Buyer-Optimal Distribution*

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ABSTRACT

We consider the problem of how a buyer can optimize his utility if he can choose his own valuation distribution in a prior-dependent auction, such as the revenue-optimal auction [18]. The problem is motivated by and equivalent to a type of the market segmentation problem [3], where a principal tries to select a subset of agents (i.e., a market segment) from the set of all agents, each with a constant valuation, to attend a posted price auction for selling multiple identical items, in order to maximize the total utilities of the agents selected into the market segment. Our results are closed-form solutions in both the single buyer case as well as the multi-buyer case where several buyers best response to each other. Interestingly, in the two-buyer case, essentially all commitments that satisfy a certain condition are equilibria.

CCS CONCEPTS

• **Theory of computation → Computational pricing and auctions**;

KEYWORDS

buyer-optimal distribution; principle; best response; equilibrium

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1 INTRODUCTION

In the standard theory of Bayesian mechanism design, it is common knowledge among the seller and buyers that there is a joint valuation distribution of the buyers. However, there are circumstances where the buyers are free to choose such a distribution. The central question studied in this paper is:
Given a revenue-maximizing auctioneer, what is the optimal valuation distribution for the buyers?

Consider the following single buyer example. The seller is a Myersonian monopolist [18] who commits to the use of the revenue optimal posted pricing, given the buyer's valuation distribution. The buyer can choose any type distribution on $[0, 1]$. What is the optimal distribution that maximizes the buyer's expect utility? Clearly, having a large distribution, a distribution with a large expectation, does not help in this case. For example, if the buyer has a point mass distribution at 1, he will end up with 0 utility since the seller will set a posted price at 1. One may wonder then, perhaps, the best shot for the buyer is a uniform distribution on $[0, 1]$, which yields an expected utility of 0.125 for the buyer. It turns out that this isn't optimal for the buyer either. As we shall show in this paper, the optimal distribution for the buyer is a piecewise distribution consisting of a piece of the *equal-revenue distribution* [12, 13] and a point mass at 1. Intuitively, an equal-revenue distribution is a distribution on $[c, \infty)$, where c is a positive number. When the buyer's value follows the equal-revenue distribution, no matter what price the seller sets, she will get the same revenue. It is without loss of generality to assume that the seller will set the lowest price thus the best price for the buyer. The buyer optimal distribution is in fact a truncation of the equal-revenue distribution at 1. In this case, the buyer will have an expected utility of $1/e \approx 0.36$, which is almost 3 times as higher as the uniform distribution.

The main part of the paper is a closed-form solution for the multiple-buyer case, where the buyer needs to come up with a good distribution in order to best respond to the distributions reported by the other buyers. The seller will still set a posted price (as opposed to run the Myerson auction) for the mixed distribution reported by buyers.

1.1 Motivation

The above abstract problem stems from a number of realistic scenarios. In a standard online ad scenario, where a principal, say an advertisement agency (aka, a demand side platform (DSP)) who represents a population of advertisers, each with a publicly known constant valuation, needs to select a subset of its advertisers (i.e., a market segment [3]) to attend a posted price auction for multiple online impressions. Due to the Myersonian nature (the auctioneer runs the Myerson auction) of the auction, the principal needs to carefully select its advertisers according to their valuations in order to maximize their total utility, a fixed fraction of which becomes her commission (utility). So the problem becomes the abstract maths problem described above: how to find the optimal buyer distribution? The problem also naturally extends to the case where multiple agencies, each represents a population of advertisers, attend an ad exchange [2, 16] in which a revenue maximizing posted price is set to sell multiple units of impressions. Similar examples bound, ranging from how to select an miner team to exploit a mine, to how to select a team to build a public project.

1.2 Related work

Perhaps the most relevant work is a recent independent draft [9], where the authors analyze a single-buyer single-item auction where the buyer can choose any valuation distribution, with support on

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[0,1], and the seller offers a posted price to the buyer. The problem they study is exactly our problem in the single buyer case. In contrast, our motivation admits a principal-agent interpretation, which naturally extends to the multiple-buyer case and we give a closed-form solution in this more technically challenging case.

Two papers by Burkett [6, 7] analyze a similar setting where the agent submits his bids through a principal to participate in a single item auction, due to the fact that the agent can not directly interact with the seller (say, a publisher). The agent and the principal then share the profit according to a predefined, fixed rate. In this paper, we analyze the case where each agent has a publicly known, constant value, thus the strategic behavior at the agent level is not considered.

This work also relates to a series of literature that consider the setting where the buyer can strategically interact with the seller via repeated auctions. A series of papers [1, 8, 10, 15, 17] focus on the dynamic and learning aspect of the problem where the buyer learns to play equilibrium. Roesler and Szentes [21] consider a buyer's optimal distribution with fixed expected value under a "mean-spread" condition. Bergemann et al. [3] consider the setting that an agent's prior distribution can be decomposed into a linear combination of several posterior distributions. They characterize the range of the combination of buyer and seller surplus that is implementable by a decomposition. Tang and Zeng [24] consider a series of prior-dependent auctions that each buyer can report a "fake distribution", while each buyer also has his true distribution. In contrast, our work focus on principal's optimal distribution with fixed support and the principal does not have a true valuation distribution.

A recently well-studied model, called persuasion model, is also related to this paper. A series of literature [4, 14, 19, 20, 22] study the general problem of a sender strategically revealing information based on external signals and give a method to find the optimal signaling scheme for the sender in a number of realistic scenarios. Our model allows the principal to reveal any information, thus fundamentally different from those works in which there is a prior distribution.

1.3 A summary of results

We now summarize the results we obtain with respect to the buyer optimal distribution problems we study.

