Efficient Allocation Mechanism with Endowments and Distributional Constraints

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ABSTRACT

We consider an allocation problem of multiple types objects to agents, where each type of an object has multiple copies (e.g., multiple seats of a school), each agent is endowed with an object, and some distributional constraints are imposed on the allocation (e.g., minimum/maximum quotas). We develop a mechanism that is based on the Top Trading Cycles mechanism, which is strategy-proof, feasible (always satisfies distributional constraints), Pareto efficient, and individually rational, assuming the distributional constraints are represented as an M-convex set. The class of distributional constraints we consider contains many situations raised from realistic matching problems, including individual minimum/maximum quotas, regional maximum quotas, type-specific quotas, and distance constraints. To the best of our knowledge, we are the first to develop a mechanism with these desirable properties.

KEYWORDS

controlled school choice; M-convex set; strategy-proofness; top trading cycles mechanism; distributional constraints

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1 INTRODUCTION

The objective of this paper is to develop a mechanism for allocating indivisible objects to agents without monetary transfers, where each individual has a prior claim to some object, each type of an object has multiple copies, and some distributional constraints are imposed on the allocation. Our motivation is to apply this mechanism to school choice for public schools, i.e., deciding the allocation of students to schools when a school district offers them the opportunity to attend public schools other than the one closest to where they live (where each school has multiple identical seats) under distributional constraints (e.g., the capacity limits of schools).

Our mechanism is general enough to be applied to any reallocation problem of indivisible objects with multiple supplies. For example, assume a student-laboratory assignment problem. It is often the case that the knowledge of a student is limited and she fails to choose an appropriate laboratory. One possible remedy is to apply the following three-step procedure: (i) students are assigned to laboratories using some mechanism, (ii) students experience a certain trial period, and (iii) each student has a chance to apply to another laboratory if her interest changes or her current laboratory fails to meet her expectations. Our new mechanism can be used in Step (iii). It is natural to require that no student is reallocated to a laboratory that is worse than her current assignment.

Following a seminal work by Abdulkadiroğlu and Sönmez [4], which formalizes a school choice problem in the context of the mechanism design approach, a wide range of theoretical analysis has been conducted on the existing mechanisms used in practice.[1] As the theory has been developed and applied to diverse types of environments, mechanism designers have faced a variety of forms of distributional constraints that are not considered in the standard model. For example, the work of Biró et al. [6] is motivated by the Hungarian education system where higher education institutions can declare minimum quotas for their study areas that should be satisfied to open courses. Another example is regional maximum quotas introduced by the Japanese government to control the geographical distributions of medical residents to hospitals across the country [24].

It is well-known that in the presence of distributional constraints, a stable matching may not exist. Stability of a matching is firstly defined by Gale and Shapley [16] for two-sided, one-to-one, and one-to-many matching problems. In the setting of a school choice problem, it is defined as the combination of individual rationality (IR), fairness, and nonwastefulness [5]. IR is a basic requirement that guarantees a student[2] to obtain a seat in a school that is at least as good as her initial endowment. Fairness is defined when schools also have priorities over students (in addition to students' preferences over schools). It ensures that when student s is not accepted to school c (which she considers better than her assigned school), then s is ranked lower than any student accepted to c according to c's priority. Nonwastefulness is an efficiency notion that rules out the incidents where a student can move unilaterally to her more preferred school without violating the underlying distributional constraints. Given the incompatibility of stability with distributional constraints, mechanism designers encounter a trade-off between fairness and efficiency. In the recent literature on distributional constraints [10, 11, 18, 19, 25, 28], a common approach is to modify the definition of stability and to maintain the balance between efficiency and fairness to some extent. Our approach is to investigate whether efficiency is achievable under distributional constraints. More specifically, we study Pareto efficiency (PE), a

[1] See for example Sönmez and Ünver [43] for a survey on the theoretical analysis of existing school choice mechanisms.

[2] For the sake of presentation, the rest of this paper is described in the context of a school-student allocation problem, but the obtained results in this paper are applicable to allocation problems in general.

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stronger welfare notion than nonwastefulness, that rules out the incidents where students' welfare can be improved without making others worse off, while observing the distributional constraints. It has been mostly an open question whether a PE mechanism (i.e., a mechanism that is guaranteed to obtain a PE matching) can be obtained under distributional constraints. Kamada and Kojima [24] is one of a few studies that investigates efficiency under distributional constraints, which shows that PE is achievable under regional maximum quotas. Another study by Hamada et al. [21] develops a PE mechanism when minimum (and standard maximum) quotas are imposed for each school. As described later, the class of distributional constraints we study in this work is a strict generalization of these classes.

We restrict our attention to strategy-proof (SP) mechanisms, in which no student has an incentive to misreport her preference over schools. In theory, we can restrict our attention to SP mechanisms without loss of generality due to the well-known revelation principle [17], i.e., if a certain property is achieved by a mechanism (more specifically, the property is satisfied in a dominant strategy equilibrium when using the mechanism), it can be achieved by an SP mechanism. An SP mechanism is also useful in practice since a student does not need to speculate about the actions of other students to obtain a good outcome; she only needs to truthfully report her preference.

Our paper is at the intersection of discrete mathematics and economics. In particular, we consider a class of distributional constraints that can be represented by an *M-convex set* (M stands for Matroid), a concept introduced by Murota [30, 31] in the field of discrete mathematics, which is a discrete counterpart of the framework of convex analysis. The insight from discrete mathematics, and discrete convex analysis in particular, has been used in a broad range of applications on discrete optimization such as scheduling, facility location, and structural analysis of engineering systems among others [27, 29, 40]. Recently, discrete convex analysis has been recognized as a powerful tool for analyzing economic or game theoretic applications, including exchange economies with indivisible objects [8, 32, 35, 44], systems analysis [32], inventory management [23, 47] and auction [34] (see Murota [33] for an extensive survey on recent developments). As this long, and yet partial, list of success stories suggests, techniques from this literature can be applied to a variety of economic problems. In this paper, we add allocation problems (including school choice problems) to this list.

This paper is not the first to apply discrete convex analysis to allocation problems. Fujishige and Tamura [13, 14] and Murota and Yokoi [36] apply discrete convex analysis to study two-sided matching problems. More specifically, these works deal with a many-to-many matching problem, in which a doctor/worker can work at multiple hospitals/firms. Fujishige and Tamura [13, 14] consider side payments as well.[3] Kojima et al. [26] apply the concept to two-sided matching problems with distributional constraints and show that if the preferences of schools can be represented as an M-concave function[4] then the generalized deferred acceptance mechanism [22] achieves a desirable outcome. Our motivation is different from these works, i.e., our goal is to develop an IR, PE, and SP mechanism.

We show that the M-convexity of the underlying distributional constraints is sufficient to guarantee the existence of a mechanism that satisfies IR[5], PE, and SP. We require one additional assumption: if every student is assigned to her initial endowment school, then the underlying distributional constraints are satisfied. This is an innocent requirement in the context of school choice since every student would go to her local school if there were no school choice program; assuming this default allocation satisfies distributional constraints is reasonable. We also show by a counterexample that the three properties easily become incompatible if the underlying distributional constraints are not M-convex.

