L}, \cdot \rangle$ be a logic and $S$ a set of $L$-formulas.

- an $L$-sequent (sequent for short) is an expression of the form $\Gamma \Rightarrow \Delta$, where $\Gamma$ and $\Delta$ are finite sets of formulas in $L$ and $\Rightarrow$ is a symbol that does not appear in $L$,
- an $L$-argument (argument for short) is an $L$-sequent $\Gamma \Rightarrow \psi$, where $\Gamma \Rightarrow \psi$. $\Gamma$ is called the support set of the argument and $\psi$ its conclusion,
- an $L$-argument based on $S$ is an $L$-argument $\Gamma \Rightarrow \psi$, where $\Gamma \subseteq S$. We denote by $\text{Arg}_S(L)$ the set of all the $L$-arguments based on $S$.

Given an argument $a = \Gamma \Rightarrow \psi$, we denote $\text{Sup}(a) = \Gamma$ and $\text{Con}(a) = \psi$. We say that $a'$ is a sub-argument of $a$ iff $\text{Sup}(a') \subseteq \text{Sup}(a)$. The set of all the sub-arguments of $a$ is denoted by $\text{Sub}(a)$.

The formal systems used for the constructions of sequents (and so of arguments) for a logic $L = \langle \mathcal{L}, \cdot \rangle$, are sequent calculi [15], denoted here by $C$. In what follows we shall assume that $C$ is sound and complete for $L = \langle \mathcal{L}, \cdot \rangle$, i.e., $\Gamma \Rightarrow \psi$ is provable in $C$ iff $\Gamma \Rightarrow \psi$.

One of the advantages of sequent-based argumentation is that any logic with a corresponding sound and complete sequent calculus can be used as the core logic. The construction of arguments from simpler arguments is done by the inference rules of the sequent calculus [15].

Argumentation systems contain also attacks between arguments. In our case, attacks are represented by sequent elimination rules. Such a rule consists of an attacking argument (the first condition of the rule), an attacked argument (the last condition of the rule), conditions for the attack (the conditions in between) and a conclusion (the eliminated attacked sequent). The outcome of an application of such a rule is that the attacked sequent is ‘eliminated’. The elimination of a sequent $a = \Gamma \Rightarrow \Delta$ is denoted by $\nabla$ or $\Gamma \not\Rightarrow \Delta$.

**Definition 2.3.** A sequent elimination rule (or attack rule) is a rule $\mathcal{R}$ of the form:

\[
\frac{\Gamma_1 \Rightarrow \Delta_1 \quad \ldots \quad \Gamma_n \Rightarrow \Delta_n}{\Gamma_n \not\Rightarrow \Delta_n} \quad \mathcal{R}
\]

1Set signs in arguments are omitted.

2See [7] for further advantages of this approach.

3That is, the eliminated sequent should not be used as a condition of later applications of rules in the derivation, nor is it considered a valid conclusion of the derivation.

Let $\Gamma \Rightarrow \psi, \Gamma' \Rightarrow \psi' \in \text{Arg}_S(L)$ and let $\mathcal{R}$ be an elimination rule. If $\Gamma \Rightarrow \psi$ is an instance of $\Gamma_1 \Rightarrow \Delta_1, \Gamma' \Rightarrow \psi'$ is an instance of $\Gamma_n \Rightarrow \Delta_n$ and all the other conditions of $\mathcal{R}$ are provable in $C$, we say that $\Gamma \Rightarrow \psi \mathcal{R}$-attacks $\Gamma' \Rightarrow \psi'$.

**Example 2.4.** We refer to [7, 24] for a definition of many sequent elimination rules. Below are three of them (assuming that $\Gamma_1 \not\Rightarrow \psi$):

**Undercut (Ucut):**

\[
\Gamma_1 \Rightarrow \psi_1 \Rightarrow \psi_1 \lor \neg \Delta_2 \quad \Gamma_2, \Gamma_2' \Rightarrow \psi_2
\]

**Direct Ucut (DUcut):**

\[
\Gamma_1 \Rightarrow \psi_1 \Rightarrow \psi_1 \lor \neg \Delta_2 \quad \Gamma_2, \Gamma_2' \Rightarrow \psi_2
\]

**Consistency Ucut (ConUcut):**

\[
\Gamma_2 \Rightarrow \psi_1
\]

A sequent-based framework is now defined as follows:

**Definition 2.5.** A sequent-based argumentation framework for a set of formulas $S$ based on a logic $L = \langle \mathcal{L}, \cdot \rangle$ and a set $\mathcal{R}$ of sequent elimination rules, is a pair $\mathcal{AF}_{L, \mathcal{R}}(S) = \langle \text{Arg}_S(L), \mathcal{R} \rangle$, where $\mathcal{R} \subseteq \text{Arg}_S(L) \times \text{Arg}_S(L)$ and $(a_1, a_2) \in \mathcal{R}$ iff there is an $\mathcal{R} \in \mathcal{R}$ such that $a_1 \mathcal{R}$-attacks $a_2$.

In what follows, to simplify notation, we will omit the subscript $L$ and/or $\mathcal{R}$, when this is known or arbitrary.

**Example 2.6.** Let $S = \{ p, q, \neg p \lor \neg q \}$ and let $\mathcal{AF}_{L, \{Ucut\}}(S)$ be a framework for $S$, induced by classical logic $CL$, its corresponding sound and complete sequent calculus $LK$, and Ucut as the only attack rule. Some of the arguments are:

\[
\begin{align*}
\text{a}_1 &= p \Rightarrow p \quad \text{a}_4 &= p \Rightarrow \neg \neg p \lor \neg q \\
\text{a}_2 &= q \Rightarrow q \quad \text{a}_5 &= q \Rightarrow \neg \neg p \lor \neg q \\
\text{a}_3 &= \neg p \lor \neg q \Rightarrow \neg p \lor \neg q
\end{align*}
\]

See Figure 1 for a graphical representation of these arguments and the attacks between them.

Given a (sequent-based) framework, Dung-style semantics [14] can be applied to it to determine what combinations of arguments (called extensions) can collectively be accepted from it.

**Definition 2.7.** Let $\mathcal{AF}_{L, \mathcal{R}}(S) = \langle \text{Arg}_S(L), \mathcal{R} \rangle$ be an argumentation framework and $S \subseteq \text{Arg}_S(L)$ a set of arguments.

- $S$ attacks an argument $a$ if there is an $a' \in S$ such that $(a', a) \in \mathcal{R}$.
- $S$ defends an argument $a$ if $S$ attacks every attacker of $a$.

![Figure 1: Part of the sequent-based argumentation graph for \( S = \{ p, q, \neg p \lor \neg q \} \) from Example 2.6](image-url)
• S is conflict-free if there are no arguments \(a_1, a_2 \in S\) such that \((a_1, a_2) \in \mathcal{A}^T\);
• S is admissible if it is conflict-free and it defends all of its elements.

An admissible set that contains all the arguments that it defends is a complete extension of \(\mathcal{AF}_L(S)\). Below are definitions of some other extensions of \(\mathcal{AF}_L(S)\):

- a preferred extension of \(\mathcal{AF}_L(S)\) is a maximal (with respect to \(\subseteq\)) complete extension of \(\text{Arg}_L(S)\);
- a stable extension of \(\text{Arg}_L(S)\) is a complete extension of \(\text{Arg}_L(S)\) that attacks every argument not in it;
- the grounded extension of \(\mathcal{AF}_L(S)\) is the minimal (with respect to \(\subseteq\)) complete extension of \(\text{Arg}_L(S)\).

