Local Fairness in Hedonic Games via Individual Threshold Coalitions

Nhan-Tam Nguyen
Institut für Informatik
Heinrich-Heine-Universität Düsseldorf
Düsseldorf, Germany
nguyen@cs.uni-duesseldorf.de

Jörg Rothe
Institut für Informatik
Heinrich-Heine-Universität Düsseldorf
Düsseldorf, Germany
rothe@cs.uni-duesseldorf.de

ABSTRACT

Hedonic games are coalition formation games where players only specify preferences over coalitions they are part of. We introduce and systematically study local fairness notions in hedonic games by suitably adapting fairness notions from fair division. In particular, we introduce three notions that assign to each player a threshold coalition that only depends on the player's individual preferences. A coalition structure (i.e., a partition of the players into coalitions) is considered locally fair if all players' coalitions in this structure are each at least as good as their threshold coalitions. We relate our notions to previously studied concepts and show that our fairness notions form a proper hierarchy. We also study the computational aspects of finding threshold coalitions and of deciding whether fair coalition structures exist in additively separable hedonic games. At last, we investigate the price of fairness.

Keywords

Coalition formation; hedonic games; fairness; game theory

1. INTRODUCTION

Coalition formation plays a crucial role in multiagent systems when agents have to cooperate. A commonly studied model of coalition formation is the model of hedonic game. These are coalition formation games with nontransferable utility, which were first studied by Drèze and Greenberg [21] and later on by Banerjee et al. [8] and Bogomolnaia and Jackson [11]. A key feature of hedonic games is that the players' preferences depend only on coalitions they are part of. Since players specify their preferences over an exponential-size domain (in the number of players), various compact representations have been proposed, which either are fully expressive but may still have an exponential size in the worst case or restrict the preference domain [7, 24, 11, 20, 3, 30, 17]. Most of these studies are concerned with stability issues. Intuitively, they capture incentives of (groups of) players to deviate by joining a different coalition so as to increase their individual utility values. Thus stability-related questions address a decentralized aspect of hedonic games.

A more recent approach to hedonic games is welfare maximization [14, 4, 5]. This idea is different because welfare maximization usually presupposes a central authority guiding the maximization process by eliciting preferences and suggesting or enforcing an optimal solution. This enforcement may be necessary because the optimality of a solution is determined by a global criterion, such as utilitarian or egalitarian social welfare, and may affect some players' utility values negatively compared with the status quo.

In this paper we will focus on the concept of local fairness. Fairness is an important aspect besides stability and efficiency (see the related work section and, e.g., [19] for a discussion of fairness in multiagent systems and [12, 31] for fair division of indivisible goods). The only work that we are aware of considering fairness in hedonic games is due to Bogomolnaia and Jackson [11], Aziz et al. [4], Wright and Vorobeychik [38], and Peters [34, 35]. Fairness is related to both centralized approaches and stability issues. On the one hand, the center may want to ensure a certain utility level for each player. This goal can be achieved by a global fairness condition. However, fairness does not per se presuppose the existence of a center. On the other hand, players may not consider their current coalition fair, given their individual preferences. While we agree with Bogomolnaia and Jackson [11] that stability has a " 'restricted fairness' flavor," we add that one can also take the complementary view that lack of fairness can be a major cause of instability.

To make this more concrete, consider a situation where all players except a single player in some coalition consider this coalition their favorite one, yet for that single player this coalition is actually only marginally better than being alone. However, because everyone else prefers this coalition and thus is much better off than that player, she rejects this coalition. This can be considered an unfair situation and is comparable to an ultimatum game situation, where the proposal is very imbalanced and the second player (responder) rejects the proposal because it is below her fair share (see, again, [19]). Note that we would have to contrast the single player's utility to the other players' utility values in order to explain the predicament. This approach of balanced utility values and inequality reduction requires either a center that knows all players' utility values or that players look at coalitions (and even other players' well-being) outside of their own. To some extent, however, this is at odds with the idea of hedonic game because players in such games should only be interested in their own coalition.

The traditional fairness notion of envy-freeness requires players to inspect other coalitions. If there is a large number of coalitions, that is something we would like to avoid. Therefore, we propose and study notions of *local* fairness—restricted fairness notions with the additional constraint that players only compare their current coalition to some bound that *solely* depends on their individual preferences.[1] We feel that this is in the general spirit of the decentralized aspect of hedonic games.

[1] In case of envy-freeness the bound would also depend on the whole coalition structure.