Our developed mechanism is based on *Top Trading Cycles* (TTC) mechanism of Shapley and Scarf [41], due to David Gale, which improves students' welfare by trading their initial endowments. They introduce a housing market problem, where objects are initially owned by agents, who have strict preferences over them, and there are no copies of an object in the market. TTC is further generalized to the Hierarchical Exchange mechanism [37] and to the Trading Cycles mechanism[6] by Pycia and Ünver [39]. TTC has been applied to a school choice problem [1], as well as assigning teachers to schools [7, 46]. Recently, TTC has attracted increasing attention from AI researchers [15, 42, 45].

Abdulkadiroğlu and Sönmez [3] and Guillen and Kesten [20] consider a housing market problem with existing tenants, where some agents may not initially own a house and some houses are not initially owned by agents.[7] The differences between their setting and ours are that we consider multiple copies of an object and impose distributional constraints. Our work is a strict extension of Hamada et al. [21], who only consider individual minimum and maximum quotas.

When TTC is applied to a housing market problem, agents sequentially form trading cycles to exchange their initial endowments. In the allocation problem we consider, some school seats may be vacant, i.e., they are not initially owned by any students, and distributional constraints are imposed on the final allocation. The main difficulty in this setting is how to utilize such vacant seats to improve students' welfare without violating distributional constraints. For example, let us assume the following complex distributional constraints (regional maximum quotas) are imposed; schools are partitioned into regions, and the total number of students allocated within a region must not exceed the maximum quota of the region. Then, even if a school has a vacant seat, allocating it to a student may violate the maximum quota of the region to which the school belongs. By utilizing a common priority order over students, our TTC-based mechanism, which we call TTC with M-convex set constraints (TTC-M), can allocate vacant seats efficiently without

[3] See also an earlier contribution by Fleiner [9] who applies matroid theory to matching. His analysis is a special case of a more recent contribution by Fujishige and Tamura [14].

[4] More precisely, they use a concept called $M^\natural$-convexity, an essentially equivalent variant of M-convexity.

[5] In the later section, we call a mechanism feasible if it always gives an IR outcome that does not violate distributional constraints.

[6] Pycia and Ünver [38] present an extension of the Trading Cycles mechanism such that each object has multiple copies. Our work is different from Pycia and Ünver [38]; they consider only standard maximum quotas, while we deal with more general distributional constraints that can be represented as an M-convex set.

[7] We can easily modify our model to describe a situation where some students do not initially own a school seat; we can assume such a student initially owns the seat of null school $c_\emptyset$, and for each student s, where $\omega(s) \neq c_\emptyset$, $\omega(s) \succ_s c_\emptyset$ holds.

violating the underlying distributional constraints. To the best of our knowledge, no mechanism with these desirable properties has ever been found in this setting.

This paper is organized as follows. In Section 2, we introduce a general model of an allocation problem with initial endowments and distributional constraints and define the desirable properties. In Section 3, the notion of an M-convex set is described to specify the domain of the distributional constraints we focus on, together with an impossibility result for the general model. In Section 4, some properties on an M-convex set are provided that are used in the proof of our main theorems. In Section 5, TTC-M is introduced and shown to satisfy the desirable properties in our setting. Finally, Section 6 concludes this paper.

2 MODEL

In this section, we introduce our model and several desirable properties. A market is a tuple $(S, C, \omega, \succ_S, F)$, where

- $S = \{s_1, \dots, s_n\}$ is the set of n students,
- $C = \{c_1, \dots, c_m\}$ is the set of m schools,
- $\omega : S \to C$ is an initial endowment function; $\omega(s) = c$ is the initial endowment school of s,
- $\succ_S = (\succ_s)_{s \in S}$ is a profile of students' strict preferences over C, and
- $F \subseteq \mathbb{Z}_+^m$ is a set of school feasible vectors that reflects distributional constraints.

Given a market, $\mu \subseteq \{(s, c) \mid s \in S, c \in C\}$ is a *matching*, where $\mu_s \in C$ denotes the school to which s is matched and $\mu_c = \{s \mid (s, c) \in \mu\}$ is the set of students matched to c. We denote $c \succeq_s c'$ if either $c \succ_s c'$ or $c = c'$.

Let $\tilde{\mu}$ denote the initial endowment matching, that is, $\tilde{\mu}_s = \omega(s)$ for all $s \in S$. We say μ is *individually rational* (IR) if $\mu_s \succeq_s \omega(s)$ holds for all s, that is, each student is matched to a school that is at least as good as her initial endowment school.

Given matching μ, let $\nu(\mu)$ denote the m-dimensional distribution vector of μ, where its i-th element $\nu_i(\mu)$ is $|\mu_{c_i}|$. We sometimes write $\nu_{c_i}(\mu)$ instead of $\nu_i(\mu)$. We say μ is *school-feasible* if $\nu(\mu) \in F$. We require μ is school-feasible only if $\sum_{i=1}^m |\mu_{c_i}| = n$, i.e., each student must be matched to some school. We assume $\nu(\tilde{\mu}) \in F$ and thus $\tilde{\mu}$ is school-feasible. We say matching μ is *feasible* if it is IR and school-feasible, and therefore $\tilde{\mu}$ is a feasible matching.

We say μ Pareto dominates μ' if $\mu_s \succeq_s \mu'_s$ holds for all s and there exists s with $\mu_s \succ_s \mu'_s$. We say school-feasible matching μ is *Pareto efficient* (PE) if no other school-feasible matching Pareto dominates it.[8]

For $s \in S$, let $(\succ_s, \succ_{-s})$ denote the preference profile of all the students, where the preference of student s is $\succ_s$ and the profile of the preferences of the other students is $\succ_{-s} = (\succ_{s'})_{s' \in S \setminus \{s\}}$.

Mechanism φ is a function that takes a profile of the preferences of students $\succ_S$ and returns matching $\varphi(\succ_S)$. Let $\varphi_s(\succ_S)$ denote the school to which s is matched, and $\varphi_c(\succ_S)$ denote the set of students matched to c.

The goal of this paper is to find a mechanism equipped with the following desirable properties. We say φ is feasible if $\varphi(\succ_S)$ is feasible for all $\succ_S$. We say φ is *strategy-proof* (SP) if for all $s, \succ_s, \succ'_s$, and $\succ_{-s}$, $\varphi_s(\succ_s, \succ_{-s}) \succeq_s \varphi_s(\succ'_s, \succ_{-s})$ holds. In words, this property requires that a student cannot be allocated to a strictly better school by misreporting her preference. We say φ is PE if $\varphi(\succ_S)$ is PE for all $\succ_S$.

Finally, let us introduce some notions that are used to describe our mechanism. A directed graph is a pair (V, E), where V is a set of vertices and $E \subseteq \{(i, j) \mid i, j \in V\}$ is a collection of directed edges. A directed edge, $e = (i, j) \in E$, is an ordered pair of vertices. A sequence of distinct vertices, $(i_1, \dots, i_k)$, $k \geq 2$, is a directed path in (V, E) from i_1 to i_k if $(i_h, i_{h+1}) \in E$ for $h = 1, \dots, k-1$. A sequence of vertices, $(i_1, \dots, i_k, i_1)$ is a *cycle*, if $(i_1, \dots, i_k)$ is a directed path and $(i_k, i_1) \in E$.