In what follows we shall refer to either complete (cmp), grounded (grd), preferred (prf) or stable (stb) semantics as completeness-based semantics. We denote by \(\text{Ext}_{\text{sem}}(\mathcal{AF}_L(S))\) the set of all the extensions of \(\mathcal{AF}_L(S)\) under the semantics \(\text{sem} \in \{\text{cmp}, \text{grd, prf}, \text{stb}\}\).

The subscript is omitted when this is clear from the context.

**Definition 2.8.** Given a sequent-based argumentation framework \(\mathcal{AF}_L(S)\), the semantics as defined in Definition 2.7 induces corresponding (nonmonotonic) entailment relations:

- **Skeptical entailment:** \(S \vdash^\phi_{\text{L-sem}} \phi\) iff for every extension \(E \in \text{Ext}_{\text{sem}}(\mathcal{AF}_L(S))\), there is \(\Gamma \models \phi \in E\) for \(\Gamma \subseteq S\);
- **Credulous entailment:** \(S \vdash^\phi_{\text{L-sem}} \phi\) iff for some extension \(E \in \text{Ext}_{\text{sem}}(\mathcal{AF}_L(S))\), there is \(\Gamma \models \phi \in E\) for \(\Gamma \subseteq S\);
- **Weakly skeptical entailment:** \(S \vdash^\phi_{\text{L-sem}} \phi\) iff there is an \(a \in \text{Arg}_L(S)\) with \(\text{Con}(a) = \phi\) such that \(a \in E\) for every \(E \in \text{Ext}_{\text{sem}}(\mathcal{AF}_L(S))\).

**Example 2.9.** Consider again the framework of Example 2.6. It holds that \(S \vdash^\phi \mathcal{L}_{\text{prf}} \rho\) and \(S \vdash^\phi \mathcal{L}_{\text{prf}} \neg \rho\), while \(S \vdash^\phi \mathcal{L}_{\text{prf}} \rho\) and \(S \vdash^\phi \mathcal{L}_{\text{prf}} \neg \rho\) for \(\star \in \{\wedge, \rightarrow\}\). Moreover, \(S \vdash^\psi \mathcal{L}_{\text{grd}} \psi\) if and only if \(\psi\) is a tautology in classical logic. On the other hand, it is easy to see that \(S \cup \{r\} \vdash^\phi \mathcal{L}_{\text{prf}} \neg r\) and \(S \cup \{t\} \vdash^\phi \mathcal{L}_{\text{prf}} \neg r\), since \(\mathcal{L}_{\text{grd}}\neg r\) is in the grounded extension of \(S \cup \{r\}\).

### 3 PREFERENCE FUNCTIONS AND PRIORITIZED ARGUMENTATION

We now formulate a general setting for prioritized sequent-based argumentation, allowing to make preferences among different arguments.

**Definition 3.1.** A priority function for a language \(\mathcal{L}\) is a function \(\pi : \mathcal{L} \mapsto \mathbb{N}^\ast\). Given a set of \(\mathcal{L}\)-formulas \(S\), we denote: \(\pi(S) = \{\pi(\phi) \mid \phi \in S\}\).

We now use \(\pi\) for defining a preference relation \(\leq_\pi\) on \(\mathcal{L}\)-sequents. The next example illustrates some ways of doing so. We shall write \(a_1 \leq_\pi a_2\) to intuitively indicate that the sequent \(a_1\) is at least as preferred as the sequent \(a_2\).

**Example 3.2.** The following are possible conditions for letting \(a_1 \leq_\pi a_2\):

1. \(\min(\pi(S(a_1))) \leq \min(\pi(S(a_2)))\). In this case only the most preferred formulas in the support of the sequents are compared.
2. \(\max(\pi(S(a_1))) \leq \max(\pi(S(a_2)))\). Here, for every formula in the support of \(a_2\) there is a more preferred formula in the support of \(a_1\).
3. \(\max(\pi(S(a_1))) \leq \pi(S(a_2))\). In this case all the formulas in the support of \(a_1\) are at least as preferred as the formulas in the support of \(a_2\).
4. \(\min(\pi(S(a_1)) \setminus \text{Sup}(a_2)) \leq \min(\pi(S(a_2)) \setminus \text{Sup}(a_1))\).

In the first item, the most preferred formulas are compared, but now only the formulas that are not part of the support of the other argument.

5. \(f(S(a_1)) \leq f(S(a_2))\), where \(f\) is an aggregation function on \(S(a_1)\) (like the average, median, summation of the \(\pi\)-values of the supports, or the max/min function on the support, as in the previous items).

6. \(S(a_1) \leq_\pi S(a_2)\) if either \(S(a_1) = 0\) or \(S(a_1) = S(a_2)\) and there is an \(i \in \mathbb{N}\) such that:
   - \(|\psi \in S(a_1) \mid \pi(\psi) = i\}| \geq |\psi \in S(a_2) \mid \pi(\psi) = i\}|\)
   - \(|\psi \in S(a_1) \mid \pi(\psi) = j\}| \geq |\psi \in S(a_2) \mid \pi(\psi) = j\}|\)
   - for every \(j < i\).
7. \(S(a_1) \leq_\pi S(a_2)\) if either \(S(a_1) = 0\) or there is an \(i \in \mathbb{N}\) such that:
   - \(|\psi \in S(a_1) \mid \pi(\psi) = i\}| \geq |\psi \in S(a_2) \mid \pi(\psi) = i\}|\)
   - \(|\psi \in S(a_1) \mid \pi(\psi) = j\}| \geq |\psi \in S(a_2) \mid \pi(\psi) = j\}|\)
   - for every \(j < i\).

Remark 1. The last two items of Example 3.2 are inspired by Brewka’s approach to reasoning with preferred theories [11]. This approach is adjusted to our case by viewing the arguments’ support sets as stratified theories, where each stratification consists of the formulas with the same \(\pi\)-value. Accordingly \(\leq_\pi\) is a subset-inclusion comparison, and \(\leq_\pi\) is a comparison by cardinality.

Remark 2. The Items 1, 2, 4, 6 and 7 of Example 3.2 are pre-orders, that is: \(\leq_\pi\) is reflexive (\(a \leq_\pi a\)) and transitive (if \(a \leq_\pi b\) and \(b \leq_\pi c\) then \(a \leq_\pi c\)). Whether the relation in Item 5 is a pre-order depends on the function \(f\).

The orders in Items 1, 4, 6 and 7 and their strict counterparts are also left monotonic: if \(a \leq_\pi b\) (resp. \(a <_\pi b\)) and \(\text{Sup}(a) \subseteq \text{Sup}(a')\) then \(a' \leq_\pi b\) (resp. \(a' <_\pi b\)).