(1) We show that in the single buyer setting, the optimal distribution consists of an equal-revenue distribution starting from $\frac{1}{e}$ and truncated with a point mass at 1. The result is consistent with the [9] but with a potentially clearer proof.
(2) For the multi-buyer case, we give a closed-form of the symmetric equilibrium, in which each buyer's distribution consists of an equal-revenue distribution starting at $ne^{-\frac{1}{n}}$, and truncated with a point mass at 1.
(3) For the multi-buyer case, we give a characterization of one's best response distribution, given any other buyers' distributions. Based on characterization, we give a necessary and sufficient condition of a distribution profile to be an equilibrium. In other words, we characterize all equilibria in the multi-buyer setting.
(4) In the two-buyer case, we show an interesting result that all commitments [23, 25] with median value $e^{-\frac{1}{2}}$ are all equilibria.

2 SETTINGS

Consider an auction where the seller sells a single item to a buyer. The buyer chooses his cumulative valuation distribution F with domain $\mathbb{S}$, subject to $\mathbb{S} \subseteq [0, 1]$. The seller set a take-or-leave price p that maximizes his expected revenue according to F. The goal of the buyer is to choose F that maximizes his expected utility.

For the ease of representation, we redefine the cumulative function of a random variable:

Definition 2.1. Given a random variable v, define its cumulative function F to be

$$F(v') = Pr_v[v < v']$$

where the right-hand side represents the probability that the random variable v takes on a value less than v'.

We assume that the buyer will buy the item if his value is exactly p. So by our redefinition, the seller's expected revenue is always $p(1 - F(p))$, no matter whether F has a point mass at p or not. If there is no point mass, our redefinition is the same as the standard definition, $F(v') = Pr_v[v \leq v']$. If there is a point mass on p, only the value $F(p)$ is different from the standard definition, and the set of such inconsistencies is measured zero. In this paper, unless specified otherwise, all of our results and proofs hold everywhere except on a measure zero set of values.

Based on Definition 2.1, we are able to define the single buyer problem:

The single buyer problem.

$$\begin{aligned} \max \quad & \int_{v \in \mathbb{S}} (v - p)^+ \, \mathrm{d}F(v) \\ s.t. \quad & (1 - F(v))v \leq p(1 - F(p)), \ \forall v \in \mathbb{S} \\ & 0 \leq v \leq 1, \ \forall v \in \mathbb{S} \end{aligned}$$

where $x^+ = \max\{x, 0\}$.

By the definition above, we know that if there are multiple optimal reserve prices, the seller will choose the minimum one.

As mentioned, the single buyer problem is motivated by the following multi-item auction problem:

Problem 1. *The seller has M (sufficiently large) identical items for sale. A principal can choose any number of unit-demand agents whose valuations are constant and in* $[0, 1]$. *Upon observing the set of selected buyers and their values, the seller sets a price p for one unit of item in order to maximize revenue. Agents with valuation v_i greater than p are allocated one item and a utility $v_i - p$. Others get nothing and pay nothing.*

The principal aims to maximize the total agents' surplus it selects.

It is assumed that the whole population of the set of agents is large enough so that the principal can freely choose agents with any valuation distribution, either discrete or continuous. The types of all agents are normalized such that the highest value is 1.

It can be shown that the problem 1 above is equivalent to the single buyer problem defined earlier. Suppose n agents are chosen, without loss of generality, one can assume that $M = n$: if $n > M$, agents whose value are less than the M-th largest value will never buy an item, and if $n < M$, we can always hire additional $M - n$ agents with zero valuation. The seller's revenue is $Mp(1 - F(p))$ and the total utility of agents is $\mathbb{E}_v[(v - p)^+]$, if the price is at p and the agents' valuations form distribution F. So if we take the distribution in the single buyer problem as the principal's action in problem 1, the two problems are identical.

We then introduce a convenient representation of a distribution called *quantile* [5], which we are essential for our technical derivations.

Definition 2.2. (Quantile). Given a random variable v with cumulative distribution $F(v)$ and support $\mathbb{S}$, denote $q = 1 - F(v) \in [0, 1]$ as the quantile of v. The function $V(q) = F^{-1}(1 - q) : [0, 1] \mapsto \mathbb{S}$ is a mapping from the quantile space to value space, and is uniquely determined by distribution F. [1]

The quantile q of a value $V(q)$ can be regarded as the rank of the value in the distribution and is uniformly distributed in [0,1]. $V(q)$ is decreasing with q, i.e. a larger quantile corresponds to a smaller value. When we use V, we represent a distribution in the quantile space, i.e. $V(q), q \in [0, 1]$, while we sometimes still use F to represent the cumulative distribution in the value space, i.e $F(v), v \in \mathbb{S}$.

We rewrite the single buyer problem in the quantile space: Note that the seller's optimal price p is always in the support of the buyer's distribution, i.e. $p \in \mathbb{S}$, so we can assume that $p = V(q^*)$, where q^* is called the reserve quantile.[2]

$$\begin{aligned} \max \quad & \int_0^1 (V(q) - V(q^*))^+ \,\mathrm{d}q \qquad (1) \\ s.t. \quad & qV(q) \le q^* V(q^*),\ \forall q \in [0, 1] \\ & 0 \le V(q) \le 1,\ \forall q \in [0, 1] \end{aligned}$$

We then give a definition of the equal-revenue distribution as mentioned in the introduction, of which most of our results take the form.

Definition 2.3. The equal-revenue distribution [13] has cumulative function

$$F(v) = 1 - c/v, v \in [c, +\infty)$$

where $c > 0$ is a constant.