Appears in: *Proceedings of the 15th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2016), J. Thangarajah, K. Tuyls, S. Marsella, C. Jonker (eds.), May 9–13, 2016, Singapore.*

Contribution: In order to achieve this goal of local fairness criteria, we introduce new fairness notions for hedonic games that are inspired by ideas from the field of fair division of indivisible goods. Our main contributions are the following:

1. We introduce the idea of local fairness and three specific such fairness notions in hedonic games. We show that these concepts form a (strict) hierarchy and relate them to previously studied concepts. Surprisingly, the hierarchy strikingly differs from the scale proposed by Bouveret and Lemaître [13] in the context of fair division of indivisible goods.

2. We systematically study the complexity of finding "threshold coalitions" and of determining whether a fair coalition structure exists in an additively separable hedonic game. We also find that two of our notions coincide in such games.

3. We initiate the study of price of fairness in hedonic games. In addition, we strengthen a result by Brânzei and Larson [14] on coalition structures maximizing social welfare in symmetric additively separable hedonic games.

Related Work: Surveys and book chapters on hedonic games are, for example, due to Aziz and Savani [6], Elkind and Rothe [23], and Hajdukova [26]. Bogomolnaia and Jackson [11] already mention envy-freeness in their work, but they focus on studying stability notions. Aziz et al. [4] study the complexity of determining the existence of stable coalition structures in additively separable hedonic games. They also consider the welfare maximization approach and the notion of envy-freeness. The work by Wright and Vorobeychik [38] is related to ours. They study hedonic games under the perspective of mechanism design and propose mechanisms for solving the team formation problem. A key difference is that they consider additively separable hedonic games with non-negative values only. Since in this case the grand coalition is most preferred by every player, Wright and Vorobeychik introduce cardinality constraints on feasible coalition sizes. They also consider *envy bounded by a single teammate*, which for the aforementioned reasons is not suitable for our goals. In addition, they introduce the *maximin share guarantee for team formation*, which is based on the idea of replacing players. This, however, leads to a provably different notion than ours (see Theorem 6). More recently introduced stability notions include *strong Nash stability*, proposed by Karakaya [28], and *strictly strong Nash stability*, due to Aziz and Brandl [2]. Brânzei and Larson [14] study social welfare maximization and core stability in additively separable hedonic games. Moreover, they consider the so-called stability gap. Bilo et al. [9, 10] study similar notions in fractional hedonic games.

Pareto optimality can be considered a notion of stability [32] as well. Elkind et al. [22] investigate the price of Pareto optimality in various representations of hedonic games. Peters and Elkind [36] give conditions on when certain classes of hedonic games admit intractable problems regarding the existence of stable coalition structures. Peters [35] considers restrictions of hedonic games that admit fast algorithms and he models allocating indivisible goods as hedonic game.

Fair division of indivisible goods and hedonic games are closely related because both fields deal with partitions of sets. In fair division, a set of goods needs to be partitioned into n subsets, where n is the number of agents. Usually, it is assumed that goods cannot be shared. This is a departure point from hedonic games because the number of coalitions in a partition is only bounded above by n. The no-externality assumption, however, is prevalent (or even defining) in both fields where an agent's utility depends only on the subset of goods that they receive (fair division) or the coalition that they are part of (hedonic games).

Surveys and book chapters on fair division are due, for example, to Chevaleyre et al. [18], Nguyen et al. [33], Lang and Rothe [31], and Bouveret et al. [12]. Inspired by the cut-and-choose protocol from cake cutting, Budish [15] introduced the max-min fair-share criterion. Since then, it has been studied by Procaccia and Wang [37], Kurokawa et al. [29], Amanatidis et al. [1], and Heinen et al. [27]. Bouveret and Lemaître [13] introduce min-max fair share and propose a scale of even more demanding fairness criteria. Caragiannis et al. [16] study the price of fairness in fair division.

Organization of the Paper: In Section 2, we formally define hedonic games and relevant notions of stability. In Section 3, we introduce our notions of fairness and relate them to other stability, fairness, and optimality concepts. In Section 4, we study our notions in additively separable hedonic games under computational aspects. The price of fairness is considered in Section 5, followed by a discussion of our findings and the conclusions in Section 6.