**Example 3.3.** In Example 2.6, let \(\pi(\rho) = 1\), \(\pi(\psi) = 2\) and \(\pi(\neg \psi) \cup \neg \psi\) = 3. Consider each of the seven instances for \(\leq_\pi\) from Example 3.2:

1. When the most preferred supports are compared we have that \(a_1 \leq_\pi a_2 \leq_\pi a_3\), \(a_1 \leq_\pi a_7\), \(a_5 <_\pi a_2\), \(a_6 <_\pi a_3\), and \(a_6 <_\pi a_8\).
2. When the least preferred supports are compared we still have \(a_1 \leq_\pi a_2 \leq_\pi a_3\), \(a_1 \leq_\pi a_7\), and \(a_6 <_\pi a_3\), but now \(a_2 <_\pi a_6\) and only \(a_6 <_\pi a_8\).
3. The max-min-comparison yields again \(a_1 \leq_\pi a_2 <_\pi a_3\), \(a_1 <_\pi a_7\) and \(a_6 <_\pi a_3\), but this time \(a_2 <_\pi a_6\) and \(a_6 <_\pi a_8\).
Definition 3.6. Let $L = (\mathcal{L}, \vdash)$ be a core logic, $C$ a corresponding sound and complete sequent calculus, $\arg$ a set of attack rules, $\pi$ a priority function on $\mathcal{L}$, and $\leq_{\pi}$ a preference order on $\mathcal{L}$-sequent. The prioritized sequent-based argumentation framework for the set $\mathcal{S}$ of formulas (induced by $L$, $C$, $\arg$, and $\leq_{\pi}$), is a triple: $\mathcal{A} = (\mathcal{L}, \vdash, \arg, \leq_{\pi})$, where $\arg \subseteq \mathcal{L}\arg$, and $(a_1, a_2) \in \arg$ if $a_1 \leq_{\pi} \overline{a_2}$ for some $R \in \mathcal{R}$.

Like before, we will omit the subscripts $L$, $\arg$ and/or $\pi$ if these are known or arbitrary.

The Dung-style semantics from Definition 2.7 are defined equivalently for $\mathcal{A} = (\mathcal{L}, \vdash, \arg, \leq_{\pi})$, now with respect to both $\arg$ and $\leq_{\pi}$. Based on this, we define the entailment relations for $\mathcal{A} = (\mathcal{L}, \vdash, \arg, \leq_{\pi})$ with respect to the different semantics as in Definition 2.8. For a given semantics $\arg$ and $\pi \in \{\cap, \cup, \setminus, \exists\}$, the relation is denoted by $\models_{\mathcal{L}, \arg, \pi}$ (super/subscripts are omitted when they are clear from the context).

Example 3.7. The flat case (without priorities) of $\mathcal{A} = (\mathcal{L}, \vdash, \arg, \leq_{\pi})$ and $\arg$ as the sole attack rule is the same as the framework of Example 2.6. The grounded extension only contains sequents with empty support sets, since there are complete extensions that contain only two of the arguments $a_1, a_2$ and $a_3$. When considering the priority function $\pi$ from Example 3.3, in any of the definitions for $\leq_{\pi}$ from Example 3.2, $a_1$ cannot be attacked. Thus $S \vdash_{\mathcal{L}, \arg, \pi} q$. For $q$ the result depends on the choice of $\leq_{\pi}$.

- When using the first instance of $\leq_{\pi}$ from Example 3.2, $a_8 \leq_{\pi} a_1$ for any $a \in \{\Gamma \Rightarrow \psi \} \subseteq L \subseteq S$. Moreover, $a_8$ and $a_9$ attack each other, one can therefore construct two different admissible sets, one in which $a_8$ attacks $a_3$ and one in which it does not. Therefore, $S \not\models_{\mathcal{L}, \arg, \pi} q$.
- According to the fourth and sixth instance of $\leq_{\pi}$ from Example 3.2, $a_8$ does not attack $a_9$, thus $a_6$ is no longer attacked, and so it defends $a_2$. Hence $S \not\models_{\mathcal{L}, \arg, \pi} q$ in this case.

4 SOME BASIC PROPERTIES

Next, we consider some basic properties of prioritized argumentation frameworks and the entailment relations induced by them.

4.1 Conservativity

For every preferential ordering from Example 3.2, prioritized reasoning is a conservative extension of the flat case:

**Proposition 4.1.** If $\leq_{\pi}$ is degenerated (i.e., $\pi$ is uniform) then $\models_{\mathcal{L}, \arg, \pi}$ and $\models_{\mathcal{L}, \arg, \pi}$ are the same for every $\pi \in \{\cap, \cup, \setminus, \exists\}$ and $\pi \in \{\arg, \vdash, \setminus, \cap, \cup, \exists\}$.

**Proof.** (Sketch) Immediate from Definition 3.4 of $\mathcal{R}_{\leq_{\pi}}$-attacks since no arguments are $\leq_{\pi}$-preferred over others, thus $\mathcal{R}_{\leq_{\pi}}$-attacks coincide with $\mathcal{R}$-attacks.

4.2 Rationality postulates

Caminada and Amgoud [1, 12] propose several postulates for argumentation reasoning. Below we consider those postulates using the next definitions.

**Definition 4.2.** Let $L = (\mathcal{L}, \vdash)$ be a propositional logic and $S$ a set of $\mathcal{L}$-formulas.
• The transitive closure of $S$ with respect to the logic $L$ is the set $CN_L(S) = \{\psi | S \vdash \psi\}$.
• $S$ is $L$-consistent if there is no $\Gamma \subseteq S$ such that $\Gamma \vdash \neg \Delta$.
• A subset $T \subseteq S$ is an $L$-minimal conflict of $S$, if it is not $L$-consistent, but $T \setminus \{\psi\}$ is $L$-consistent for every $\psi \in T$.
• $Free(S)$ is the set of formulas that are not part of any minimal conflict of $S$.

**Definition 4.3.** [1, 12] The postulates below refer to a prioritized sequent-based argumentation framework $AF^\prec_L(S) = \langle Arg_L(S), AF^\prec_L, \leq, \Delta, \leq, \psi \rangle$, a semantics sem of it (i.e., one of those in Definition 2.7), every extension $E \in Ext_{sem}(AF^\prec_L(S))$, and arbitrary argument $a \in Arg_L(S)$.

- **Closure of extensions:** $Con(E) = CN_L(Con(E))$.
- **Closure under sub-arguments:** if $a \in E$ and $b \in Sub(a)$ then $b \in E$.
- **Weak Closure under sub-arguments:** if $a \in E$, $b \in Sub(a)$ and $b \leq a$, then $b \in E$.
- **Consistency:** $Con(E)$ is consistent.
- **Exhaustiveness:** if $Sup(a) \cup \{Con(a)\} \subseteq Con(E)$, $a \in E$.
- **Weak exhaustiveness:** if $Sup(a) \subseteq \bigcup_{b \in E} Sup(b)$, $a \in E$.
- **Free precedence:** $Arg_L(Free(S)) \subseteq E$.

Below, we shall consider these postulates under the following assumptions:

1. The core logic is **non-trivial** (there is no $\phi$ such that both $\vdash \phi$ and $\vdash \neg \phi$) and **contrapositive** ($\vdash \phi \rightarrow \neg \phi$ implies that $(\Gamma \setminus \Gamma')$, $\Delta \vdash \neg \gamma \rightarrow (\neg \gamma \rightarrow \Gamma)$, for $\Delta' \leq \Delta$ and $\Gamma' \leq \Gamma$).
2. The preferential order $\leq_{\pi}$ and its strict counterpart $<_{\pi}$ are **left monotonic:** if $a \leq_{\pi} b$ (resp. $a <_{\pi} b$) and $Sup(a) \subseteq Sup(b)$ then $a' \leq_{\pi} b$ (resp. $a' <_{\pi} b$).
3. Given a priority function $\pi$ for $L$, we will consider preference orders $\leq_{\pi}$ on sets of $L$-formulas for which $\leq_{\pi}$ and $<_{\pi}$ are monotonic, reflexive and transitive relations (as in Items 1, 4, 6, and 7 in Example 3.2). We lift $\leq_{\pi}$ to sequents as follows: $a \leq_{\pi} b$ if $Sup(a) \leq_{\pi} Sup(b)$.