Equivalently, in the quantile space,

$$V(q) = c/q, q \in (0, 1].$$

The equal-revenue distribution has the property that the seller's revenue is indifferent for any posted-price in its support.

3 THE SINGLE BUYER CASE

In this section, we solve the single buyer problem.

Theorem 3.1. *The following V maximize the buyer's utility for the single-buyer problem.*

- $V(q) = 1,\ \forall 0 \le q \le \frac{1}{e}$
- $V(q) = \frac{1}{eq},\ \forall \frac{1}{e} < q \le 1$

Equivalently, we write the distribution in the value space:

- $F(v) = 0,\ \forall v \le \frac{1}{e}$
- $F(v) = 1 - \frac{1}{ev},\ \forall \frac{1}{e} < v \le 1$
- $F(v) = 1,\ \forall v > 1$

To prove the theorem, we first show that it is without loss of generality to only consider distributions such that the seller sets the price at $V(1)$ (the minimum value of the support). Then we prove that the optimal distribution V is only determined by the value of $V(1)$. That is, the buyer's utility can be written as a function of $V(1)$, so we can compute the optimal $V(1)$ by the first-order condition.

Lemma 3.2. *It is without loss of generality to only consider the buyer's distribution V such that the seller set the price at $V(1)$ (the minimum value in the support)*

Proof. Suppose that the buyer chooses the distribution V and we assume that the reserve quantile $q^* < 1$. Then we construct a new distribution V^* as follows:

- $V^*(q) = V(q),\ \forall 0 \le q \le q^*$
- $V^*(q) = \frac{q^* V(q^*)}{q},\ \forall q^* < q \le 1$

First the monotonicity of $V^*(q)$ is satisfied so V^* is a distribution. Since $q^* \in \arg\max_q qV(q)$, so $1 \in \arg\max_q qV^*(q)$ and by definition, the monopoly price is $V^*(1)$ for V^*. Also, V^* produces the same utility as V for any $q \in [0, q^*]$, while for $q \in (q^*, 1)$, the utility from V is 0 and from V^* is a non-negative amount $V^*(q) - V^*(1)$. Thus the function V^* weakly dominates V. □

Lemma 3.3. *Given that the seller sets the price at p, it is without loss of generality to only consider the buyer's distribution V with the following form:*

- $V(q) = 1,\ \forall 0 \le q \le p$
- $V(q) = \frac{p}{q},\ \forall p < q \le 1$

Proof. First the monotonicity of $V(q)$ is satisfied so V is a distribution, and by Lemma 3.2 it is without loss of generality to assume $p = V(1)$, so

$$qV(q) \le V(1) \Rightarrow V(q) \le V(1)/q,\ \forall q \in (0, 1].$$

Otherwise the seller prefers to set a price at some $V(q)$ other than $V(1)$.

By definition we also have $V(q) \le 1, \forall q \in [0, 1]$, so $V(q) \le \min\{1, v(1)/q\}$

Note that the utility function (1) is monotone increasing with each $V(q)$ for any $q \in [0, 1]$, so the lemma is proved. □

With the above two lemmas, we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. By Lemma 3.2 and Lemma 3.3, we only need to determine $V(1)$

The objective function in (1), denoted by U, is

$$\begin{aligned} U &= \int_0^1 (V(q) - V(1))^+ \,\mathrm{d}q \\ &= \int_0^{V(1)} (1 - V(1)) \,\mathrm{d}q + \int_{V(1)}^1 (V(1)/q - V(1)) \,\mathrm{d}q \\ &= V(1) \ln V(1). \end{aligned}$$

[1]For the case where the distribution has a point mass at value v_0 with probability p_0, $V(q)$ is defined to be v_0 for all $q \in [1 - F(v_0), 1 - F(v_0) + p_0]$

[2]This formalization is well-defined even when there is a point mass. However in the value space F could be non-differentiable. We can still assume F to be differentiable because one can get the same results using Lebesgue integral. We omit the details here.

To maximize U, we must have $\frac{\mathrm{d}U}{\mathrm{d}V(1)} = 0 \Rightarrow V(1) = \frac{1}{e}$. Thus the theorem is proved. □

We call this distribution the *buyer-optimal distribution*, which consists of an equal-revenue distribution starting from $\frac{1}{e}$ and a point mass at 1. Figure 1 shows the redefined cumulative function of the buyer-optimal distribution.

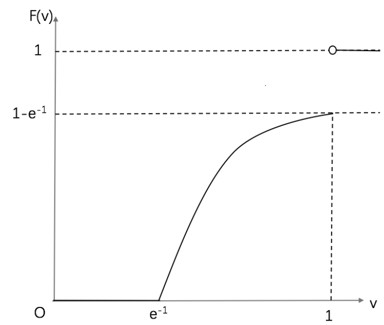

Figure 1: Cumulative function of buyer-optimal distribution

We derive the following interesting corollaries of the buyer-optimal distribution:

Corollary 3.4. *In a single item auction with one buyer, if the buyer chooses the buyer-optimal distribution, then the buyer's utility equals to the seller' revenue.*

In this case, both the buyer's utility and the seller's revenue is $1/e$ fraction of the maximum value in the support.

From the proof of Theorem 3.1, we know the closed-form of the buyer's optimal distribution given the seller's monopoly price. Thus we derive the following:

Corollary 3.5. *In a single item auction with one buyer, if the seller's monopoly reserve price is no less than* $1/e$ *fraction of the maximum value in the support, then the seller's revenue is no less than the buyer's utility. The number* $1/e$ *is tight.*

Proof. We assume, without loss of generality, that the maximum value in the support is 1. Suppose that the buyer's distribution is V, the monopoly reserve price is p, and q_0 is the reserve quantile. So the seller's revenue is $q_0 p$.