2. PRELIMINARIES

We denote by $N = \{1,\dots,n\}$ the set of *players*. A *coalition* is a subset of N and a *coalition structure* π is a partition of N. The set of all coalition structures over N is $\Pi(N)$. We denote by $\pi(i)$ the unique coalition with player i in coalition structure π and by $\mathscr{N}_i = \{C \subseteq N \mid i \in C\}$ all coalitions that player i is part of. Every player i has a weak and complete preference order $\succeq_i$ over $\mathscr{N}_i$. For $A,B \in \mathscr{N}_i$, we write $A \succeq_i B$ if player i *weakly prefers* coalition A to B; we write $A \succ_i B$ if player i *(strictly) prefers* coalition A to B, i.e., $A \succeq_i B$ but not $B \succeq_i A$; and we write $A \sim_i B$ if $A \succeq_i B$ and $B \succeq_i A$ (i.e., i is indifferent between A and B). Let $\succeq$ be the collection of all $\succeq_i$, $i \in N$. A *hedonic game* is a pair $(N,\succeq)$. It is an *additively separable hedonic game* (ASHG) if for every $i \in N$, there is a valuation function $v_i : N \to \mathbb{Q}$ such that $\sum_{j\in A} v_i(j) \geq \sum_{j\in B} v_i(j) \iff A \succeq_i B$. We write (N,v) for an additively separable hedonic game, where v is the collection of all v_i, $i \in N$. We assume normalization of the valuation functions, that is, $v_i(i) = 0$. We overload the notation to mean $v_i(A) = \sum_{j\in A} v_i(j)$ for each coalition $A \in \mathscr{N}_i$.

Now we define previously studied notions of stability that are relevant for this work. We distinguish, as is common, between group deviations, individual deviations, and other notions.

We consider the following notions of group deviations:

(1) A nonempty coalition $C \subseteq N$ *blocks* a coalition structure π if every $i \in C$ prefers C to $\pi(i)$. A coalition structure π is *core-stable* (CS) if no coalition blocks π.

(2) A coalition $C \subseteq N$ *weakly blocks* a coalition structure π if every $i \in C$ weakly prefers C to $\pi(i)$ and there is some $j \in C$ that prefers C to $\pi(j)$. A coalition structure π is *strictly core-stable* (SCS) if no coalition weakly blocks π.

(3) Given a coalition $H \subseteq N$, coalition structure π' is *reachable from coalition structure* $\pi \neq \pi'$ *by coalition* H if for all $i, j \in N \setminus H$, we have $\pi(i) = \pi(j) \iff \pi'(i) = \pi'(j)$. A nonempty coalition $H \subseteq N$ *weakly Nash-blocks* coalition structure π if there exists some coalition structure π' that is reachable from π by coalition H such that every $i \in H$ weakly prefers $\pi'(i)$ to $\pi(i)$ and there is some $j \in H$ that prefers $\pi'(j)$ to $\pi(j)$. We say π is *strictly strong Nash-stable* (SSNS) if there is no coalition that weakly Nash-blocks π.

As to individual deviations, we need the following definitions:

(1) A coalition structure π is *Nash-stable* (NS) if every $i \in N$ weakly prefers $\pi(i)$ to $C \cup \{i\}$ for every $C \in \pi \cup \{\emptyset\}$.

(2) A coalition structure π is *contractually individually stable* (CIS) if for every $i \in N$, the existence of a coalition $C \in \pi \cup \{\emptyset\}$ with

$C \cup \{i\} \succ_i \pi(i)$ implies that there exists some $j \in C$ such that $C \succ_j C \cup \{i\}$ or there exists some $k \in \pi(i)$ such that $\pi(k) \succ_k \pi(k) \setminus \{i\}$.

Of the remaining notions we need the following:

(1) A coalition structure π is *perfect* if every $i \in N$ weakly prefers $\pi(i)$ to C for every $C \in \mathscr{N}_i$. We refer to this property by PERFECT.

(2) A coalition structure π' *Pareto-dominates* coalition structure π if every $i \in N$ weakly prefers $\pi'(i)$ to $\pi(i)$ and there is some $j \in N$ that prefers $\pi'(j)$ to $\pi(j)$. A coalition structure π is *Pareto-optimal* (PO) if no coalition structure Pareto-dominates it.

(3) A coalition structure π is *envy-free by replacement* (EF-R) if $\pi(i) \succeq_i (\pi(j) \setminus \{j\}) \cup \{i\}$ for every $i, j \in N$.

(4) A coalition $C \in \mathscr{N}_i$ is acceptable for $i \in N$ if $C \succeq_i \{i\}$. A coalition structure π is *individually rational* (IR) if $\pi(i)$ is acceptable for every $i \in N$.