**Proposition 4.4.** Let $AF^\prec_L(S)$ be a prioritized framework in which the core logic and the preferential order satisfy the three conditions specified above. Suppose also that $DCut$ is the attack rule. Then, for every completeness-based semantics, $AF^\prec_L(S)$ satisfies closure of extensions, weak closure under sub-arguments, consistency, and weak exhaustiveness. When $ConUCut$ is also an attack rule, $AF^\prec_L(S)$ satisfies free precedence as well.

The following lemmas are required to prove Proposition 4.4.

**Lemma 4.5.** Let $E$ be a complete extension of $AF^\prec_L(S)$. If (1) $a = \Delta \vdash \gamma \in E$ and $b = \Gamma \vdash \gamma \in E$, (2) $a < b$, (3) $\Gamma' \subseteq \Gamma$, (4) $\Delta', \Delta' \vdash \psi$, and (5) $\Delta \cup \Gamma'$ is consistent, then $\Delta \vdash \psi \in E$.

**Proof.** Suppose that $d = \Theta \vdash \psi$ attacks $c = \Gamma', \Delta \vdash \psi$. Since $\Gamma' \cup \Delta$ is consistent (by Condition (5)), $d$ $DCut$ attacks $c$. Hence $c \leq \Delta$. By left monotonicity $c \leq d$. Assume for a contradiction that $b < d$. Since $a < b$ (by Condition (2)), also $a < d$, which is a contradiction. Thus, $b \not< d$. Since $d$ $DCut$ attacks $c$, there is a $\beta \in \Gamma' \cup \Delta$ for which both $\gamma \Rightarrow \neg \beta$ and $\beta \Rightarrow \neg \gamma$ are derivable. If $\beta \in \Delta$ then $d$ $DCut$ attacks $a$. If $y \in \Gamma'$ then $d$ $DCut$ attacks $b$. By the admissibility of $E$ there is an $f \in E$ that attacks $d$, and so $f$ defends $c$ from $d$.

**Lemma 4.6.** If $a \not< b$ and $Sup(b) \subseteq Sup(b')$, then $a \not< b'$.

**Proof.** Suppose that $a \not< b$ and $Sup(b) \subseteq Sup(b')$. Assume for a contradiction that $a \not< b'$. By reflexivity, $b \leq b'$, and so by left monotonicity, $b' \leq b$. By transitivity, $a < b$, which contradicts our supposition. □

**Lemma 4.7.** Let $E$ be a complete extension of $AF^\prec_L(S)$. If $\Gamma \vdash \gamma$ and $\Delta \Rightarrow \delta$ are in $E$ and $\Gamma, \Delta \Rightarrow \phi \in Arg_L(S)$, then $\Gamma, \Delta \Rightarrow \phi$ is also in $E$.

**Proof.** We start with assuming that $DUcut$ is the only attack rule. Suppose that $a = \Theta \Rightarrow \tau \in Arg_L(S)$ attacks $b = \Gamma, \Delta \Rightarrow \phi$. Then $\Rightarrow \tau \Rightarrow \neg \tau'$ is derivable in $C$ for some $\tau' \in \Gamma \cup \Delta$ and $\Gamma \cup \Delta \vdash \phi \not\in \Theta$. By left monotonicity, also $\Gamma \not\vdash \pi \Theta$ and $\Delta \not\vdash \pi \Theta$. Thus $b \not\vdash \pi a$, $\Delta \Rightarrow \delta \not\vdash \pi a$, and $\Gamma \Rightarrow \gamma \not\vdash \pi a$. Suppose, without loss of generality, that $\tau' \in \Delta$. Then $a$ attacks $\Delta \Rightarrow \delta$. Since $E$ is admissible there is some $a' \in E$ that attacks $a$. This shows that $E$ defends $b$ and since $E$ is complete, $b \in E$.

Now, suppose that $ConUCut$ is one of the attack rules and assume for a contradiction that $\Gamma, \Delta \Rightarrow \phi$ is $ConUCut$-attacked. In this case $\Gamma \cup \Delta$ is inconsistent. We have that either $\Delta \not\vdash \Gamma$ or $\Gamma \not\vdash \Delta$. Without loss of generality we suppose $\Delta \not\vdash \Gamma$.

Since $\Gamma \cup \Delta$ is inconsistent, there is a maximal $\Delta' \subseteq \Delta$ for which $\Gamma \cup \Delta'$ is consistent. (Note for this that $\Gamma$ is consistent since otherwise $y \vdash \gamma$ is $ConUCut$ attacked and cannot be defended, which is impossible since $\Gamma \Rightarrow \gamma \in E$. ) Thus, there is a $\delta' \in \Delta \setminus \Delta'$ for which $\Gamma, \Delta' \Rightarrow \delta'$. Let $d = \Gamma, \Delta' \Rightarrow \delta' \in Arg_L(S)$ and $a = \Delta \Rightarrow \delta$. Note that $a \not< d$ by Lemma 4.6. So, $d$ $DCut$ attacks $a$.

Hence, there is a $\epsilon = \Theta \Rightarrow \psi \in E$ that defends $a$ from this attack by attacking $d$. Note that $\epsilon$ does not $ConUCut$-attack $d$ since $\Gamma \cup \Delta'$ is consistent. Hence, $d \not< \epsilon$ and there is a $\beta \in \Gamma \cup \Delta'$ for which both $\psi \Rightarrow \beta$ and $\beta \Rightarrow \neg \psi$ are derivable.

We have two cases: (a) $\beta \in \Gamma$ and (b) $\beta \in \Delta'$. We now show that both lead to a contradiction.

If (a) holds, then by left monotonicity $b = \Gamma \Rightarrow \gamma \not\vdash \epsilon$ which means that $\epsilon$ attacks $b$ in contradiction to the conflict-freeness of $E$.

If (b) holds, then $a < \epsilon$, since otherwise $\epsilon$ $DCut$ attacks $a$ in contradiction to the conflict-freeness of $E$. Since $\Theta \not\vdash \neg \beta$ there is a maximal $\Theta' \subseteq \Theta$ for which $\Delta \cup \Theta' \vdash \neg \epsilon$ is consistent and for which $\Delta \cup \Theta' \vdash \neg \epsilon$ for some $\epsilon \in \Theta \setminus \Theta'$. By Lemma 4.5, $\Delta' = \Delta, \Theta' \Rightarrow \neg \epsilon \in E$. By left monotonicity $\epsilon' < \epsilon$. But then $\epsilon'$ attacks $b$ in contradiction to the conflict-freeness of $E$.

**Proposition 4.8.** We show each postulate:

**Weak Closure under sub-argument:** Suppose that $a = \Gamma \Rightarrow \psi \in E$ and $b = \Delta \Rightarrow \psi \in Sub(a)$ and $b \leq a$. Thus $\Delta \subseteq \Lambda$. Suppose some $c$ attacks $b$. Thus, $b \not< c$ and since $b \leq a$, also $a \not< c$. Hence, $c$ also attacks $a$. Since $E$ defends $a$, it attacks $c$ and so also $b$ is defended by $E$. Since $E$ is complete, $b \in E$.