3 M-CONVEX SET AS A CLASS OF DISTRIBUTIONAL CONSTRAINTS

In this section we describe the class of distributional constraints that is considered in our model. Let χ_i denote an m-element unit vector, whose i-th element is 1 and all other elements are 0. We sometimes write χ_{c_i} instead of χ_i.

Definition 3.1. Set of m-element vectors F is an *M-convex set* if for all $\nu, \nu' \in F$ and i with $\nu_i < \nu'_i$, there exists j with $\nu_j > \nu'_j$ such that $\nu + \chi_i - \chi_j \in F$ and $\nu' - \chi_i + \chi_j \in F$ hold.

This property characterizes an M-convex set and is called *(simultaneous) exchange property* [30]. The notion of an M-convex set is analogous to that of maximum elements of a convex set in a continuous domain, i.e., there is no hollow in a set. Next, we show that several distributional constraints introduced in the literature can be represented by an M-convex set.

Individual minimum/maximum quotas [10, 21]: Consider a market[9] where, for each school $c \in C$, there is maximum quota q_c and minimum quota p_c. The distributional constraints of this market can be expressed as F where

$$F = \{\nu \in \mathbb{Z}_+^m \mid \sum_{c \in C} \nu_c = n \text{ and } p_c \leq \nu_c \leq q_c \ \forall c \in C\},$$

and it can be verified that F is M-convex.

Regional maximum quotas [18, 24]: In addition to the individual minimum/maximum quotas, capacity constraints are imposed on *regions*. Set of regions $R \subseteq 2^C \setminus \{\emptyset\}$ partitions set of schools C into regions, and for each $r \in R$, there is a regional minimum quota of p_r and a maximum quota of q_r. The distributional constraints of this market can be expressed as F where

$$\begin{aligned} F \quad = \quad & \{\nu \in \mathbb{Z}_+^m \mid \sum_{c \in C} \nu_c = n,\ p_c \leq \nu_c \leq q_c \\ & \quad \text{and } p_r \leq \sum_{c \in r} \nu_c \leq q_r \ \forall c \in C, \forall r \in R\}, \end{aligned}$$

and it can be verified that F is M-convex.

[8] Sometimes this property is referred to as constrained Pareto efficient, since the set of matchings considered is restricted to the set of school-feasible ones.

[9] The standard model where there are only maximum quotas, e.g., Abdulkadiroğlu and Sönmez [4], can be seen as $p_c = 0$ for all $c \in C$.

Type-specific quotas [2, 11]: In addition to the individual maximum quotas, there are additional type-specific quotas for each *type* of students. A type of a student may represent race, gender, or any socioeconomic status. There is a set of types $T \subseteq 2^S \setminus \{\emptyset\}$ that partitions the set of students S into types, and for each $c \in C$ and $t \in T$, there is a type-specific minimum quota of $p_{c,t}$ and the maximum quota of $q_{c,t}$. We represent a distribution vector v as $m \times |T|$ matrix, where $v_{c,t}$ denotes the number of type t students allocated to school c. Distributional constraints of this market can be expressed as F where

$$F = \{v \in \mathbb{Z}_+^{m\times|T|} \mid \sum_{c\in C} v_c = n, \sum_{t\in T} v_{c,t} \le q_c \text{ and } p_{c,t} \le v_{c,t} \le q_{c,t} \ \forall c \in C, \ \forall t \in T\},$$

and it can be checked that F is M-convex.

Distance constraints [26]: When allocating n students among m schools, suppose there exists an ideal distribution vector from the viewpoint of the mechanism designer and he considers a distribution vector feasible if it is 'close enough' to the ideal vector. The distance constraints is defined by ideal vector v^* and distance k describing what is close enough from the ideal distribution. The set of feasible vectors under such constraints is expressed as F where

$$F = \{v \in \mathbb{Z}_+^m \mid \sum_{c\in C} v_c = n \text{ and } \delta(v, v^*) \le k\},$$

and it can be verified that F is M-convex assuming distance function δ is given by either (i) the Manhattan distance (or L^1 distance), which is defined as $\delta(v, v') = \sum_{c\in C} |v_c - v'_c|$, or (ii) the Chebyshev distance (or L^∞ distance), which is defined as $\delta(v, v') = \max_{c\in C} |v_c - v'_c|$.

In the next theorem, we show that three basic properties (feasibility, PE, and SP) can be incompatible when the set of feasible vectors is not an M-convex set.

Theorem 3.2. *There exists a market where the set of feasible vectors does not form an M-convex set such that no mechanism simultaneously satisfies feasibility, PE, and SP. This is true even for a market with two students, three schools, and the set of feasible vectors becomes an M-convex set by just adding one vector.*

Proof. Consider the following market.

- $S = \{s_1, s_2\}$,
- $C = \{c_1, c_2, c_3\}$,
- $\omega(s_1) = \omega(s_2) = c_1$,
- $c_3 \succ_{s_1} c_2 \succ_{s_1} c_1$ and $c_3 \succ_{s_2} c_2 \succ_{s_2} c_1$, and
- $F = \{(2, 0, 0), (1, 1, 0), (0, 1, 1)\}$.

This market can be interpreted as follows: within the school district, there are three schools c_1, c_2, and c_3, where c_1 has a larger capacity than the others. Due to logistic constraints on resource of the district, two schools can operate at the same time only if they are close to each other. Note that F is not an M-convex set. For $v = (2, 0, 0)$ and $v' = (0, 1, 1)$, where $v_3 < v'_3$, $v_j > v'_j$ holds only for $j = 1$. However, $v + \chi_3 - \chi_1 = (1, 0, 1)$ is not in F. On the other hand, $F \cup \{(1, 0, 1)\}$ is an M-convex set.

In this market, there are two feasible and PE matchings, $\{(s_1, c_3), (s_2, c_2)\}$ and $\{(s_1, c_2), (s_2, c_3)\}$. Suppose feasible and PE mechanism φ chooses $\varphi(\succ_S) = \{(s_1, c_3), (s_2, c_2)\}$. Then s_2 can misreport $\succ'_{s_2}$ where $c_3 \succ'_{s_2} c_1 \succ'_{s_2} c_2$. With this misreport, the only feasible and PE matching is $\varphi(\succ_{s_1}, \succ'_{s_2}) = \{(s_1, c_2), (s_2, c_3)\}$ and $\varphi_{s_2}(\succ_{s_1}, \succ'_{s_2}) \succ_{s_2} \varphi_{s_2}(\succ_{s_1}, \succ_{s_2})$. Similarly, s_1 has an incentive to misreport if $\varphi(\succ_S) = \{(s_1, c_2), (s_2, c_3)\}$ holds. In both cases, since a student can benefit by misreporting, φ cannot satisfy SP. □

This theorem implies that violation of M-convexity easily leads to the nonexistence of any fruitful mechanism. In the rest of this paper, we show that if the distributional constraints of a market can be represented as an M-convex set, there exists a mechanism that satisfies the three properties. Therefore, in a sense, M-convexity is the most general class of distributional constraints under which we can still have a mechanism with these desirable properties.