The following notions are defined only for ASHGs (N, v): A coalition structure $\pi \in \Pi(N)$ maximizes

(1) *utilitarian social welfare* (USW) if for every $\pi' \in \Pi(N)$, $\sum_{i \in N} v_i(\pi(i)) \geq \sum_{i \in N} v_i(\pi'(i))$;

(2) *egalitarian social welfare* (ESW) if for every $\pi' \in \Pi(N)$, $\min_{i \in N} v_i(\pi(i)) \geq \min_{i \in N} v_i(\pi'(i))$.

For the last two definitions we make the common assumption (see, for example, [4, 14]) in coalition formation that values are interpersonally comparable.

We also say that a coalition structure π *satisfies some notion X* if π is X or maximizes X.

Figure 1 shows the relationships between these notions. We refer to the surveys, book chapters, and papers mentioned in the related work section for more explanations of these definitions and their interrelations. The notions are chosen such that our separation results in the next section also apply to intermediate notions such as *contractual strict core stability* and *individual stability*.

3. FAIRNESS IN HEDONIC GAMES

We now introduce our fairness criteria, starting with the weakest one. Individual rationality is the most basic notion of stability. It is also the weakest fairness criterion. Similarly to the example in the introduction, a player who is in a coalition not acceptable to her is exploited by the other players in that coalition if this coalition is acceptable to them. In other words, this player has to be in a disliked coalition just for other players to benefit. In this case, a coalition structure consisting of singletons is more preferable for this player. Lowering the bound of acceptable coalitions would make the situation even worse. Note that individual rationality and perfectness are examples of local fairness criteria that only propose a threshold coalition a player has to be part of. In a sense, we look for criteria situated between these two notions. Because all fairness criteria have to satisfy individual rationality necessarily, we consider such fairness criteria only. Following the definition of envy-freeness by replacement [11], it is not immediately clear which players to replace in the criteria that we will propose and how to motivate this. Therefore, we will focus on definitions that are based on players *joining* coalitions (without replacing any players). This is also another reason for why we do not consider the well-known fairness notion of envy-freeness (based on joining) because it coincides with Nash stability. Similarly, the maximin share guarantee for team formation by Wright and Vorobeychik [38] is defined in terms of replacing a player (and, therefore, is different from max-min fairness considered here).

3.1 Min-Max Fairness

Before we formally define the min-max threshold, we illustrate it with the following situation: A player is arriving late and all other players have already formed a coalition structure without her (where the specific form of the coalition structure is irrelevant for this argument). Because the player could not participate in the coalition formation process, the player is allowed to join any coalition. Clearly, this player joins her most preferred coalition. This describes a fairness criterion because someone who was neglected should be allowed to adapt to the situation in the best possible way.

Definition 1. The *min-max threshold* of $i \in N$ is defined as

$$\text{MinMax}_i = \min_{\pi \in \Pi(N \setminus \{i\})} \max_{C \in \pi \cup \{\emptyset\}} C \cup \{i\},$$

where minimization and maximization are with respect to $\succeq_i$. A coalition structure π satisfies *min-max fairness* (MIN-MAX) if

$$\pi(i) \succeq_i \text{MinMax}_i$$

for every $i \in N$.

This notion is the hedonic-games variant of min-max fair share, originally proposed by Bouveret and Lemaître [13] in fair division. We relate min-max fairness to previously known notions of stability. By definition, min-max fairness satisfies individual rationality. Clearly, since USW is not IR, it cannot satisfy min-max fairness. Similarly, EF-R, PO, and CIS cannot imply min-max fairness. Later (in Section 3.3 on max-min fairness) we will see that min-max fairness is independent of most stability notions in the sense that it does not imply them. Now, we check which stability notions except for perfectness imply min-max fairness.

PROPOSITION 1. *A strictly core-stable coalition structure does not necessarily satisfy min-max fairness.*

PROOF. Consider Example 2 in [11]:

$$\{1,2\} \succ_1 \{1\} \succ_1 \{1,2,3\} \succ_1 \{1,3\},$$
$$\{1,2\} \succ_2 \{2\} \succ_2 \{1,2,3\} \succ_2 \{2,3\},$$
$$\{1,2,3\} \succ_3 \{2,3\} \succ_3 \{1,3\} \succ_3 \{3\}.$$

We compute each player's min-max threshold coalition. Player 1 considers coalition structures $\pi = \{\{2\},\{3\}\}$ and $\pi' = \{\{2,3\}\}$. The best acceptable coalition for player 1 with respect to π is $\{1,2\}$ and with respect to π' it is $\{1\}$. Thus $\text{MinMax}_1 = \{1\}$. Analogously, $\text{MinMax}_2 = \{2\}$ and $\text{MinMax}_3 = \{2,3\}$.