**Closure of extensions:** $Con(E) \subseteq CN_L(Con(E))$ holds by the reflexivity of $L$. For $Con(E) \supseteq CN_L(Con(E))$, let $\phi \in CN_L(Con(E))$. Since $L$ is finitary, there are $\phi_1, \ldots, \phi_n \in Con(E)$ for which $\phi_1, \ldots, \phi_n \Rightarrow \phi$ is derivable in $C$. Thus, there are $a_1 = \Gamma_1 \Rightarrow \phi_1, \ldots, a_n = \Gamma_n \Rightarrow \phi_n \in E$. By $n$ applications of cut, $\Gamma_1, \ldots, \Gamma_n \Rightarrow \phi \in E$. Hence $\phi \in Con(E)$.

**Consistency:** Suppose that $Con(E)$ is inconsistent. There, thus are $\phi_1, \ldots, \phi_n \in Con(E)$ for which $\Rightarrow \neg \bigwedge_{i=1}^n \phi_i$ is derivable in
C. Since there are $a_1 = \Gamma_1 \Rightarrow \phi_1, \ldots, a_n = \Gamma_n \Rightarrow \phi_n \in \mathcal{E}$, by Lemma 4.7 $\Gamma_1, \ldots, \Gamma_n \Rightarrow \phi_1, \ldots, \phi_n \in \mathcal{E}$. By Contraposition $\bigwedge_{i=1}^n \phi_i$ is derivable in $C$. Now, by Cut and Lemma 4.7, $\Gamma_1, \ldots, \Gamma_n \Rightarrow \neg \gamma \in \mathcal{E}$. By Contraposition, Monotonicity and Lemma 4.7, $b = \Gamma_1, \ldots, \Gamma_n \Rightarrow \neg \gamma \in \mathcal{E}$ where $\gamma \in \Gamma_1$. Clearly, $b$ attacks $a_i$, which contradicts the conflict-freeness of $\mathcal{E}$.

**Weak exhaustiveness:** Suppose that $a = \Delta \Rightarrow \psi \in \mathcal{AF}_{\mathcal{E}}(S)$ such that $\Delta \subseteq \bigcup_{b \in \mathcal{E}} \text{Sup}(b)$. Since $\Delta$ is finite, there are $b_1, \ldots, b_n \in \mathcal{E}$ such that $\Delta = \text{Sup}(b_1) \cup \ldots \cup \text{Sup}(b_n)$. By $n - 1$ applications of Lemma 4.7, $a \in \mathcal{E}$.

**Free precedence:** Assume that ConUcut is part of the attack rules as well. Let $a = \Gamma \Rightarrow \phi$ where $\Gamma \in \text{Free}(\mathcal{S})$. In particular, $\Gamma$ is consistent, and so $a$ cannot be ConUcut-attacked. Suppose that $b = \Delta \Rightarrow \delta$ attacks $a$. Then $\delta \leftrightarrow \neg \gamma$ is derivable in $C$ for some $\gamma \in \Gamma$. By Cut, $\Delta \Rightarrow \neg \gamma$ is also derivable and Cut and Contraposition again show that $\neg \gamma \Rightarrow \neg (\Delta \cup \{\gamma\})$ is derivable in $C$. Since $\gamma$ is not a member of a minimally inconsistent subset of $\mathcal{S}$, there is a $\Theta \subseteq \Delta$ for which $\neg \gamma \Rightarrow \neg \Theta$ is derivable in $C$. Thus, $b$ is attacked by $c$. Since $c$ has no attackers, $c \in \mathcal{E}$. Thus, $\mathcal{E}$ defends $a$ and thus $a \in \mathcal{E}$ by the completeness of $\mathcal{E}$. □

Some negative results are reported next:

(1) Exhaustiveness is not satisfied by every framework that satisfies the requirements of Proposition 4.4 (just Weak exhaustiveness is satisfied):

**Example 4.8.** Let $S = \{p \land q, s, t, \neg \gamma \land \neg p\}$ and assume $\pi(p \land q) = 1, \pi(q) = 3, \pi(s) = \pi(\neg s) = \pi(t \land \neg s) = \pi(t \land \neg s) = 2$ where $\pi(\gamma) = \pi(\neg \gamma) = 2$ as in Example 3.2. Here, $\mathcal{E} = \{\neg \phi \mid \phi \in \mathcal{S}\} \cup \{p \land q \Rightarrow \neg \phi \land \neg p \land q \mid \phi \in \mathcal{S}\}$ is a complete extension. Note that $q \Rightarrow q \notin \mathcal{E}$. The reason is that $s, t \land \neg p \Rightarrow \neg q$ attacks $q \Rightarrow q$, while no argument in $\mathcal{E}$ attacks $s, t \land \neg p \Rightarrow \neg q$. Moreover $\mathcal{E}$ does not defend any other argument in $\mathcal{AF}_{\mathcal{E}}(\mathcal{S}) \setminus \mathcal{E}$.

(2) Consistency does not hold for Undercut:

**Example 4.9.** Consider the flat framework $\mathcal{AF}_{\mathcal{CL}}(\mathcal{S})$ of Example 2.6, for $S = \{p, q, \neg p \land q\}$. It can be shown that $S = \{a_1, a_2, a_3, a_4, a_5\}$ is admissible in $\mathcal{AF}_{\mathcal{CL}}(\mathcal{S})$, however, $\text{Con}(\mathcal{S})$ is inconsistent.

(3) Sub-argument closure for complete extensions does not hold when ConUcut is part of the system:

**Example 4.10.** Let $S = \{p \land q, s, r, \neg p \land \neg s\}$ and assume $\pi(p \land q) = 1, \pi(r \land \neg p \land \neg s) = 2$ and $\pi(s) = 3$ where $\leq \pi$ is as in Example 3.2. Item 1. Note that there is a complete extension with $a = p \land q, s \Rightarrow s$ but without $b = s \Rightarrow s$. This follows since the only attacker of $a$ is $c = p \land q, r \land \neg s \Rightarrow \neg s$, but $c$ is ConUcut-attacked and thus cannot be defended.  

### 4.3 Conflict preservation

An attack in a sequent-based argumentation framework $\mathcal{AF}_{\mathcal{L}}(\mathcal{S})$ will not always be successful in the prioritized argumentation framework $\mathcal{AF}_{\mathcal{L}}(\mathcal{S})$, because the attacked argument might be ${\leq}_p$-stronger than the attacking argument. This way, it might be that attacks and conflicts are lost. This is sometimes avoided by requiring that attacks are always symmetric (see, e.g., [16]) or by reversing the attacks and rejecting the attacking argument instead of the attacked argument (see, e.g., [13]). In the ASPIC+ framework [19] this is handled taking the structure of arguments into account.

The next proposition shows that argumentation frameworks with priorities are conflict preserving: extensions of the prioritized framework are conflict-free in the non-prioritized case.