4 PROPERTIES OF M-CONVEX SET

In this section, we present several properties related to M-convexity that are used in later sections. These properties are either already known in the literature or proving them is rather straightforward. To be self-contained, however, we provide proofs.

Lemma 4.1 (Murota [32, Lemma 9.23], Fujishige [12, Lemma 4.5]). *Let F be an M-convex set on $M = \{1, \ldots, m\}$. Suppose for some $v \in F$, there exist $i_1, j_1, \ldots, i_r, j_r$, all are in M and distinct, such that*

$$\begin{cases} v + \chi_{i_h} - \chi_{j_k} \in F & \text{if } h = k \\ v + \chi_{i_h} - \chi_{j_k} \notin F & \text{if } h > k \end{cases} \qquad (h, k \in \{1, \ldots, r\}). \tag{1}$$

Then it holds that $v + \sum_{k=1}^r (\chi_{i_k} - \chi_{j_k}) \in F$.

Proof. The proof is done by induction on r. When $r = 1$, it obviously holds. Assume the supposition is true up to $r = \ell$ and consider a case where $r = \ell + 1$. Take two vectors $a := v + (\chi_{i_1} - \chi_{j_1})$ and $b := v + \sum_{k=2}^r (\chi_{i_k} - \chi_{j_k})$. It holds that $a \in F$ by assumption. It also holds that $b \in F$ from the induction argument, because $i_2, j_2, \ldots, i_r, j_r$ are 2ℓ distinct elements in M, which satisfy (1). Since $i_1, j_1, \ldots, i_r, j_r$ are distinct, it holds that $a_{i_1} > b_{i_1}$. It also holds that $\{k \in M \mid a_k < b_k\} = \{j_1, i_2, i_3, \ldots, i_r\}$. From the M-convexity of F, there must exist $j \in \{j_1, i_2, i_3, \ldots, i_r\}$ such that $a + (\chi_j - \chi_{i_1}) = v + (\chi_j - \chi_{j_1}) \in F$. It follows that j_1 is the only candidate, since $v + \chi_{i_h} - \chi_{j_1} \notin F$ for any i_h, $h > 1$. It then follows that $b - (\chi_{j_1} - \chi_{i_1}) = b + (\chi_{i_1} - \chi_{j_1}) = v + \sum_{k=1}^r (\chi_{i_k} - \chi_{j_k}) \in F$. □

In words, Lemma 4.1 means that we can apply unilaterally feasible moves simultaneously if they are *sorted properly* in some sense.

Lemma 4.2. *Let F be an M-convex set on $M = \{1, \ldots, m\}$, $v \in F$ and $J = \{1, \ldots, r\}$. For a fixed $q \in J$, if we are given elements $i_1, j_1, \ldots, i_r, j_r \in M$ such that $\{i_1, \ldots, i_r\} \cap \{j_1, \ldots, j_r\} = \emptyset$ and*

$$\begin{cases} v + \chi_{i_h} - \chi_{j_k} \in F & \text{if } h = k \ne q \\ v + \chi_{i_h} - \chi_{j_k} \notin F & \text{if } h = k = q \\ v + \chi_{i_h} - \chi_{j_k} \notin F & \text{if } h > k \end{cases} \qquad (h, k \in \{1, \ldots, r\}) \tag{2}$$

hold, then we have

$$v + \sum_{\ell \in J} (\chi_{i_\ell} - \chi_{j_\ell}) \notin F.$$

Proof. Assume to the contrary that $v' = v + \sum_{\ell\in J} (\chi_{i_\ell} - \chi_{j_\ell})$ is in F. If $r > q$, by the exchange property for v', v and i_r with

$v'_{i_r} > v_{i_r}$, by $\{i_1, \ldots, i_r\} \cap \{j_1, \ldots, j_r\} = \emptyset$, there exists k such that $v'_{j_k} < v_{j_k}$ and

$$v' - \chi_{i_r} + \chi_{j_k},\ v + \chi_{i_r} - \chi_{j_k} \in F.$$

Furthermore, from (2), k must be equal to r. Then, we have $v + \sum_{\ell \in J \setminus \{r\}} (\chi_{i_\ell} - \chi_{j_\ell}) \in F$. By repeating the same argument, we can assume that $q = r$. Since $v'_{i_q} > v_{i_q}$ holds, by the exchange property for v', v and i_q, there exists k such that $v'_{j_k} < v_{j_k}$ and

$$v' - \chi_{i_q} + \chi_{j_k},\ v + \chi_{i_q} - \chi_{j_k} \in F.$$

However, all of the elements of J are less than or equal to q, which contradicts (2). □

In words, Lemma 4.2 means that if a move is infeasible, it remains infeasible after applying a set of feasible moves.

Lemma 4.3. *Assume for $v \in F$, there exist $I, J \subset M$, $I \cap J = \emptyset$ such that $\forall i \in I$, $\forall j \in J$, the following condition holds:*

$$v + \chi_i - \chi_j \notin F. \tag{3}$$

Then there exists no $v' \in F$ such that the following condition holds:

$$\forall i \in M \setminus J,\ v'_i \geq v_i \text{ and } \exists j \in I,\ v'_j > v_j. \tag{4}$$

Proof. Assume to the contrary that there exists v' satisfying (4). Since F is an M-convex set, for $v', v \in F$ and $i \in I$ with $v'_i > v_i$, there exists j such that $v'_j < v_j$ and

$$v' - \chi_i + \chi_j,\ v + \chi_i - \chi_j \in F.$$

By (4), j must belong to J. However, (3). □

In words, Lemma 4.3 means that if we cannot move one student from J to I, then we cannot increase the number of students in I without decreasing the number of students in $M \setminus J$.

5 PROPOSED MECHANISM (TTC-M)

In this section, we introduce an SP mechanism, which we call Top Trading Cycles mechanism with M-convex set constraints (TTC-M), that achieves a feasible and PE outcome in our settings.

Let us explain the outline of TTC-M. It repeats several rounds. At the beginning of round k, let μ^{k-1} denote the matching of students who have already left the market, and let $\tilde{\mu}^{k-1}$ denote the initial endowment matching of the remaining students. Let S^k denote $\{s \mid (s, c) \in \tilde{\mu}^{k-1}\}$, i.e., the set of remaining students at round k.

Let $\widehat{\mu}^{k-1} := \mu^{k-1} \cup \tilde{\mu}^{k-1}$ denote the tentative matching at the beginning of round k. We say student $s \in S^k$, whose initial endowment school is c_j, is *acceptable for school c_i at round k* if $v(\widehat{\mu}^{k-1}) + \chi_i - \chi_j \in F$ holds. If school c has no acceptable student in S^k, it leaves the market at the beginning of round k. Let C^k denote the set of remaining schools at round k. Note that from this definition, if $\omega(s) = c$, s is always acceptable for c at any round k as long as $v(\widehat{\mu}^{k-1}) \in F$ holds.