Overall, coalition structure $\{\{1,2\},\{3\}\}$ is SCS but does not satisfy min-max fairness. □

COROLLARY 1. *An individually rational or core-stable coalition structure does not necessarily satisfy min-max fairness.*

On the other hand, Nash stability does imply min-max fairness.

THEOREM 1. *Every Nash-stable coalition structure satisfies min-max fairness.*

PROOF. Let π be a Nash-stable coalition structure and $i \in N$. Then $\pi(i) \succeq_i C \cup \{i\}$ for every $C \in \pi \cup \{\emptyset\}$. Since MinMax_i is a best coalition in a *worst* coalition structure for i, $\pi(i) \succeq_i \text{MinMax}_i$. □

The following example shows that min-max fair coalition structures do not always exist (which is to be expected from any reasonable notion of fairness; envy-freeness is a classic fairness condition in fair division of indivisible goods, but in conjunction with completeness or Pareto optimality such partitions do not always exist either). This also shows that coalition structures that maximize egalitarian social welfare do not necessarily satisfy min-max fairness.

Example 1. Consider the following additively separable hedonic game, defined via the values $v_i(j)$:

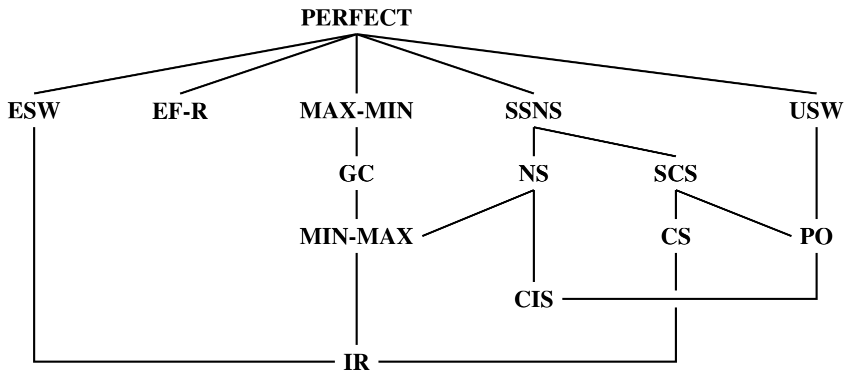

Figure 1: Relations between notions. A line from notion *A* to a lower notion *B* means that every coalition structure that is *A* is also *B*. For example, every strictly core-stable (SCS) coalition structure is core-stable (CS) and Pareto-optimal (PO).

$i \backslash j$	1	2	3
1	0	−10	15
2	−100	0	20
3	10	20	0

The individual min-max thresholds are $\text{MinMax}_1 = 5$, $\text{MinMax}_2 = 0$, and $\text{MinMax}_3 = 20$. Therefore, player 1 has to be in coalition $\{1,3\}$ or $\{1,2,3\}$, player 2 in $\{2\}$ or $\{2,3\}$, and player 3 in $\{2,3\}$ or $\{1,2,3\}$. Hence, there is no min-max fair coalition structure.

Conversely, consider the following hedonic game:

$$\{1,2\} \succ_1 \{1\} \succ_1 \{1,2,3\} \succ_1 \{1,3\},$$
$$\{1,2,3\} \succ_2 \{2,3\} \succ_2 \{1,2\} \succ_2 \{2\},$$
$$\{1,2,3\} \succ_3 \{2,3\} \succ_3 \{1,3\} \succ_3 \{3\}.$$

The players' min-max threshold coalitions are $\text{MinMax}_1 = \{1\}$, $\text{MinMax}_2 = \{2,3\}$, and $\text{MinMax}_3 = \{2,3\}$. Thus $\{\{1\},\{2,3\}\}$ satisfies min-max fairness but not grand-coalition fairness. □

It follows that a coalition structure that satisfies USW, ESW, EF-R, PO, or CIS does not necessarily satisfy grand-coalition fairness (otherwise it would satisfy min-max fairness). Later we will see that grand-coalition fairness is independent of all other considered notions except for perfectness. For now we show that these notions do not imply grand-coalition fairness.