**Proposition 4.11.** Let $\mathcal{AF}_{\mathcal{L}}^{\leq}(\mathcal{S}) = \langle \mathcal{AF}_{\mathcal{L}}(\mathcal{S}), \mathcal{AT}, \leq \rangle$ be a prioritized sequent-based argumentation framework with Ucut and/or DUCut that satisfies the requirements of Proposition 4.4 and let $\mathcal{AF}_{\mathcal{L}}(\mathcal{S}) = \langle \mathcal{AF}_{\mathcal{L}}(\mathcal{S}), \mathcal{AT} \rangle$ be the corresponding flat (i.e., preference-free) sequent-based framework. For any completeness-based semantics $\text{sem}(\mathcal{S})$ (Definition 2.7), we have that:

1. Every $\mathcal{E} \in \text{Ext}_{\text{sem}}(\mathcal{AF}_{\mathcal{L}}^{\leq}(\mathcal{S}))$ is conflict-free in $\mathcal{AF}_{\mathcal{L}}(\mathcal{S})$.
2. Every $\mathcal{E} \in \text{Ext}_{\text{sem}}(\mathcal{AF}_{\mathcal{L}}(\mathcal{S}))$ is conflict-free in $\mathcal{AF}_{\mathcal{L}}^{\leq}(\mathcal{S})$.

**Proof.** Let $\mathcal{E} \subseteq \{\text{cmp}, \text{grp}, \text{prf}, \text{stb}, \}$.

1. Let $\mathcal{E} \in \text{Ext}_{\text{sem}}(\mathcal{AF}_{\mathcal{L}}^{\leq}(\mathcal{S}))$. Suppose that there are $a = \Gamma \Rightarrow \gamma$ and $b = \Delta \Rightarrow \delta \in \mathcal{E}$ such that $(a, b) \in \mathcal{AT}$. Assume first that DUCut is the attack rule. By Lemma 4.7 and the monotonicity of $\mathcal{L}$ we have that $a' = \Gamma \Rightarrow \gamma \in \mathcal{E}$. Since $(a', b') \in \mathcal{AT}^{\leq}$ (by left monotonicity of $\leq$) this is a contradiction to $\mathcal{E} \in \text{Ext}_{\text{sem}}(\mathcal{AF}_{\mathcal{L}}^{\leq}(\mathcal{S}))$.

Suppose now that Ucut is the attack rule. Then $\nabla \Rightarrow \gamma \Rightarrow \neg \gamma \Rightarrow \neg \gamma \Rightarrow \neg \gamma \Rightarrow \neg \gamma \Rightarrow \neg \gamma \Rightarrow \neg \gamma$ is derivable in $C$ for some $\gamma \subseteq \Delta$. By Cut $\Gamma \Rightarrow \neg \gamma \Rightarrow \neg \gamma \Rightarrow \neg \gamma \Rightarrow \neg \gamma \Rightarrow \neg \gamma \Rightarrow \neg \gamma \Rightarrow \neg \gamma \Rightarrow \neg \gamma \Rightarrow \neg \gamma \Rightarrow \neg \gamma \Rightarrow \neg \gamma$. Thus, $\mathcal{E}$ attains $a$ and thus $a \in \mathcal{E}$ by the completeness of $\mathcal{E}$.

(2) This follows immediately from the fact that every $\mathcal{R}_{\leq}$-attack is in particular an $\mathcal{R}$-attack. □

By Proposition 4.11 any completeness-based extension of the prioritized framework is still conflict-free in the flat case, and so no conflicts are lost, although, as shown in Example 3.7, the extensions in both frameworks are not the same.

An example, discussed in [19] for the ASPIC+ framework, is the following:

**Example 4.12.** Let $\mathcal{AF}_{\mathcal{L}}^{\leq}(\mathcal{S}) = \langle \mathcal{AF}_{\mathcal{L}}(\mathcal{S}), \mathcal{AT}, \leq \rangle$ be a prioritized sequent-based argumentation framework based on classical logic as the core logic, the attack rules DUCut and ConUcut, and the formulas $S = \{p, q, \neg p\}$, such that $\pi(q) = 1, \pi(\neg p) = 2, \pi(p) = 3$. Some of the arguments of $\mathcal{AF}_{\mathcal{L}}(\mathcal{S})$ are the following:

$a_1 = p \Rightarrow p$
$a_2 = q \Rightarrow q$
$a_3 = \neg p \Rightarrow \neg p$
$a_4 = p \land q \Rightarrow p$
$a_5 = p \land q \Rightarrow q$
$a_6 = p \land q \Rightarrow p \land q$

The preference-based argumentation frameworks (PAFs) of [3, 4] result in a stable extension that contains both $a_3$ and $a_6$, and so consistency is not preserved by PAFs. In our case, when $e.g., l \leq_p \gamma \leq_p (the sixth item in Example 3.2) is taken as the preference ordering, this problem is avoided, since every stable extension that contains
A well-known method for handling inconsistent sets of formulas is
\(T\) (since [10] it extends the results in [6, 8] to the prioritized case.

A REASONING WITH MAXIMALLY
CONSISTENT SUBSETS

A well-known method for handling inconsistent sets of formulas is

**Definition 5.1.** Let \(L = \langle L, \vdash \rangle\) be a propositional logic and \(S\) a set
of \(L\)-formulas. We denote by \(\text{MCS}_L(S)\) the set of all the maximally
consistent subsets of \(S\) (with respect to \(\subseteq\)).

- \(S \models_{\text{mcs}} \psi\) if and only if \(\psi \in \text{CN}_L(\bigcap \text{MCS}_L(S))\),
- \(S \not\models_{\text{mcs}} \psi\) if and only if \(\psi \not\in \bigcap \text{MCS}_L(S)\).
- \(S \not\models_{\text{mcs}} \psi\) if and only if \(\psi \not\in \bigcup T \in \text{MCS}_L(S)\).

It has been shown that sequent-based argumentation is a useful
platform for representing and reasoning with maximally consistent
subsets [6, 8]. Here we extend these results to the prioritized case.

We continue to use \(\leq\) as a preference order, determined by \(\pi\),
on sets of \(L\)-formulas. In what follows we shall abbreviate \(\leq\) [resp.
\(\leq\)] by \(\leq\) [resp. \(\leq\)] and write \(T \prec S\) to denote that \(T \leq S\) and \(S \not\prec T\). Accordingly, the set of the \(\leq\)-most preferred maximally
consistent subsets of an \(S\) is defined as follows:

**Definition 5.2.** \(\text{MCS}_L^\pi(S) = \{T \in \text{MCS}_L(S) | \not\models T \prec T\} \subseteq \text{MCS}_L(S)\)
such that \(T \not\prec T\).

**Example 5.3.** Consider again the set \(S = \{p, q, \neg p \lor \neg q\}\) from
Example 2.6 and the priority assignment \(\pi\) from Example 3.3 on
\(S\). We have that: \(\text{MCS}_L(S) = \{p, q\}, \{p, \neg p \lor \neg q\}, \{q, \neg p \lor \neg q\}\).
When \(\leq\) is the preference order as in Items 2, 4, 5 (when e.g. \(f\) is
the average function), 6 and 7 of Example 3.2, we get: \(\text{MCS}_L^\pi(S) = \{p, q\}\). When \(\leq\) is as in Item 1 of Example 3.2 we have that
\(\text{MCS}_L^\pi(S) = \{p, q\}, \{p, \neg p \lor \neg q\}\).

Now we can consider the prioritized versions of the entailment
relations from Definition 5.1.