Similar to the proof of Lemma 3.2, for any $0 < q \le q_0$, we have

$$qV(q) \le pq_0 \Rightarrow V(q) \le \frac{q_0 p}{q}.$$

So $V(q) \le \min\left\{1, \frac{q_0 p}{q}\right\}$.

In this case, the buyer's optimal utility is

$$\begin{aligned}
& \int_0^1 (V(q) - p)^+ \,\mathrm{d}q \\
= & \int_0^{q_0 p} (1-p)\,\mathrm{d}q + \int_{q_0 p}^{q_0} \left(\frac{q_0 p}{q} - p\right) \mathrm{d}q \\
= & \; q_0 p(1-p) - (q_0 - q_0 p)p + \int_{q_0 p}^{q_0} \frac{q_0 p}{q}\,\mathrm{d}q \\
= & \; q_0 p \ln p.
\end{aligned}$$

So when the seller's revenue is no less than the buyer's utility:

$$q_0 p \ge qp \ln p \Leftrightarrow p \ge \frac{1}{e}.$$

Thus the corollary is proved. □

So if the seller ignores all values of the buyer that less than $1/e$ fraction of the maximum value in the support and runs Myerson auction, then the seller can guarantee his profits to be at least the buyer's. It is obvious that this mechanism is truthful and more efficient than the trivial mechanism that always sets the price equals to half of the maximum value in the support, which can achieve the same goal.

4 THE MULTI-BUYER CASE

In this section, we generalize the single buyer case to the multi-buyer case:

Consider an auction where the seller sells n identical items to n buyers. Each buyer chooses his cumulative valuation distribution F_i on $[0, 1]$. The seller chooses an anonymous price p for each item. Agents with value larger that p buy one unit of item and pay the price. The goal of the seller is to maximize his expected revenue subject to the total allocation probability over n buyers no more than 1. Formally,

The multi-buyer game. *Given* $F_1, F_2, ..., F_n$ *with domain* $\mathbb{S} \subseteq [0, 1]$,

$$p = \arg\max_{p'} \; p' \sum_{i=1}^{n} (1 - F_i(p')) \tag{2}$$

subject to

$$\sum_{i=1}^{n} (1 - F_i(p)) \le 1.$$

Principle i's utility $U_i(F_1, ..., F_n)$ *is*

$$U_i(F_1, ..., F_n) = \int_0^1 (V_i(q) - p)^+ \,\mathrm{d}q. \tag{3}$$

Definition 4.1. A buyer i's distribution F_i is a *best respond* over other buyers' distributions F_{-i} if for all distribution F_i' on $[0, 1]$, in the multi-buyer game,

$$U_i(F_i, F_{-i}) \ge U_i(F_i', F_{-i}).$$

Definition 4.2. A distribution profile $(F_1, ..., F_n)$ is a *Nash-equilibrium* if for each buyer i, F_i is a best response.

Recall that F_i represents the cumulative function of buyer i's distribution in the value space and V_i represent the same distribution in the quantile space. Here the function $1 - F_i$ is the inverse function of V_i. Actually, for any value v, $V_i^{-1}(v) = Pr_{v' \sim F_i}[v' \ge v] = 1 - F_i(v)$.

Also note that

$$\arg\max_{p'} \; p' \sum_{i=1}^{n} (1 - F_i(p')) = \frac{1}{n} p' \left(1 - \frac{\sum_{i=1}^{n} F_i(p')}{n}\right).$$

so for the seller it is equivalent to set the optimal posted price for the mixed distribution reported by buyers, but subject to the constraint that the allocation probability is no more than $\frac{1}{n}$.

The multi-buyer game is motivated by the following problem:

Problem 2. *The seller has at most M (sufficient large) identical items for sale. There are n principals. Each principal can choose any number of unit-demand agents whose values are between* 0 *and* 1. *Upon observing all agents from all principals, the seller then sets a common price p for one unit of item in order to maximize revenue, subject to resource feasibility. Agents (not necessarily belong to the same principal) with valuation greater than p are allocated one item and produces a utility for his principal that equals his value minus p. Others get nothing and pay nothing.*

Similar to the single buyer case, it is without loss of generality to only consider that each principal chooses exact M agents in problem 2. For a sufficiently large population, each principal's action in Problem 2 is to choose a distribution F_i.

When the seller sets the price p, $M(1 - F_i(p))$ agents from principal i are allocated, so the seller's revenue get from buyer i is $Mp(1 - F_i(p))$. Thus the total revenue of the seller is

$$Mp\sum_{i=1}^{n}(1 - F_i(p)).$$

So the objective function (2) follows.

The constraint in the multi-buyer problem, called *feasibility constraint*, is due to the resource feasibility: in problem 2, since the total number of winners is at most M, we have:

$$M\sum_{i=1}^{n}(1 - F_i(p)) \le M.$$

Thus the constraint follows.

Note that if $\sum_i(1 - F_i(1)) > 1$, the feasibility constraint can be never satisfied. This case is trivial as the seller may randomly allocate the item to agents with value 1. We suppose that it is not the case.

4.1 Equilibrium analysis

In this subsection, we give an equilibrium of the multi-buyer game. This equilibrium is the unique symmetric equilibrium if we ignore the values that less than the price given by the seller. (These values do not affect the outcome.)