The mechanism utilizes a common serial order over students[10] denoted $\succ$. Based on this order, the mechanism constructs $\succ_c$, i.e., the priority order of school c, which is basically identical to $\succ$ but the initial endowment students of c are prioritized. More specifically, $s \succ_c s'$ holds if and only if one of the following condition holds: (i) $\omega(s) = \omega(s') = c$ and $s \succ s'$, (ii) $\omega(s) \neq c$, $\omega(s') \neq c$, and $s \succ s'$, or (iii) $\omega(s) = c$ and $\omega(s') \neq c$. Without loss of generality, we assume $s_1 \succ s_2 \succ \cdots \succ s_n$ holds.

Now we are ready to introduce TTC-M, which is described as Mechanism 1. Intuitively, we can assume in TTC-M, at each round k, each school chooses one student to give the right to use its seat. Then, students with such rights can trade the seats among themselves by constructing trading cycles in G^k by the standard TTC mechanism. Therefore, a student can join a trade only when she is chosen by some school. By definition of $\succ_c$, the priority right of school c is first given to its initial endowment students, where a tie is broken by common priority order $\succ$. When all of its initial endowment students have left the market, school c gives the right to a remaining student who is acceptable (meaning allocating that student to c unilaterally from the current situation does not violate the distributional constraints), where a tie is broken by $\succ$.

Since a student considers her initial endowment school acceptable, and the school considers her acceptable at any round as long as she remains in the market, she eventually gets the right to use a seat of her initial endowment school. Thus, every student is included in a cycle at some round before TTC-M terminates, since she thinks her initial endowment school acceptable.

One particular feature of TTC-M is how it deals with the underlying distributional constraints. At round k of TTC-M, the right of a school is given to a student according to $\succ$, $v(\widehat{\mu}^{k-1})$, and F. If the distributional constraints are on individual maximum quotas, then a school gives its right to a student as far as the number of allocated students is less than the quota. Under more complex distributional constraints such as minimum quotas and/or regional quotas, just looking at the current status and its own quota is insufficient for a school to determine which student it should prioritize. For example, accepting a student, who is initially owned by another school c, decreases the number of students allocated to c, i.e., $v_c(\widehat{\mu}^k)$. It may lead to the violation of distributional constraints. The next example describes how TTC-M works with regional quotas.

Example 5.1. Consider the following market.

- $S = \{s_1, s_2, s_3, s_4, s_5\}$,
- $C = \{c_1, c_2, c_3, c_4\}$,
- $c_2 \succ_{s_1} c_1 = \omega(s_1)$, $c_3 \succ_{s_2} c_2 = \omega(s_2)$, $c_2 \succ_{s_3} c_3 = \omega(s_3)$, $c_3 \succ_{s_4} c_4 = \omega(s_4)$, $c_2 \succ_{s_5} c_4 = \omega(s_5)$ (here, for each student s, we describe $\succ_s$ only for her acceptable schools, i.e., schools weakly better than her initial endowment), and
- $F = \left\{ v \in \mathbb{Z}_+^4 \;\middle|\; \begin{array}{l} \sum_{i \in M} v_i = 5, \\ 0 \leq v_i \leq 2\ \forall i \in M, \\ 2 \leq v_3 + v_4 \leq 3 \end{array} \right\}$.

Set F represents a situation where the schools in the same region (c_3 and c_4) are jointly subject to the regional minimum and maximum quotas, in addition to the individual maximum quota of 2.

First, $\tilde{\mu}^0$ is determined as follows: $\tilde{\mu}^0 = \tilde{\mu} = \{(s_1, c_1), (s_2, c_2), (s_3, c_3), (s_4, c_4), (s_5, c_4)\}$. Note that $v(\tilde{\mu}^0) = (1, 1, 1, 2) \in F$.

At Step 1 of Round 1, since every school still has its initial endowment student in the market, all of the schools remain in the market. Each school c points to a student according to $\succ_c$, $v(\widehat{\mu}^0)$,

[10] In some applications, schools (as well as students) can agree on such an order, e.g., using GPA. If no agreeable order exists, we can rely on a lottery to decide an order randomly. Such an order is also used in a serial dictatorship mechanism [19], which is school-feasible, SP and PE, but not IR. If no student has her initial endowment school, our TTC-M becomes identical to a serial dictatorship mechanism.

Mechanism 1 (TTC-M)

Initialize $\tilde{\mu}^0 \leftarrow \tilde{\mu}, \mu^0 \leftarrow \emptyset, k \leftarrow 1$.
Round k:
Step 1: Construct directed graph G^k as follows.
- Each school c leaves the market if it has no acceptable student in S^k.
  Otherwise, c points to the acceptable student who is highest according to $\succ_c$ in S^k.
- Each student in S^k points to her most preferred school in C^k.
- This creates directed graph $G^k = (V^k, E^k)$, where $V^k = S^k \cup C^k$,
  $(s, c) \in E^k$ represents the fact that student s points to school c,
  and $(c, s) \in E^k$ represents the fact that school c points to student s.

Step 2: Let $\mathscr{C}^k$ be the set of all directed edges that forms cycles in G^k.
Since V^k is finite, there exists at least one cycle in G^k and thus $\mathscr{C}^k$ is nonempty.
Step 3: $\tilde{\mu}^k \leftarrow \tilde{\mu}^{k-1} \setminus \{(s, \omega(s)) \mid (s, c) \in \mathscr{C}^k\}$ and $\mu^k \leftarrow \mu^{k-1} \cup \{(s, c) \mid (s, c) \in \mathscr{C}^k\}$.
Each student s such that $(s, c) \in \mathscr{C}^k$ leaves the market.
Step 4: If $\tilde{\mu}^k = \emptyset$, then return μ^k. Otherwise, $k \leftarrow k + 1$ and go to the next round.

and F. In this round, each school points to its initial endowment student who is highest according to $\succ$. Each student points to her best school that remains in the market. This results in G^1, as shown in Figure 1. There is one cycle: $(c_2, s_2, c_3, s_3, c_2)$. At Step 2, $\mathscr{C}^1$ is $\{(c_2, s_2), (s_2, c_3), (c_3, s_3), (s_3, c_2)\}$. At Step 3, (s_2, c_2) and (s_3, c_3) are removed from $\tilde{\mu}^0$ and (s_2, c_3) and (s_3, c_2) are added to μ^0. $\tilde{\mu}^1$ and μ^1 are determined as follows:

$$\begin{aligned}\tilde{\mu}^1 &= \{(s_1, c_1), (s_4, c_4), (s_5, c_4)\},\\ \mu^1 &= \{(s_2, c_3), (s_3, c_2)\}.\end{aligned}$$

Note that $\nu(\widehat{\mu}^1) = \nu(\widehat{\mu}^0) = (1, 1, 1, 2)$, since at this round s_2 and s_3 'exchange' the seats of their initial endowment schools and thus the distributional vector does not change. At Step 4, TTC-M goes to Round 2 because $\tilde{\mu}^1 \neq \emptyset$.