**Definition 5.4.** For a propositional logic \(L = \langle L, \vdash \rangle\), a set \(S\)
of \(L\)-formulas, and a priority function \(\pi\) on \(L\), we define:

- \(S \models_{\pi, \text{mcs}} \psi\) if and only if \(\psi \in \text{CN}_L(\bigcap \text{MCS}_L(S))\);
- \(S \not\models_{\pi, \text{mcs}} \psi\) if and only if \(\psi \not\in \bigcap \text{MCS}_L(S)\);
- \(S \not\models_{\pi, \text{mcs}} \psi\) if and only if \(\psi \not\in \bigcup T \in \text{MCS}_L(S)\).

**Example 5.5.** In Example 5.3 we have that \(S \not\models_{\pi, \text{mcs}} \psi\) for
every \(\psi \in S\), but \(S \models_{\pi, \text{mcs}} \psi\) for \(\psi \in \{p, q\}\).

The main result of this section is given in the next proposition.
It extends the results in [6, 8] to the prioritized case.

**Proposition 5.6.** Let \(L = \langle L, \vdash \rangle\) be a contraposition propositional
logic, \(S\) a finite set of \(L\)-formulas, and \(\pi\) a priority relation on \(L\).

Let \(\leq\) be a monotonic and transitive preference relation on sets of
formulas that is induced by \(\pi\), and let \(a \leq b\) iff \(\text{Sup}(a) \leq \text{Sup}(b)\) be
the induced preference relation on arguments. Denote by \(\text{MCS}_L^\pi(S) = \langle \text{Arg}_L(S), \text{AT}, \leq \rangle\) the corresponding prioritized framework where
\(\text{AT}\) is based on the rules \(\text{DUC}\) and \(\text{ConUC}\). Then:

1. \(S \models_{\text{mcs}} \phi\) iff \(S \models_{\text{mcs}} \phi\) iff \(S \models_{\text{mcs}} \phi\);
2. \(S \not\models_{\text{mcs}} \phi\) iff \(S \models_{\text{mcs}} \phi\) iff \(S \models_{\text{mcs}} \phi\);
3. \(S \not\models_{\text{mcs}} \phi\) iff \(S \models_{\text{mcs}} \phi\) iff \(S \models_{\text{mcs}} \phi\).

We sketch here the proof of the first item (the proofs of the other
items are similar). First, some lemmas.

**Lemma 5.7.** If \(T \in \text{MCS}_L^\pi(S)\) and \(S' \subseteq S\) is a consistent set,
then \(S' \not\prec T\).

**Proof.** Since \(S'\) is consistent, there is a \(S'' \in \text{MCS}_L(S)\) such that
\(S' \subseteq S''\). By the left monotonicity of \(\leq\), \(S'' \not\prec S'\). Since \(T \in \text{MCS}_L^\pi(S)\), \(S'' \not\prec T\) and by the transitivity of \(\leq\) also \(S' \not\prec T\).

**Lemma 5.8.** If \(S\) is finite and \(T \in \text{MCS}_L^\pi(S)\), then \(\text{Arg}_L(T) \subseteq \text{Stb}(\text{AF}_L^\pi(S))\).

**Proof.** Suppose that \(T \in \text{MCS}_L^\pi(S)\) and \(E = \text{Arg}_L(T)\). We show that \(E\) is stable.

Assume for a contradiction that there are \(a = \Gamma \Rightarrow \gamma\) and \(b = \Delta \Rightarrow \delta\) in \(E\) such that \(a\) attacks \(b\). Then \(\Rightarrow \gamma \leftrightarrow \delta\), where \(\delta' \in C\), is derivable in \(C\). But then \(\Rightarrow \gamma \wedge \Theta \Rightarrow \delta'\) is derivable in \(C\) by Cut and Contraposition. Since \(\Gamma \cup \{\delta'\} \subseteq T\) this is a contradiction to the consistency of \(T\).

Suppose that \(b = \Theta \Rightarrow \tau \in \text{Arg}_L(S) \setminus E\). Thus, \(\Theta \not\setminus T\).
Suppose first that \(\Theta\) is inconsistent. Then \(\Rightarrow \gamma \wedge \Theta \Rightarrow \delta'\) is derivable in \(C\), it attacks \(b\) and it is in \(E\) since it has no attackers.

Now suppose that \(\Theta\) is consistent. By Lemma 5.7, \(\Theta \not\prec T\). Let \(r \in \Theta \setminus T\). Thus, there is a finite \(\Gamma' \subseteq T\) for which \(\Gamma' \Rightarrow \tau\) is derivable. By monotonicity, \(a = T \Rightarrow \tau\in E\).
Since \(\Theta \not\prec T\), \(a\) attacks \(b\).

Thus, whether \(\Theta\) is consistent or not, we have shown that \(E\)
attacks any argument in \(\text{Arg}_L(S) \setminus E\), and so \(E\) is stable.

**Lemma 5.9.** If \(E \in \text{Cmp}(\text{AF}_L^\pi(S))\), there is a \(T \subseteq S\) for
which \(E = \text{Arg}_L(T)\).

**Proof.** Let \(T = \bigcup \{\sup(a) | a \in T\} \subseteq T\) and \(b = \Gamma \Rightarrow \phi \in \text{Arg}_L(S)\).
By weak exhaustiveness (Proposition 4.4), \(b \in E\). Thus, \(E = \text{Arg}_L(T)\).

**Lemma 5.10.** If \(S\) is finite and \(E \in \text{Prf}(\text{AF}_L^\pi(S))\), there is some
\(T \in \text{MCS}_L^\pi(S)\) for which \(E = \text{Arg}_L(T)\).

**Proof.** By Lemma 5.9 there is a \(T \subseteq S\) for which \(E = \text{Arg}_L(T)\).
Assume first for a contradiction that \(T\) is inconsistent. Thus, there is a \(\Gamma \subseteq T\) for which \(a = \Rightarrow \gamma \wedge \Gamma\) is derivable in \(C\). By weak exhaustiveness (Proposition 4.4), \(b = \Gamma \Rightarrow \gamma \wedge \Gamma \in E\). It is attacked by \(a\) and cannot be defended, which is a contradiction to the fact
that \(E\) is admissible. Thus, \(T\) is consistent.

Suppose now for a contradiction that there is a \(T' \in \text{MCS}_L^\pi(S)\)
for which \(T' \not\prec T\). By Lemma 5.8, \(\text{Arg}_L(T') \subseteq \text{Stb}(\text{AF}_L^\pi(S))\).

---

10As noted in [19], in ASPIC+ this problem is avoided as well.
Since $E \in \text{Prf}(\mathcal{A}\mathcal{F}_{L}^{=}(S))$, $\text{Arg}_{\text{GL}}(T) \setminus \text{Arg}_{\text{GL}}(T') \neq \emptyset$, thus there is a $\gamma \in T' \setminus T$. Then there is a $\Delta \subseteq T'$ for which $\Delta \Rightarrow \neg \gamma$ is derivable in $C$. By monotonicity, also $c = T' \Rightarrow \neg \gamma$ and $d = C \Rightarrow \gamma$ are derivable in $C$ (note that $\gamma \Rightarrow \gamma$ is derivable as well). Since $T' \subset T$, $c$ attacks $d$. Thus, there is a $\epsilon = \emptyset = \theta \in E$ which attacks $c$. Hence, $T' \not\subseteq \Theta$. By left monotonicity and since $T \subseteq T', T \subseteq \Theta$. By transitivity, $T' \subseteq \Theta$, which is a contradiction. Altogether, this shows that $T \in \text{MCS}_{L}^{=}(S)$.