Theorem 4.3. *Each buyer chooses the following distribution V^* is an equilibrium of the multi-buyer game.*

- $V^*(q) = 1,\ \forall\ 0 \le q \le \frac{v_0}{n}$
- $V^*(q) = \frac{v_0}{nq},\ \forall\ q > \frac{v_0}{n}$

where $v_0 = e^{-\frac{1}{n}}$

Equivalently, we write the distribution in the value space:

- $F(v) = 0,\ \forall v \le \frac{v_0}{n}$
- $F(v) = 1 - \frac{v_0}{nv},\ \forall \frac{v_0}{n} < v \le 1$
- $F(v) = 1,\ \forall v > 1$

Intuitively, by the symmetry, at the equilibrium each buyer should consume $1/n$ fraction of the goods, i.e., $1 - F_i(p) = \frac{1}{n}$. We prove the optimality by the first-order condition.

Proof of Theorem 4.3. Suppose that all V_{-i}'s are equal to V^* in Theorem 4.3, we prove that buyer i's distribution $V_i = V^*$ is a best response. First, given that the seller's optimal price is $p = V^*(q_0)$ (obviously $p \le 1$), then the following two constrains must hold:

(1) $1 - F_i(p) \le 1 - (n-1)q_0$
(2) $V_i(q)(q + (n-1)(1 - F^*(V_i(q)))) \le p((n-1)q_0 + 1 - F_i(p))$, for any q

Where F^*, F_i represent the distribution V^*, V_i in the value space respectively. Constraint (1) is due to fact that the total allocation probability is no more that 1 in the multi-buyer game, and constraint (2) is to ensure that the seller's optimal price is p.

By definition in Theorem 4.3, for any $v < 1$, $1 - F^*(v) = \frac{v_0}{nv}$. By combining constraint (1) and (2) we have

$$\begin{aligned}
&V_i(q)(q + (n-1)[1 - F^*(V_i(q))]\\
=&\ qV_i(q) + \frac{(n-1)v_0}{n}\\
\le&\ p[(n-1)q_0 + 1 - F_i(p)]\\
\le&\ p = V^*(q_0) = \frac{v_0}{nq_0}.
\end{aligned}$$

So

$$V_i(q) \le \frac{v_0}{qn}\left(\frac{1}{q_0} - (n-1)\right).$$

Agent i's utility $U_i(V_i, V^*_{-i})$ is

$$\begin{aligned}
&\int_0^{1-F_i(p)} (V_i(q) - p)\,\mathrm{d}q\\
\le&\int_0^{1-F_i(p)} \left(\min\left\{\frac{v_0}{qn}\left(\frac{1}{q_0} - (n-1)\right), 1\right\} - p\right)\mathrm{d}q\\
\le&\int_0^{1-(n-1)q_0} \left(\min\left\{\frac{v_0}{qn}\left(\frac{1}{q_0} - (n-1)\right), 1\right\} - p\right)\mathrm{d}q.
\end{aligned}$$

In particular, the equalities are reached if

$$V_i(q) = \min\left\{\frac{v_0}{qn}\left(\frac{1}{q_0} - (n-1)\right), 1\right\}$$

. We assume this is the case.

Define q_t to be the largest quantile such that $V_i(q_t) = 1$, so

$$\frac{v_0}{q_t n}\left(\frac{1}{q_0} - (n-1)\right) = 1.$$

We get

$$q_t = \frac{v_0}{n}\left(\frac{1}{q_0} - (n-1)\right).$$

So far we have obtained the closed-form of the best response when the optimal price is given. Next we determine the optimal q_0 for buyer i.

Now buyer i's utility U_i is

$$\begin{aligned}
U_i =& \int_0^{\frac{v_0}{n}\left(\frac{1}{q_0}-(n-1)\right)} \left(1 - \frac{v_0}{nq_0}\right)\mathrm{d}q\\
&+ \int_{\frac{v_0}{n}\left(\frac{1}{q_0}-(n-1)\right)}^{1-(n-1)q_0} \left(\frac{v_0}{nq}\left(\frac{1}{q_0} - (n-1)\right) - \frac{v_0}{nq_0}\right)\mathrm{d}q\\
=&\frac{v_0}{n}\left(\frac{1}{q_0} - (n-1)\right)\ln\frac{nq_0}{v_0}.
\end{aligned}$$

Take derivative of U_i with respect to q_0:

$$\frac{\mathrm{d}U_i}{\mathrm{d}q_0} = \frac{v_0}{n}\left(\left(\frac{1}{q_0} - (n-1)\right)\frac{1}{q_0} - \frac{1}{q_0^2}\ln\frac{nq_0}{v_0}\right).$$

Let the formula above equal to zero and we get $q_0 = \frac{1}{n}$. This solution is unique since the right side times q_0^2 is decreasing with q_0. By the first-order condition we conclude that in buyer i's best response $q_0 = \frac{1}{n}$. (We omit the verification of the second-order condition).

Plug $q_0 = \frac{1}{n}$ into the formula of $V_i(q)$ and we get $V_i(q) = \min\left\{\frac{v_0}{qn}, 1\right\}$, so we conclude that $V_i = V^*$ is a best response. Thus by the symmetry the theorem is proved. □

Figure 1 shows the revenue curve [5] of the value distribution of all M winners (buyers who buy an item), in the symmetric equilibrium. If $n = 1$, the distribution is the buyer-optimal distribution. As $n \to \infty$, the distribution converges to the point mass distribution at 1 and the seller's revenue approaches 1.

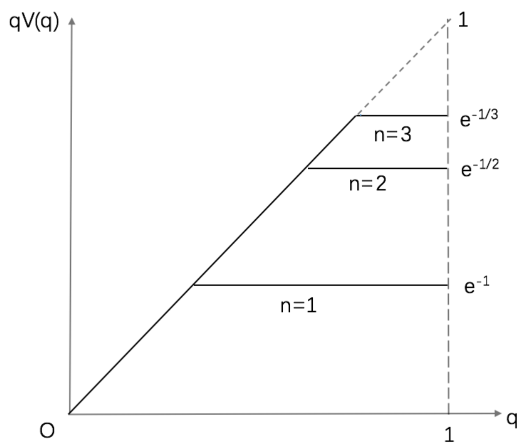

Figure 2: Revenue curves of the winners' distribution

4.2 Best response analysis

In this section we give a characterization of a buyer's best response given other buyers' distributions. Based on this, we are able to find all equilibria. The idea is that, given the seller's optimal price p, we can get a closed-form best response of the buyer. Then we determine p that maximize the buyer's utility by the first-order condition.