At Step 1 of Round 2, schools c_2 and c_3 do not have their initial endowment students. School c_2 points to s_1 because she is highest according $\succ_{c_2}$ among the remaining students and allocating her to c_2 from her initial endowment school c_1 does not violate distributional constraints ($(1, 1, 1, 2) + \chi_{c_2} - \chi_{c_1} = (0, 2, 1, 2) \in F$), that is, she is acceptable for c_2 at Round 2. For school c_3, however, s_1 is not acceptable because $(1, 1, 1, 2) + \chi_{c_3} - \chi_{c_1} = (0, 1, 2, 2) \notin F$ due to the maximum quota of the region containing c_3 and c_4. On the other hand, moving a student from c_4 to c_3 is feasible. Thus, c_3 points to s_4 according to $\succ_{c_3}$. Therefore, G^2 is determined as shown in Figure 1. There are two cycles: (c_2, s_1, c_2) and (c_3, s_4, c_3). At Step 2, $\mathscr{C}^2$ is $\{(c_2, s_1), (s_1, c_2), (c_3, s_4), (s_4, c_3)\}$. $\tilde{\mu}^2$ and μ^2 are determined as follows:

$$\begin{aligned}\tilde{\mu}^2 &= \{(s_5, c_4)\},\\ \mu^2 &= \{(s_2, c_3), (s_3, c_2), (s_1, c_2), (s_4, c_3)\}.\end{aligned}$$

Observe that $\nu(\widehat{\mu}^2) = (0, 2, 2, 1) \in F$. Note that c_2's decision to give its right to student s_1 is based on the fact that moving s_1 from c_1 to c_2, i.e., $(0, 2, 1, 2)$ is in F. Also, c_3's decision to give the right to student s_4 is based on the fact that moving s_4 from c_4 to c_3, i.e., $(1, 1, 2, 1)$ is in F. The fact that merging these moves still gives a feasible vector is guaranteed by M-convexity, as we show in the proof of Theorem 5.3. At Step 4, TTC-M goes to Round 3 because $\tilde{\mu}^2 \neq \emptyset$.

At Step 1 of Round 3, G^3 is determined as shown in Figure 1. Since there is no student acceptable for c_2 and c_3, these schools leave the market. There is one cycle: (c_4, s_5, c_4). At Step 2, $\mathscr{C}^3$ is $\{(c_4, s_5), (s_5, c_4)\}$. Therefore, $\tilde{\mu}^3$ and μ^3 are determined as follows:

$$\begin{aligned}\tilde{\mu}^3 &= \emptyset,\\ \mu^3 &= \{(s_2, c_3), (s_3, c_2), (s_1, c_2), (s_4, c_3), (s_5, c_4)\}.\end{aligned}$$

At Step 4, TTC-M returns μ^3 because $\tilde{\mu}^3 = \emptyset$.

We use the following property to prove that TTC-M is feasible.

Lemma 5.2. *In TTC-M, for $k \geq 1$, if $\nu(\widehat{\mu}^k) \neq \nu(\widehat{\mu}^{k-1})$, there exist some $r \geq 1$ and $\{\tilde{c}_1, \omega(\tilde{s}_1), \ldots, \tilde{c}_r, \omega(\tilde{s}_r)\} \subseteq C^k$ such that*

$$\nu(\widehat{\mu}^k) = \nu(\widehat{\mu}^{k-1}) + \sum_{\ell=1}^{r} (\chi_{\tilde{c}_\ell} - \chi_{\omega(\tilde{s}_\ell)}), \tag{5}$$

where $\tilde{s}_1, \ldots, \tilde{s}_r$ are ordered such that $\tilde{s}_1 \succ \cdots \succ \tilde{s}_r$ and $\tilde{c}_1, \ldots, \tilde{c}_r$ are ordered such that $(\tilde{c}_\ell, \tilde{s}_\ell) \in E^k$ for all $\ell = 1, \ldots, r$. Furthermore, the schools in $\{\tilde{c}_1, \omega(\tilde{s}_1), \ldots, \tilde{c}_r, \omega(\tilde{s}_r)\}$ are distinct, no schools in $\{\tilde{c}_1, \ldots, \tilde{c}_r\}$ have initial endowment students in S^k, and

$$\begin{cases} \nu(\widehat{\mu}^{k-1}) + \chi_{\tilde{c}_h} - \chi_{\omega(\tilde{s}_\ell)} \in F & \text{if } h = \ell \\ \nu(\widehat{\mu}^{k-1}) + \chi_{\tilde{c}_h} - \chi_{\omega(\tilde{s}_\ell)} \notin F & \text{if } h > \ell \end{cases} \quad (h, \ell \in \{1, \ldots, r\}). \tag{6}$$

Proof. In TTC-M, the fact that $(c, s) \in \mathscr{C}^k$ means that "school c accepts a student (who is pointing to c), while student s moves from her initial endowment school $\omega(s)$ to a school (to which s is pointing)." Therefore, $\nu(\widehat{\mu}^k)$ can be expressed as:

$$\nu(\widehat{\mu}^k) = \nu(\widehat{\mu}^{k-1}) + \sum_{(c,s) \in \mathscr{C}^k} (\chi_c - \chi_{\omega(s)}).$$

From this expression, it is clear that having $(c, s) \in \mathscr{C}^k$ with $\omega(s) = c$ does not affect the resulting vector. When $\nu(\widehat{\mu}^k) \neq \nu(\widehat{\mu}^{k-1})$, there

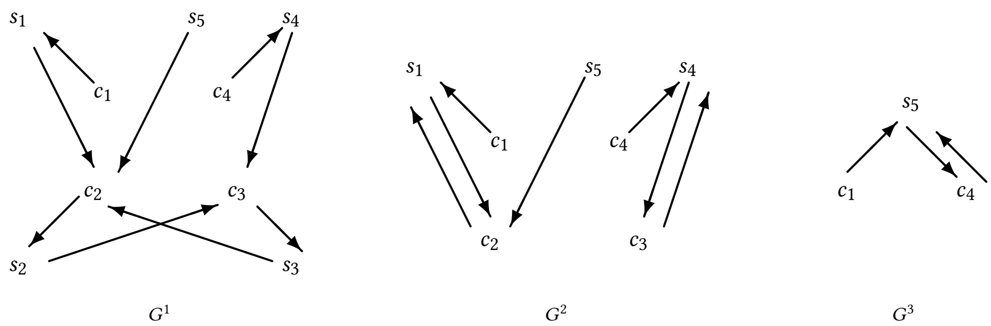

Figure 1: G^1, G^2 and G^3 obtained from Example 1.

exist some $r \geq 1$ and $\{\tilde{c}_1, \omega(\tilde{s}_1), \ldots, \tilde{c}_r, \omega(\tilde{s}_r)\} \subseteq C^k$ such that

$$\begin{aligned} \nu(\widehat{\mu}^k) =& \nu(\widehat{\mu}^{k-1}) + \sum_{(c,s)\in \mathscr{C}^k} (\chi_c - \chi_{\omega(s)}) \\ =& \nu(\widehat{\mu}^{k-1}) + \sum_{(c,s)\in \mathscr{C}^k,\ \omega(s)\neq c} (\chi_c - \chi_{\omega(s)}) \\ =& \nu(\widehat{\mu}^{k-1}) + \sum_{\ell=1}^{r} (\chi_{\tilde{c}_\ell} - \chi_{\omega(\tilde{s}_\ell)}), \end{aligned}$$

