A Geometric Least Squares Method for Peer Assessment

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ABSTRACT
In the peer assessment problem, a set of agents give evaluations to each other, and we are going to combine these peer assessments together to construct an overall evaluation. In this paper, we propose a geometric least squares method (GLS) to find an aggregate scoring overall agents for the peer assessment problem. Our method is based on the following observation. Since each agent has a missing score that should be given by itself, we consider the missing score as a variable and then each agent can be regarded as a line in an $n$-dimensional vector space. The final aggregate scores of the agents can be regarded as points on a line vector, called the projection vector. Thus, we treat the peer assessment problem as an optimization problem of selecting a projection vector with minimum total squared distance to all the lines representing the agents. We will see that this aggregate method has some advantages compared with the simple average method. One advantage is that, when the scores given by each agent (even ignoring the magnitude of the agent) are close to a groundtruth, the new method finds the groundtruth with the highest expectation.

KEYWORDS
Social Choice; Score Aggregation; Peer Assessment

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1 INTRODUCTION
The peer assessment problem has been well known for many years in social choice [1]. In the setting, each agent is both a candidate and a voter, and each agent is asked to assign numeric scores from its own point of view regarding the overall performance of other agents. The object is to get a combined overall evaluation of the agents. This problem has many important application scenarios [2, 7–12] and has been extensively studied in the literature [4, 5, 12, 13]. Peer assessments suffer from several fundamental problems, such as, how to set an incentive or define a mechanism to ensure that agents report their evaluations truthfully; after gaining the scores, how to aggregate the scores into a coherent assessment; and so on. In this paper, we focus on aggregating scores for the peer assessment problem.

A related problem is to obtain an aggregate rank of the agents based on the rankings given by the agents. For rank aggregation problems, there are some famous protocols, such as Kemeny [3] and Condorcet [16], which is to minimize the number of disagree pairs.

However, for score aggregation problems, there are few well-known protocols used to evaluate the aggregation results. Not all score aggregation problems in real life can be changed to rank aggregation problems, for example, to compute GPA of students. Most of references focus on the mechanism designs or how to normalize the scores. After doing these, they simply apply the sum-up or weight sum-up method to aggregate. In this paper, we study the aggregate models based on the given scores from agents and build a protocol for peer assessments inspired by the idea in [14].

2 THE PROBLEM AND DEFINITIONS
In the peer assessment problem, there is a set of agents $N = \{1, 2, \ldots, n\}$. Each agent $j \in N$ has evaluated each other agent $i \in N \setminus \{j\}$ by giving a nonnegative score $s_{ij}$. We need to aggregate the $n$ evaluations into a final overall vector of scores for the agents. The evaluation given by an agent $i$ can be denoted by a column vector $[s_{i1}, s_{i2}, \ldots, s_{i(n-1)}, -, s_{i(n+1)}, \ldots, s_{in}]^T$. Here we use ‘-’ to denote the missing score at the $i$th site. All the evaluations form a matrix $S_{n \times n}$, called the score matrix. In the score matrix $S_{n \times n}$, all the elements on the diagonal are ‘-’ denoting missing elements. We may also view each missing element as a variable and use $t_i$ to denote them.

3 THE MODEL AND ALGORITHM
We consider the peer assessment problem from the geometric perspective. We look at the $n$-dimensional vector space $A$ with base vectors, denoted by $e_1, e_2, \ldots, e_n$, representing the $n$ agents. Each agent has a missing score that should be given by itself and we view it as a variable. Thus, each agent can be represented by a line $l_i$ in the $n$-dimensional vector space $A$ and this line $l_i$ is parallel with the base vector $e_i$ corresponding to itself. The final aggregate result is a vector, denoted by $e$ and the final score for each agent $i$ is the $i$th element of $e$. The aggregation problem can be interpreted as a geometric problem to “project” $n$ lines $l_i$ in an $n$-dimensional vector space $A$ to $n$ points in a 1-dimensional unit vector space $e/\|e\|_2$. We still need to give the precise
We will call this method the geometric least squares method where the lines, which can be computed by being 1 and all other elements being 0. Let the line) and \( p \) parameter, the problem of selecting a projection vector to project the core. In the geometric model, the average method is exactly the minimum distance between any two points lying on and \( l_0 = (0, l_1, l_2, \ldots) \) representing the object of GLS is to find a line \( l \) such that the total squared distance from \( e' \) to the \( n \) lines \( l_i \) is minimized. We will call this method the geometric least squares method (GLS). Next, we describe the detailed steps.