We then state our main theorem of this section. For simplicity, we only state our results for the case where $n = 2$. All techniques can be naturally extend to more general cases.

Theorem 4.4. *Suppose buyer 2's distribution is V_2, if $V_2(0) > 1/e$ then the following distribution is a best response of buyer 1:*

$$\begin{aligned} &V_1(q)[q + 1 - F_2(V_1(q))] = V_2(q_0^*),\ \forall 1 - q_0^* \le q \le q_t^* \\ &V_1(q) = 1,\ \forall 0 \le q \le q_t^* \\ &V_1(q) = 0,\ otherwise \end{aligned}$$

where q_0^* is given by $1 - q_0^* = -\ln V_2(q_0^*)$ and q_t^* is given by $q_t^* = V_2(q_0^*) + F_2(1) - 1$. F_2 is the cumulative function of distribution V_2 in the value space.

If $V_2(0) \le 1/e$, then the best response of buyer 1 is the buyer-optimal distribution

(Note that when $V_2(0) > e^{-1}$, the function $1 - x + \ln V_2(x)$ is monotone and $1 + \ln V_2(0) > 0$, so there is a unique root q_0. We will later prove that the first equation has a unique solution of $V_1(q)$ subject to the feasibility constraint.)

In order to prove Theorem 4.4, we first define a function $X(q), q \in [0, 1]$, which is essentially the best responds of buyer i.

Definition 4.5. Given continuous distributions V_{-i} and R, let $X(q)$ to be the solution to the following optimization problem:

$$\begin{aligned} \max \quad & x \\ s.t. \quad & x\left(q + \sum_{-i}(1 - F_{-i}(x))\right) \le R \qquad (4) \\ & x \le 1 \end{aligned}$$

Proposition 4.6. *$X(q)$ has following properties*

- *property 1: $X(q)$ is continuous and monotone decreasing with respect to q*
- *property 2: If*
$$X(q)\left(q + \sum_{-i}(1 - F_{-i}(X(q)))\right) < R,$$
then $X(q) = 1$
- *property 3: If there exists a distribution V_i such that R is the optimal revenue in the multi-buyer game with $V_1, \ldots, V_n$, then $X(q) \ge V_i(q)$, for any $q \in [0, 1]$.*

Proof. Note that when $x = 0$ the left side of (4) is 0, and by the continuity of F_i's there always exists a optimal solution $X(q)$ for any q.

Proof of property 1: The continuity follows from the continuity of F_i. For the monotonicity, suppose by contradiction that $X(q_1) < X(q_2)$ and $q_1 < q_2$, we have

$$\begin{aligned} &X(q_2)\left(q_1 + \sum_{-i}[1 - F_{-i}(X(q_2))]\right) \\ <&X(q_2)\left(q_2 + \sum_{-i}[1 - F_{-i}(X(q_2))]\right) \\ \le&R, \end{aligned}$$

which mean that when $q = q_1$, it is better to choose $X(q)$ to be $X(q_2)$ rather than $X(q_1)$, a contradiction.

Proof of property 2: If the condition is not stratified, it is better to replace x with $x + \epsilon$ and (4) still holds, since left side of (4) is continuous with respect to x.

Proof of property 3: For any q, when $x = V_i(q)$ the left side of (4) is the seller's revenue when the price equals to $V_i(q)$. By the definition of R, (4) is satisfied when $x = V_i(q)$. So by the maximality of $X(q)$ we have $X(q) \ge V_i(q)$. □

We then prove that it is without loss of generality to only consider the case where the feasibility constraint is binding:

Lemma 4.7. *If V_i is a best response to V_{-i}, then*

$$\sum_{i=1}^{n}(1 - F_i(p)) = 1,$$

where p is a maximizer of (2), F_i represents the distribution V_i in the value space.

Proof. Given V_{-i} and the seller's optimal price p, we assume on the contrary that V_i is a best response to V_{-i} and $\sum_{i=1}^{n}(1-F_i(p)) < 1$. We then prove that buyer i can deviate from V_i and his utility increases. Here we assume that each F_{-i} is a continuous function. [3]

Consider the following function

$$G(q) = q + \sum_{-i}[1 - F_{-i}(X(q))],\ q \in [0, 1].$$

By property 1 in Proposition 4.6, $G(q)$ is strictly increasing in q. By property 3, $X(1 - F_i(p)) \ge V_i(1 - F_i(p)) = p$, So

$$G(1 - F_i(p)) \le 1 - F_i(p) + \sum_{-i}[1 - F_{-i}(p)] < 1.$$

Also note that $G(1) \ge 1$, by the continuity of function G, there exists a $q_0 \in (1 - F_i(p), 1]$ such that $G(q_0) = 1$. We set q_0 to be the largest quantile such that $G(q_0) = 1$.

Then we reconstruct buyer i's distribution V_i^* to be:

$$V_i^*(q) = X(q),\ \forall 0 \le q \le 1. \tag{5}$$

i) First, by property 3 we have $V_i^*(q) \ge V_i(q)$,$\forall q \in [0, 1]$, and by property 1 we have $V_i^*(q)$ is decreasing in q, so V_i^* is a distribution.

ii) We then prove that the seller sets a new optimal price at $V_i^*(q_0)$. By property 2 when $q = q_0$ and $x = X(q_0)$ either the left side of (4) equals to R or $X(q_0) = 1$.