where $\tilde{s}_1, \ldots, \tilde{s}_r$ are ordered such that $\tilde{s}_1 \succ \cdots \succ \tilde{s}_r$ and $\tilde{c}_1, \ldots, \tilde{c}_r$ are ordered such that $(\tilde{c}_\ell, \tilde{s}_\ell) \in \mathscr{C}^k$ for all $\ell = 1, \ldots, r$. Next, we show that the elements in $\{\tilde{c}_1, \omega(\tilde{s}_1), \ldots, \tilde{c}_r, \omega(\tilde{s}_r)\}$ are distinct. Since each school may receive at most one student in a round, it is clear that elements in $\{\tilde{c}_1, \ldots, \tilde{c}_r\}$ are distinct. Also, it is clear that elements in $\{\omega(\tilde{s}_1), \ldots, \omega(\tilde{s}_r)\}$ are distinct, since if $\omega(\tilde{s}_i) = \omega(\tilde{s}_h)$ holds where $i < h$, $\tilde{c}_h$ should have pointed to $\tilde{s}_i$ (not $\tilde{s}_h$) according to $\succ_{\tilde{c}_h}$. Furthermore, in G^k, each school in $\{\tilde{c}_1, \ldots, \tilde{c}_r\}$ points to a student who is not its initial endowment student, while each school in set $\{\omega(\tilde{s}_1), \ldots, \omega(\tilde{s}_r)\}$ points to its initial endowment student. Therefore, all the schools in set $\{\tilde{c}_1, \omega(\tilde{s}_1), \ldots, \tilde{c}_r, \omega(\tilde{s}_r)\}$ are distinct. Besides, each school in set $\{\tilde{c}_1, \ldots, \tilde{c}_r\}$ points to its best acceptable student according to the common order $\succ$. Thus (6) holds. □

Now, we are ready to prove TTC-M is feasible.

Theorem 5.3. *TTC-M is feasible.*

Proof. We show that TTC-M always obtains a school-feasible and IR outcome. First, we show by induction that $\widehat{\mu}^k$ is school-feasible for any k. For $k = 0$, it is clear from the assumption that $\widehat{\mu}^0 = \tilde{\mu}$ is school-feasible. By assuming that $\nu(\widehat{\mu}^{k-1}) \in F$ is true for some $k \geq 1$, the induction is completed by showing $\nu(\widehat{\mu}^k) \in F$. Since $\nu(\widehat{\mu}^k)$ is represented as (5), by (6) and Lemma 4.1, $\nu(\widehat{\mu}^k) = \nu(\widehat{\mu}^{k-1}) + \sum_{\ell=1}^{r} (\chi_{\tilde{c}_\ell} - \chi_{\omega(\tilde{s}_\ell)})$ must be in F.

The outcome is IR since (i) each student belongs to a cycle precisely once, and (ii) a student never points to a school that she reports to be worse than her initial endowment school. (i) follows from the definition of TTC-M. (ii) holds because $\nu(\widehat{\mu}^k) + \chi_i - \chi_j \in F$ is satisfied for any k if $j = i$, that is, as long as a student is in the market, her initial endowment school remains in the market and considers her acceptable. □

Next, let us show a lemma related to the following property of TTC-M, i.e., if student s is unacceptable for school c at round k, then s remains unacceptable for c at any round after k. This is a key property to show SP and PE.

Lemma 5.4. *Let s' be a student in S^{k+1} and $c' = \omega(s')$. For $c \in C$ having no initial endowment students in S^k (c may not be in the market at round k), if $\nu(\widehat{\mu}^{k-1}) + \chi_c - \chi_{\omega(s)} \notin F$ for all students $s \in S^k$ with $s \succeq s'$ then $\nu(\widehat{\mu}^k) + \chi_c - \chi_{c'} \notin F$.*

Proof. If $\nu(\widehat{\mu}^k) = \nu(\widehat{\mu}^{k-1})$ then the assertion obviously holds. Let us assume that $\nu(\widehat{\mu}^k) \neq \nu(\widehat{\mu}^{k-1})$. By Lemma 5.2, there exist some $r \geq 1$ and $\{\tilde{c}_1, \omega(\tilde{s}_1), \ldots, \tilde{c}_r, \omega(\tilde{s}_r)\} \subseteq C^k$ satisfying (5) and (6). We remark that $s' \in S^{k+1}$ and no student in $\{\tilde{s}_1, \ldots, \tilde{s}_r\}$ is included in S^{k+1}. We can assume a strict order on $\{\tilde{s}_1, \ldots, \tilde{s}_r, s'\}$ as $\tilde{s}_1 \succ \cdots \succ \tilde{s}_p \succ s' \succ \tilde{s}_{p+1} \succ \cdots \succ \tilde{s}_r$. According to this order, we consider orders on $\{\tilde{c}_1, \ldots, \tilde{c}_r, c\}$ and $\{\omega(\tilde{s}_1), \ldots, \omega(\tilde{s}_r), \omega(s')\}$ as below: $\tilde{c}_1, \ldots, \tilde{c}_p, c, \tilde{c}_{p+1}, \ldots, \tilde{c}_r$, and $\omega(\tilde{s}_1), \ldots, \omega(\tilde{s}_p), \omega(s'), \omega(\tilde{s}_{p+1}), \ldots, \omega(\tilde{s}_r)$.

We will apply Lemma 4.2 to the above schools in order to show the assertion. We have the following properties:

- $\nu(\widehat{\mu}^{k-1}) \in F$ by Theorem 5.3,
- $\{\tilde{c}_1, \ldots, \tilde{c}_r, c\} \cap \{\omega(\tilde{s}_1), \ldots, \omega(\tilde{s}_r), \omega(s')\} = \emptyset$ by Lemma 5.2,
- $\{\tilde{c}_1, \omega(\tilde{s}_1), \ldots, \tilde{c}_r, \omega(\tilde{s}_r)\}$ satisfies (6) by Lemma 5.2,
- $\nu(\widehat{\mu}^{k-1}) + \chi_c - \chi_{\omega(s')} \notin F$ by the hypothesis,
- $\nu(\widehat{\mu}^{k-1}) + \chi_c - \chi_{\omega(\tilde{s}_\ell)} \notin F$ for $\ell \in \{1, \ldots, p\}$ by the hypothesis, and
- $\nu(\widehat{\mu}^{k-1}) + \chi_{\tilde{c}_h} - \chi_{\omega(s')} \notin F$ for $h \in \{p+1, \ldots, r\}$; since otherwise, $\tilde{c}_h$ should have pointed to s' (not $\tilde{s}_h$).

From Lemma 4.2 and the above facts, $\nu(\widehat{\mu}^k) + \chi_c - \chi_{c'} = \nu(\widehat{\mu}^{k-1}) + \sum_{\ell=1}^{r} (\chi_{\tilde{c}_\ell} - \chi_{\omega(\tilde{s}_\ell)}) + (\chi_c - \chi_{\omega(s')})$ is not contained in F. □

Next, we show TTC-M satisfies SP and PE.

Theorem 5.5. *TTC-M is SP.*

Proof. We first show the followings:

(i) $(s, c) \in E^k$ and $s, c \in V^{k+1}$ imply $(s, c) \in E^{k+1}$,
(ii) $(c, s) \in E^k$ and $s \in V^{k+1}$ imply $c \in V^{k+1}$ and $(c, s) \in E^{k+1}$,
(iii) $(s, c) \in E^k$, $(c, s') \in E^k$, and $s' \in V^{k+1}$ imply $s \in V^{k+1}$.