\[
\text{dist}(l_i, l_j) = \min_{c_i, c_j} \|c_i - c_j\|_2.
\]

The distance \( \text{dist}(l_i, l_j) \) between two lines \( l_i = v_i t_i + q_i \) and \( l_j = v_j t_j + q_j \) in a high-dimensional space is defined to be the minimum distance between any two points lying on the lines, which can be computed by

\[
\text{dist}(l_i, t_j) = \min_{c_i, c_j} \|c_i - c_j\|_2 = \min_{c_i, c_j} \|v_i t_i - v_j t_j + q_i - q_j\|_2.
\]

Directed computation shows that

\[
\text{dist}(l_0, l_i) = \|p_i e_i^T e_0 - e_i e_0^T e_i p_i - p_i\|_2.
\]

The object of GLS is to find a line \( l_0 \) minimizing the total squared distance between \( l_0 \) and the \( n \) lines \( l_i \)

\[
E = \sum_{i=1}^{n} \text{dist}^2(l_0, l_i) = \sum_{i=1}^{n} \left\| p_i e_i^T e_0 - e_i e_0^T e_i p_i - p_i \right\|^2.
\]

Thus, it is to find a unit vector \( e_0 \) satisfying

\[
e_0 = \arg \min_{l_0} E = \arg \min_{l_0} \sum_{i=1}^{n} \text{dist}^2(l_0, l_i).
\]

We only need to solve the above optimization problem. Let \( e_0 = (e_{01}, e_{02}, \ldots, e_{0n}) \). When \( E \) achieves the minimum value, the derivatives of \( E \) with respect to \( e_0 \) is zero. Thus, the partial derivatives of \( E \) with respect to each component of \( e_0 \) is zero, i.e., \( \frac{\partial E}{\partial e_{0i}} = 0, i = \{1, 2, \ldots, n\} \).

By solving the above function set, we can get the projection vector \( e_0 \). However, the solution is too complex to obtain an explicit expression directly. There exists some numerical methods to solve these kinds of partial derivatives and implementations of them are available in most famous mathematical softwares. We implemented the algorithm in Matlab and the code is available online [15].

### 4 PROPERTY ANALYSIS

**Relations With Maximum Likelihood Estimate**

The object of GLS is to optimize the total squared distance. This criteria has a good meaning in probabilistic semantics.

Assume that there is a groundtruth score vector \( c_i \) for each agent \( i \). The actual score of agent \( i \) given by agent \( j \) is an error estimation of the \( j \)th element of \( c_i \). The actually score vector \( c_i \) of agent \( i \) is

\[
c_i = \hat{c}_i + \varepsilon_i,
\]

in which \( \varepsilon_i \) is the error vector. For groundtruth, we assume that all the agents agree with each other consistently ignoring the magnitude, i.e., the groundtruth score vectors for all agents satisfy the linear relationship.

**Property 1.** Under the above assumption, if the error obeys the Gaussian distribution, then to minimize the total squared distance is equal to maximize the likelihood function.

It is reasonable to assume the error obeys the Gaussian distribution, because the Center Limit Theorem shows if an error is the sum of many small, independent error sources, the total error will be about Normally distributed (Gaussian distribution) [6].

**A Stable Property**

**Property 2.** Given a consistent score matrix. If any agent changes its given scores by multiplying them by a positive constant \( \gamma \), then the new final aggregate score vector obtained by our method is a constant multiple of the old one.

This property says that the relationship of the agents in the final aggregate score vector will not change if one agent amplifies or shrinks its magnitude without changing the relationship of the agents in its given scores. This provides an incentive for agents to avoid grading all others low.

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REFERENCES