3.2 Grand-Coalition Fairness

Bogomolnaia and Jackson [11] proposed the grand coalition as a notion of fairness. We recover their idea in the context of local fairness. It can be seen as a special variant of proportionality in the setting of hedonic games.

Definition 2. The *grand-coalition threshold* of $i \in N$ is defined as

$$\text{GC}_i = \max\{\{i\}, N\},$$

where we maximize with respect to $\succeq_i$. A coalition structure satisfies *grand-coalition fairness* (GC) if

$$\pi(i) \succeq_i \text{GC}_i$$

for every $i \in N$.

Grand-coalition fairness is a notion of fairness because the grand coalition can be interpreted as an average: Every player has to face both her friends and her enemies. Note that a proportionality threshold is typically defined as the ratio of the valuation for the whole to the number of players. Since players "share" their coalitions, it is not clear which number the valuation of the whole should be compared to. Comparing to the number of coalitions in a coalition structure, however, violates our locality requirement: thresholds that should only depend on a player's own preference.

First, we show that grand-coalition fairness is strictly stronger than min-max fairness.

THEOREM 2. *Every grand-coalition fair coalition structure satisfies min-max fairness, yet a min-max fair coalition structure does not necessarily satisfy grand-coalition fairness.*

PROOF. Let $i \in N$. Every coalition structure serves as an upper bound of MinMax_i. Consider the coalition structure $\{N\}$. Then $\max\{\{i\},N\} \succeq_i \text{MinMax}_i$.

PROPOSITION 2. *A strictly strong Nash-stable coalition structure does not necessarily satisfy grand-coalition fairness.*

PROOF. Consider the following hedonic game:

$$\{1,2\} \succ_1 \{1,3\} \succ_1 \{1\} \succ_1 \{1,2,3\},$$
$$\{1,2,3\} \succ_2 \{1,2\} \succ_2 \{2,3\} \succ_2 \{2\},$$
$$\{2,3\} \succ_3 \{3\} \succ_3 \{1,2,3\} \succ_3 \{1,3\}.$$

Coalition structure $\{\{1,2\},\{3\}\}$ is SSNS but is not grand-coalition fair. □

COROLLARY 2. *An individually rational, Nash-stable, core-stable, or strictly core-stable coalition structure does not necessarily satisfy grand-coalition fairness.*

3.3 Max-Min Fairness

We motivate the next fairness notion with the following situation: Suppose some player is allowed to partition all players excluding herself but does not know which coalition she will be part of in the end. Since she had the right to choose a partition, she has to live with all possible consequences. In other words, she could end up in any of these coalitions, even the worst. Therefore, a player would partition all remaining players so that the worst coalition among them is as good as possible for her.

Definition 3. The *max-min threshold* of $i \in N$ is defined as

$$\text{MaxMin}_i = \max_{\pi \in \Pi(N \setminus \{i\})} \max\{\{i\}, \min_{C \in \pi} C \cup \{i\}\},$$

where maximization and minimization are with respect to $\succeq_i$. A coalition structure π satisfies *max-min fairness* (MAX-MIN) if

$$\pi(i) \succeq_i \text{MaxMin}_i$$

for every $i \in N$.

Note that we cannot include the acceptability constraint into the minimization because then the definition would be weaker than IR. Max-min fairness is the hedonic-games variant of max-min fair share due to Budish [15]. We show that max-min fairness is strictly stronger than grand-coalition fairness.

THEOREM 3. *Every max-min fair coalition structure satisfies grand-coalition fairness, yet a grand-coalition fair coalition structure does not necessarily satisfy max-min fairness.*

PROOF. Let $i \in N$. The coalition structure π consisting of the grand coalition without i is the one where $\max_{C \in \pi \cup \{\emptyset\}} C \cup \{i\}$ and $\max\{\{i\}, \min_{C \in \pi} C \cup \{i\}\}$ become equal. Since every coalition structure gives a lower bound for MaxMin_i and an upper bound for MinMax_i, we have

$$\text{MaxMin}_i \succeq_i \text{GC}_i \succeq_i \text{MinMax}_i.$$

Conversely, consider the following hedonic game:

$$\{1,2\} \succ_1 \{1,3\} \succ_1 \{1,2,3\} \succ_1 \{1\},$$
$$\{1,2,3\} \succ_2 \{1,2\} \succ_2 \{1,3\} \succ_2 \{2\},$$
$$\{2,3\} \succ_3 \{1,2,3\} \succ_3 \{3\} \succ_3 \{1,3\}.$$

Coalition structure $\{\{1,2,3\}\}$ satisfies grand-coalition fairness but not max-min fairness because player 1's max-min threshold coalition is $\{1,3\}$. $\square$

Theorems 2 and 3 give additional motivation of grand-coalition fairness: It is strictly between max-min and min-max fairness. It follows that a USW, ESW, EF-R, PO, or CIS coalition structure does not necessarily satisfy max-min fairness.