In the former case, if the seller's new optimal price $p' < V_i^*(q_0)$, then $1 - F_i^*(p') > q_0$ so $G(1 - F_i^*(p')) > 1$, which means that

$$1 - F_i^*(p') + \sum_{-i}[1 - F_{-i}(p')] > 1$$

violating the constraint.

If $V_i^*(q_0) < p \le V_i(0)$, let $q = 1 - F_i^*(p')$ in (4), since $p' \ge V_i^*(q) = X(q)$ is a solution of (4), we have

$$p'\left(1 - F_i^*(p') + \sum_{-i}[1 - F_{-i}(p')]\right) \le R.$$

If $p > V_i(0)$, then $1 - F_i^*(p') = 0$, we also have

$$p'\left(\sum_{i}(1 - F_i(p'))\right) = p'\left(\sum_{-i}(1 - F_{-i}(p'))\right) \le R$$

which means that the seller's revenue does not exceed R in this case. Since the left side of (4) is R when $q = q_0$ and $x = X(q_0) = V_i^*(q_0)$, the seller's revenue is maximized when setting the price at $V_i^*(q_0)$.

In the later case, for the same reason as above the seller can not choose any price $p' < 1$, also the seller's revenue is zero when he chooses any price $p' > 1$. So it is optimal for the seller to set the price equal to 1.

iii) We then compare $V_i^*(q_0)$ and p. By definition

$$p\left(\sum_{i=1}^{n}(1 - F_i(p))\right) = R.$$

If $p < V_i^*(q_0)$, as the seller set the price at $V_i^*(q_0)$ for (V_i^*, V_{-i}) rather than p, we have

$$p\left(1 - F_i^*(p) + \sum_{-i}[1 - F_{-i}(p)]\right) < R.$$

Combining the two formulas above we have $1 - F_i^*(p) < 1 - F_i(p)$. However, since $V_i^*(q) \ge V_i(q)$ holds for any q, we also have $1 - F_i^*(p) \ge 1 - F_i(p)$, contradiction. So $V_i^*(q_0) \le p$

Combining i), ii), iii), $q_0 > 1 - F_i(p)$ and recall the buyer's utility function in (3), buyer i is better to deviate to V_i^*, thus proves the lemma. □

From the proof of Lemma 4.7, it is straightforward to derive the following propositions:

Proposition 4.8. *Given V_{-i} and R,*
i) there exists a q_0 such that $G(q_0) = 1$.
ii) for any q such that $q_0 < q$ and $X(q) < 1$, $X(q)$ is the unique solution of the following equation:

$$x\left(q + \sum_{-i}(1 - F_{-i}(x))\right) = R$$

if $q + \sum_{-i}(1 - F_{-i}(x)) \le 1$.

Proof. The existence of q_0 follows from $G(0) < 1$ and $G(1) \ge 1$.

By Proposition 4.6 $X(q)$ is a feasible solution of the equation and is continuous.

To prove the uniqueness, we assume on the contrary that there exists a quantile q_1, such that there exists another solution x_1 of the equation when $q = q_1$. By the definition of $X(q_1)$, we have $x_1 < X(q_1)$

If $x_1 \ge R$, since $X(q_0) = R$, we can write it as $x_1 = X(q_2)$ and $q_1 < q_2 \le q_0$, so

$$X(q_2)\left(q_1 + \sum_{-i}(1 - F_{-i}(X(q_2)))\right) = R.$$

Since $X(q_2)$ is a solution of equation when $q = q_2$, we have

$$X(q_2)\left(q_2 + \sum_{-i}(1 - F_{-i}(X(q_2)))\right) = R.$$

So $q_1 = q_2$. A contradiction.

If $x_1 < R$, then $q_1 + \sum_{-i}(1-F_{-i}(x_1)) > 1$, violating the constraint.

Thus the proposition is proved. □

The uniqueness of the solution in the first equation in Theorem 4.4 is proved.

From the construction of (4), we also get the following proposition:

Proposition 4.9. *Given V_{-i} seller's optimal price p, then $p = R$, the seller's revenue, and (5) is a best response of buyer i.*

Proof. By Lemma 4.7,

$$p = p\left(\sum_{i}(1 - F_i(p))\right) = R.$$

For each quantile q, by the definition of $X(q)$, it is the maximum value in $V_i(q)$'s feasible domain, and $V_i(q) = X(q)$ is a distribution. So it is a best response. □

[3] If the continuous condition does not hold for some distribution F at value v, we replace F on the interval $(v, v + \epsilon)$ by a straight line $(v, F(v))$, $(v + \epsilon, F(v + \epsilon))$. Then the new distribution has a continuous cumulative function and converges to the original distribution as ϵ approaches to 0. Similar approaches are applied in [11]

We then prove Theorem 4.4.

Proof of Theorem 4.4. For the case $V_2(0) > e^{-1}$:

Given that the seller's the optimal price p. we construct buyer 1's best response V_1.