(i) means that, if $(s, c) \in E^k$ holds, i.e., c is the best remaining school for s at round k, and s and c remain in the market at round $k + 1$, then c is still the best one for s at this round. This is true because the schools that left the market will never come back in a later round in TTC-M.

(ii) means that, if s is the highest-ranked acceptable student for c at round k, then s remains so in the next round, given that s is still in the market. To show this, we first observe that $\nu(\widehat{\mu}^k)+\chi_c-\chi_{\omega(s)} \in F$ holds, that is, s is acceptable for c at $k+1$. If $\omega(s) = c$, it obviously holds. If $\omega(s) \neq c$, then consider a hypothetical case where c is the most preferred school of s at round k. This case could have happened, since the preference of each student is arbitrary. In this situation, exactly one cycle (c, s, c) is formed at round k in addition to the cycles formed in the original situation. From Theorem 5.3, the resulting vector at the end of k in this hypothetical setting should be feasible, that is, $\nu(\widehat{\mu}^k) + \chi_c - \chi_{\omega(s)} \in F$. From this observation, c has an acceptable student at $k+1$ and therefore $c \in V^{k+1}$. Next, we show that s remains the highest-ranked acceptable student for c at $k+1$. If $\omega(s) = c$, then it clearly holds. Assume $\omega(s) \neq c$ and let s' be a student in V^{k+1} with $s' \succ s$. The fact that $(c, s') \notin E^k$ implies that $\nu(\widehat{\mu}^{k-1}) + \chi_c - \chi_{\omega(s')} \notin F$. Then from Lemma 5.4 and $\nu(\widehat{\mu}^{k-1}) + \chi_c - \chi_{\omega(s)} \in F$, it holds that $\nu(\widehat{\mu}^k) + \chi_c - \chi_{\omega(s')} \notin F$, and thus s' is unacceptable for c at round $k+1$. Therefore, we have $(c, s) \in E^{k+1}$.

(iii) is an elementary property, which is inherited from the standard TTC mechanism. Assume $(s, c) \in E^k$, $(c, s') \in E^k$ and student s leaves the market at round k. Student s leaves the market only when (s, c) is included in a cycle. If (s, c) is included in a cycle, then (c, s') must be also included in the same cycle. Then, s' must leave the market at round k. Thus, the fact that s' remains in the market implies s also remains in the market.

From (i), (ii) and (iii), for any directed path $(c, \ldots, s)$, if $(c, \ldots, s)$ is in G^k and $s \in V^{k+1}$, then $(c, \ldots, s)$ is also in G^{k+1}. This further implies that once there is a directed path to a student in a round, it stays in any later round as long as she is in the market. By construction of TTC-M, a student can obtain a school seat only if there is a directed path from it to her. Fix s and $\succ_{-s}$. Let a school be called obtainable at k if s can obtain its seat by pointing to it at k, i.e., the schools that are on the directed paths to her in G^k. It is now clear that the set of obtainable schools is increasing in k, and how the set grows depends only on $\succ_{-s}$. Since s can obtain a school seat only from her obtainable schools at a round, what she can do at best is to choose an obtainable school at the round when it becomes her best school in the market. Reporting her true preference $\succ_s$ will do the job, and therefore TTC-M is SP. □

Theorem 5.6. *TTC-M is PE.*

Proof. Suppose we run TTC-M and feasible matching μ is obtained. Take a student who is matched at round r in TTC-M. We show that she cannot be allocated to a school that is better than her allocation in μ without making a student who is matched before r worse off. The proof is done by induction on r.

When $r = 1$, the statement is trivially true because a matched student at round 1 is allocated to her top choice.

Assume the supposition is true up to $r = k-1$ with $k \geq 2$. Consider $r = k$ and let us define the following notations.

$I^k := C \setminus C^k$: schools that are not in the market at round k.
$J^k := \{c \mid \tilde{\mu}_c^{k-1} \neq \emptyset\}$: a set of schools, each of which has at least one remaining initial endowment student in the market. Note that $J^k \subseteq C^k$ holds.

Each $c_i \in C \setminus J^k$ is filled with $\nu_i(\widehat{\mu}^{k-1}) = \nu_i(\mu^{k-1})$ students at the beginning of round k. Without loss of generality, assume $I^k \neq \emptyset$ (if $I^k = \emptyset$, every student matched at k goes to her top choice). Take student, s, who is matched to school c at round k, and assume c is not her top choice. Then all the schools that she prefers to c are in I^k. From Lemma 5.4 and the definitions of I^k and J^k, $\forall c_i \in I^k$ and $\forall c_j \in J^k$, $\nu(\widehat{\mu}^{k-1}) + \chi_i - \chi_j \notin F$ holds. Lemma 4.3 then implies that there is no feasible matching μ' such that

$$\forall c_i \in C \setminus J^k,\ \nu_i(\mu') \geq \nu_i(\mu^{k-1}) \text{ and } \exists c_i \in I^k,\ \nu_i(\mu') > \nu_i(\mu^{k-1}).$$

Put differently, for any feasible matching μ' with $\mu'_s \in I^k$ (which covers all the feasible matchings where the allocation of s is better than c), at least one of either

$$\exists c_i \in C \setminus J^k,\ \nu_i(\mu') < \nu_i(\mu^{k-1}) \text{ or } \nu_{\mu'_s}(\mu') \leq \nu_{\mu'_s}(\mu^{k-1})$$

holds. Whichever is the case, there exists a student who is matched before k in μ and has a different allocation in μ'. From the induction argument, however, such a change necessarily makes someone who is matched before k worse off. □

Finally, let us examine the time complexity of TTC-M.

Theorem 5.7. *The time complexity of TTC-M is $O(|S| \cdot |C|)$ under the assumption that we can check in $O(1)$ time whether $\nu \in F$ for an M-convex set F and a vector ν on C.*

Proof. Since there exists at least one cycle in each round, at least one student in S leaves the market with her allocation. Therefore, the number of rounds required for TTC-M is at most $|S|$. Also, for each round, there are at most $|C|$ students who are able to be a part of cycles, since each remaining school points to exactly one student, and finding the cycles can be done in $O(|C|)$. Furthermore, a school needs to check whether a student is acceptable or not. By Lemma 5.4, when a student becomes unacceptable, she remains unacceptable in future rounds. Thus, for each school, the cost required for this check (until the school leaves the market) is $O(|S|)$. Thus, the overall time complexity is $O(|S| \cdot |C|)$. □

6 CONCLUSIONS

In this paper, we considered an allocation problem of multiple types objects to agents, where each type of an object has multiple copies, each agent is endowed with an object, and some distributional constraints are imposed on the allocation. A representative application domain of this setting is a school choice problem, in which each student has a right to attend her nearby school (and moves to another school only if she prefers it over her default school), while some distributional constraints, such as minimum/maximum quotas in regions must be satisfied for schools to operate. We developed a mechanism called TTC-M, which is feasible, SP, and PE when distributional constraints are represented as an M-convex set. Our future works include developing mechanisms that can work for class of constraints that is broader than M-convex sets (with weaker efficiency conditions).

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