Max-min fairness is independent of all other considered notions except for perfectness.

PROPOSITION 3. *A max-min fair coalition structure does not necessarily satisfy contractually individual stability or core stability.*

PROOF. Consider the following hedonic game:

$$\{1,2\} \succ_1 \{1\} \succ_1 \{1,3\} \succ_1 \{1,2,3\},$$
$$\{1,2\} \succ_2 \{2\} \succ_2 \{2,3\} \succ_2 \{1,2,3\},$$
$$\{1,3\} \succ_3 \{3\} \succ_3 \{2,3\} \succ_3 \{1,2,3\}.$$

The max-min threshold coalitions are $\text{MaxMin}_i = \{i\}$, $i \in \{1,2,3\}$. Thus coalition structure $\{\{1\},\{2\},\{3\}\}$ satisfies max-min fairness but neither CIS nor CS. $\square$

COROLLARY 3.
1. *A grand-coalition fair or min-max fair coalition structure does not necessarily satisfy contractually individual stability or core stability.*
2. *A max-min fair, grand-coalition fair, or min-max fair coalition structure does not necessarily satisfy Nash stability, Pareto optimality, strictly strong Nash stability, strict core stability, utilitarian social welfare, or perfectness.*

PROPOSITION 4. *A max-min fair coalition structure does not necessarily satisfy envy-freeness by replacement or egalitarian social welfare.*

PROOF. For EF-R, consider the following hedonic game:

$$\{1,2\} \succ_1 \{1\} \succ_1 \{1,2,3\} \succ_1 \{1,3\},$$
$$\{1,2\} \succ_2 \{2\} \succ_2 \{1,2,3\} \succ_2 \{2,3\},$$
$$\{2,3\} \succ_3 \{3\} \succ_3 \{1,2,3\} \succ_3 \{1,3\}.$$

Coalition structure $\{\{1,2\},\{3\}\}$ satisfies max-min fairness but is not envy-free by replacement.

For ESW, consider the following additively separable hedonic game, defined via the values $v_i(j)$:

i \ j	1	2	3	4
1	0	10	−20	0
2	10	0	−20	0
3	−10	−10	0	10
4	−10	−10	10	0

Coalition structure $\{\{1,2\},\{3\},\{4\}\}$ satisfies max-min fairness but does not maximize egalitarian social welfare (coalition structure $\{\{1,2\},\{3,4\}\}$ has a higher egalitarian social welfare). $\square$

COROLLARY 4. *A grand-coalition fair or min-max fair coalition structure does not necessarily satisfy envy-freeness by replacement or egalitarian social welfare.*

From Proposition 2 we have the following corollary.

COROLLARY 5. *An individually rational, Nash-stable, core-stable, strictly strong Nash-stable, or strictly core-stable coalition structure does not necessarily satisfy max-min fairness.*

See Figure 1 for a summary of the results of this section.

4. LOCAL FAIRNESS IN ASHGS

In this section we study the existence of fair coalition structures, the complexity of computing fairness thresholds and of deciding whether a hedonic game admits a fair coalition structure. Since additively separable hedonic games are a well-studied (see [6] and the references therein) class of hedonic games, we will focus on this class. In addition, it will be easier to compare our complexity results to some results in fair division with additive utility functions. We begin with min-max fairness.

4.1 Min-Max Fairness

We start by computing min-max fairness thresholds. Since we have valuation functions in ASHGs, we can compare to the *value* of threshold coalitions. In particular, we consider the decision problem MIN-MAX-THRESHOLD: Given a set N of players, a player i's valuation function v, and a rational number k, does it hold that $\text{MinMax}_i \geq k$?

By considering coalition structures consisting of either the grand coalition or only of singletons, we have the following observations that show that MIN-MAX-THRESHOLD is easy to solve for certain restricted valuation functions.