If $p < V_2(0)$, we assume that $p = V_2(q_0)$[4], By Lemma 4.7, we assume, without loss of generality, that $V_1(1-q_0) = p = V_2(q_0)$. Also, by proposition 4.9, $V_1(q)$ satisfies the following:

$$V_1(q)(q + 1 - F_2(V_1(q))) = V_2(q_0),\ \forall 1 - q_0 \le q \le q_t \tag{6}$$
$$V_1(q) = 1,\ \forall 0 \le q \le q_t$$

where q_t is the smallest quantile such that $V_1(q_t) = 1$, given by the following equation:

$$q_t = V_2(q_0) + F_2(1) - 1.$$

So buyer 1's utility is

$$\begin{aligned} U_1 &= \int_0^{q_t} (1 - V_2(q_0))\,\mathrm{d}q + \int_{q_t}^{1-q_0} (V_1(q) - V_2(q_0))\,\mathrm{d}q \\ &= q_t - (1-q_0)V_2(q_0) + \int_{q_t}^{1-q_0} V_1(q)\,\mathrm{d}q. \end{aligned}$$

Take derivative of U_1 with respect to q_0 and use the Leibniz formula we have

$$\frac{\mathrm{d}U_1}{\mathrm{d}q_0} = (q_0 - 1)V_2'(q_0) + \int_{q_t}^{1-q_0} \frac{\mathrm{d}V_1(q)}{\mathrm{d}q_0}\,\mathrm{d}q. \tag{7}$$

We define the multivariate function $G(v(q), q)$ to be the left side of (6), then

$$G(V_1(q), q) = V_2(q_0).$$

Note that q_0 and q are independent, taking derivative with respect to v_0

$$\frac{\partial G}{\partial V_1(q)} \cdot \frac{\mathrm{d}V_1(q)}{\mathrm{d}q_0} = V_2'(q_0).$$

Then taking derivative with respect to q

$$\frac{\partial G}{\partial V_1(q)} \cdot V_1'(q) + \frac{\partial G}{\partial q} = 0.$$

Since $\frac{\partial G}{\partial q} = V_1(q)$, combine the two formulas above together we have

$$\frac{\mathrm{d}V_1(q)}{\mathrm{d}q_0} = -\frac{V_2'(q_0)V_1'(q)}{V_1(q)}.$$

Plug this into (7) and we get

$$\begin{aligned} \frac{\mathrm{d}U_1}{\mathrm{d}q_0} &= V_2'(q_0)\left(q_0 - 1 - \int_{q_t}^{1-q_0} \frac{V_1'(q)}{V_1(q)}\,\mathrm{d}q\right) \\ &= V_2'(q_0)(q_0 - 1 - \ln V_2(q_0)). \end{aligned}$$

The last equality holds since we assume that $V_1(1-q_0) = V_2(q_0)$ and $V_1(q_t) = 1$.

So the optimal q_0 is given by the first order condition:

$$q_0 - 1 - \ln V_2(q_0) = 0.$$

If $p \ge V_2(0)$, the problem degenerates to the single buyer case. By the proof of Theorem 3.1, buyer 1's maximum utility is decreasing with p when $p > e^{-1}$, thus less than the case when $p = V_2(0)$, which equals to the utility in this proof when $p = V_2(0)$, therefore also less than the utility when $p = V_2(q_0)$.

For the case $V_2(0) \le e^{-1}$:

If V_1 equals to the buyer-optimal distribution, of which the lowest value in the support is e^{-1}, then buyer 1 can consume all goods. Also it is obvious that one's utility in the multi-buyer game is dominated by the maximum utility in the single-buyer problem. So the buyer-optimal distribution is the best response. Thus the theorem is proved. □

[4] Note that here the q_0 is the reserve quantile of buyer 2, which is different from that in Lemma 4.7, while $1 - q_0$ corresponds to the reserve quantile of buyer 1

Note that best response $V_1(q)$ is unique for the part $q \le 1 - q_0$.

Following the proof of the Lemma 4.4, we get following results:

Proposition 4.10. *In all equilibria, the seller sets the same price and yields the same revenue. Each buyer consumes $\frac{1}{n}$ fraction of total items.*

Proof. We only prove the case where $n = 2$. By Lemma 4.4, we have

$$1 - q_0 = -\ln V_2(q_0)$$

and similarly by computing the best response of buyer 2 we have

$$q_0 = -\ln V_1(1 - q_0).$$

Since $V_2(q_0) = V_1(1-q_0)$, we have $q_0 = \frac{1}{2}$, and $p = V_2(q_0) = e^{-\frac{1}{2}}$. The seller's revenue is always $e^{-\frac{1}{2}}$. □

Theorem 4.11. *A distribution profile (F_1, F_2) is an equilibrium if and only if*

$$F_1(\frac{1}{2}) = F_2(\frac{1}{2}) = e^{-\frac{1}{2}}$$
$$v(2 - F_1(v) - F_2(v)) = e^{-\frac{1}{2}},\ \forall e^{-\frac{1}{2}} \le v \le 1$$
$$v(2 - F_1(v) - F_2(v)) < e^{-\frac{1}{2}},\ \forall v < e^{-\frac{1}{2}}.$$

Proof. Consider the following equation:

$$x(q + 1 - F_2(x)) = e^{-\frac{1}{2}}$$
$$e^{-\frac{1}{2}} \le x \le 1.$$

For any $e^{-\frac{1}{2}} \le v \le 1$, when $q = 1 - F_1(v)$, by Proposition 4.8, $x = v$ is the unique solution.

The "if" direction follows from the fact that $x = v$ is a solution and Proposition 4.9, this solution is a best response. So F_1 is a best response to F_2. Similarly, F_2 is a best response to F_1.

The "only if " direction follows from the uniqueness of the solution and Proposition 4.10. □

Corollary 4.12. *If we ignore the values that less than the reserve price, then equilibrium in Theorem 4.3 is the unique symmetric equilibrium.*

The results also show that, if we consider the Stackelberg version of this game, where the leader first choose a distribution, then the follower chooses a best response to the leader's distribution, then any distribution of the leader satisfying that the median value equals to $e^{-\frac{1}{2}}$ results in a Nash-equilibrium.

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