Now we can show Proposition 5.6.

Proof.

($\Rightarrow$) Suppose that $S \models_{L,\text{Lmc}} \phi$ and let $E \in \text{Prf}(\mathcal{A}\mathcal{F}_{L}^{=}(S))$. By Lemma 5.10, there is a $T \in \text{MCS}_{L}^{=}(S)$ for which $E = \text{Arg}_{\text{GL}}(S)$. By the assumption $T \vdash \phi$, hence $T \Rightarrow \phi \in E$. This shows that $S \models_{L,\text{Lprf}} \phi$, which also implies that $S \models_{L,\text{Lsub}} \phi$.

($\Leftarrow$) Suppose that $S \models_{L,\text{Lsub}} \phi$ and let $T \in \text{MCS}_{L}^{=}(S)$. By Lemma 5.8, $E = \text{Arg}_{\text{GL}}(T) \in \text{Stb}(\mathcal{A}\mathcal{F}_{L}^{=}(S))$. Thus, there is a $\Delta \models \phi \in E$ for which $\Delta \subseteq T$. Hence $T \vdash \phi$, which shows that $S \models_{L,\text{Lmc}} \phi$.

Remark 5. Lemma 5.10 (and so Proposition 5.6) does not hold for infinite sets (for instance, for the orderings in Example 3.2, Items 6 and 7). Here is an example: let $\mathcal{A}\mathcal{F}_{C_{1}}^{=}(S) = (\text{Arg}_{C_{1}}(S), \mathcal{A}\mathcal{T}, \leq)$ be a prioritized sequent-based argumentation framework, with $\mathcal{C}_{1}$ as core logic, $\text{DUcut}$ and $\text{ConUcut}$ as attack rules and $S = \{p_{i} \mid i \geq 1\} \cup \{q, \neg q\}$ where $\pi(p_{1}) = 1$ for all $i \geq 1$, $\pi(q) = 2$ and $\pi(\neg q) = 3$. We have two MCSs, $T = \{p_{i} \mid i \geq 1\} \cup \{q\}$ and $T' = \{p_{i} \mid i \geq 1\} \cup \{\neg q\}$ where $T < T'$. Nevertheless, $\text{Arg}(T')$ is a stable extension of $\mathcal{A}\mathcal{F}_{C_{1}}^{=}(S)$.

6 CONCLUSION

Sequent-based argumentation frameworks provide general and modular formalisms for representing arguments and reasoning with them, using different kinds of languages, logics, and attacks. The goal of this work is to carry these formalisms a step forward and to incorporate external information in the form of priorities that the reasoner might want to introduce for properly choosing the arguments that can be mutually accepted. Once the priorities have been decided, different orders may be defined for making preferences among the underlying arguments, and accordingly applying entailment relations for drawing conclusions from a given set of assertions.

Clearly, the entailment relations that are induced by a prioritized framework depend on many factors, among which are the choice of the core logics, the attack relations, and the preferences among the arguments. We have shown that in many cases these choices provide the reasoner with a robust framework, satisfying rationality postulates and enjoying other desirable properties, like conflict preservation and strong links to reasoning with the most preferred maximally consistent subsets of the premises, a well-studied approach for handling inconsistent information.

There are several other formalisms for supporting prioritized data in the context of argumentation systems. A detailed comparison to some of these formalisms will be provided in the full version of this work. Here we only mention one of them, the ASPIC+ framework [19], which also provides a general setting for (prioritized) logical argumentation. Apart of the different representations of objects (like arguments and attacks) in the two frameworks, a primary difference from ASPIC+ is that our approach is more proof theoretically oriented, using tools and methods (like sequents and their derivations by proof systems) from proof theory. Among others, in future work we plan to strengthen this characteristic of our approach and provide dynamic proof systems [9] for nonmonotonically reasoning with the prioritized data in a proof-like manner.

We conclude this paper by exemplifying some of the advantages of our approach using the puzzle given in the introduction (Example 1.1).

Example 6.1 (Example 1.1 continued). Recall the flat owner negotiating with potential tenants about the construction of a swimming pool (s), a tennis-court (t) and a private car-park (p). The consideration that the rent (r) increases if more than one facility is constructed can be represented by the formula $\psi_{1} = r \leftrightarrow ((s \land t) \lor (s \land p) \lor (t \land p))$. The preferences of the tenants not to increase the rent and not to have two sport facilities are modeled by $\neg r$ and by $\psi_{2} = s \land t$ and $\psi_{3} = t \land s$, respectively.

This situation may be represented by a prioritized sequent-based framework $\mathcal{A}\mathcal{F}_{C_{1}}^{=}(S) = (\text{Arg}_{C_{1}}(S), \mathcal{A}\mathcal{T}, \leq)$, induced by classical logic, $\text{Ucut}$ and $\text{ConUcut}$ as attack rules, and set of formulas $S = \{s, t, r, \neg r, \psi_{1}, \psi_{2}, \psi_{3}\}$, where $\pi(\neg r) = 1$, $\pi(\psi_{1}) = \pi(\psi_{2}) = \pi(\psi_{3}) = 2$ and $\pi(s) = \pi(t) = \pi(p) = 3$. We take the preference relation by the $\leq_{3}$ comparison (Item 6 of Example 3.2).

• We have that $\neg r \leftrightarrow (\neg (s \land t) \land \neg (s \land p) \land \neg (t \land p))$ classically follows from $\psi_{1}$, which implies that $\neg r, \neg \psi_{1} \Rightarrow \neg (s \land y)$ is in $\text{Arg}_{C_{1}}(S)$ for every distinct $x,y \in \{s,t\}$. It follows that every argument of the form $x, y \Rightarrow x \land y$ for each $x,y$ (suggesting to construct two facilities) is $\text{Ucut}$-attacked by a more preferred argument.

• Arguments such as $s \Rightarrow s$ and $t \Rightarrow t$, which suggest to construct swimming pool and tennis court are respectively attacked by the more preferred arguments $t, \psi_{3} \Rightarrow s$ and $s, \psi_{2} \Rightarrow t$.

• The argument $a = p \Rightarrow p$, suggesting to construct a car park, is attacked by $b = \neg r, s, \psi_{1} \Rightarrow \neg p$. However, the argument $a' = p, \neg r, \psi_{2}, \psi_{3} \Rightarrow p$ for the same conclusion is not attacked by $b$ since $a' < b$. In fact, $a'$ is only attacked by arguments whose support set is classically inconsistent, for instance $S \Rightarrow \neg p$. These attacks are counter $\text{ConUcut}$ attacked by the tautological argument $\Rightarrow \neg \land S$ and so $a'$ is defended.

From the above considerations it follows that the only sequents of the form $\Gamma \Rightarrow x$ for some $\Gamma \subseteq S$ and $x \in \{s,t,p\}$ that belong to the grounded extension of the prioritized sequent-based argumentation framework under consideration, are those in which $x = p$. That is, based on the considerations and the preferences stated above, according to the grounded semantics of the framework, the flat owner should decide to build only a parking lot.
ACKNOWLEDGEMENT

The first two authors are supported by the Israel Science Foundation (grant 817/15). The second and the third author are supported by the Alexander von Humboldt Foundation and the German Ministry for Education and Research.

REFERENCES