OBSERVATION 1. *If* $v_i(N) \leq 0$, *then* $\text{MinMax}_i = 0$.

OBSERVATION 2. *If* $v_i(j) \geq 0$ *for every* $j \in N$, *then* $\text{MinMax}_i = \max_{j \in N} v(j)$.

For general valuation functions, however, we have this result:

THEOREM 4. MIN-MAX-THRESHOLD *is* coNP-*complete.*

PROOF. We consider the complementary problem, which for the same input asks whether $\text{MinMax}_i < k$ holds. Membership of this problem in NP follows from guessing a coalition structure π and comparing the maximum value of a coalition in π to k.

To show NP-hardness, we reduce from a restricted variant of PARTITION that contains no input numbers of value 1. This problem can be shown to be NP-hard by reducing from the original PARTITION problem and multiplying all values by 2.

The reduction works as follows. Let $b_1, \ldots, b_n \in \mathbb{N}$ be a restricted PARTITION instance. We consider valuation function v_i defined as follows: $v_i(j) = b_j$ and $v_i(n+1) = v_i(n+2) = -(L-1)$, where $\sum_{j=1}^{n} b_j = 2L$. Thus $N = \{1, \ldots, n, i, n+1, n+2\}$ with i different from $1, \ldots, n, n+1, n+2$. In addition, we set the bound k to 2.

We show that there is a partition if and only if $\text{MinMax}_i < k$.

From left to right: Suppose (A, B) is a partition of the numbers $b_1, \ldots, b_n$. Consider the coalition structure $\pi = \{A \cup \{n+1\}, B \cup \{n+2\}\}$. We have $\max_{C \in \pi \cup \{\emptyset\}} v_i(C \cup \{i\}) = \max\{1, 1, 0\} = 1$. Therefore, $\text{MinMax}_i \leq 1 < 2 = k$.

From right to left: Suppose for every partition (A, B) we have $\sum_{j \in A} b_j > L$ or $\sum_{j \in B} b_j > L$. Note that for the grand coalition we have $\max_{C \in \{N \setminus \{i\}\} \cup \{\emptyset\}} v_i(C \cup \{i\}) = \max\{2L - 2(L-1), 0\} = 2$.

For every coalition structure consisting of two nonempty coalitions A', B' with $A' \cup B' = N \setminus \{i, n+1, n+2\}$, we have, without loss of generality, $v_i(A' \cup \{i\}) \geq L+1$ and $v_i(B' \cup \{i\}) \leq L-1$. Adding players $n+1$ and $n+2$ to A' would give a maximum of at least 2 because B' is nonempty and $b_j \geq 2$ by assumption. Adding both players to B' would give a maximum of at least $L+1$. Since the partition instance has at least two numbers of value 2, $L \geq 2$. Assigning these players to separate coalitions we have $v_i(A' \cup \{i\} \cup \{n+1\}) \geq L + 1 - L + 1 = 2$. Thus the maximum is at least 2.

For all coalition structures consisting of $\ell > 2$ nonempty coalitions, we have that the maximum value is at least 2 because all numbers of the partition instance are at least 2 and there are only two players with a negative value. Thus there is always a coalition that is valued at least 2.

Since all coalition structures have a coalition which is valued at least 2, we have $\text{MinMax}_i \geq 2 = k$. □

Coming now to the question of existence of fair coalition structures, we first define the decision problem that we study. The input of the problem MIN-MAX-EXIST consists of an additively separable hedonic game. The question is whether a min-max fair coalition structure exists. We start with a simple observation.

OBSERVATION 3. *If $v_i(j) \geq 0$ for every $i, j \in N$, then $\{N\}$ satisfies min-max fairness, i.e., min-max fair coalition structures always exist.*

We say an additively separable hedonic game is *symmetric* if $v_i(j) = v_j(i)$ for every $i, j \in N$. Since there always exist Nash-stable coalition structures in symmetric ASHGs [11], we have

COROLLARY 6. *Symmetric ASHGs always admit min-max fair coalition structures.*

For general additively separable hedonic games, we have NP-hardness as a lower bound and membership in Σ_2^p (the second level of the polynomial hierarchy) as an upper bound. We use a modified version of the game in Example 1 as a gadget.

THEOREM 5. MIN-MAX-EXIST *is* NP-*hard and in* Σ_2^p.

PROOF. We reduce from MONOTONE-ONE-IN-THREE-3SAT (see, e.g., the comment of Garey and Johnson [25, p. 259], on ONE-IN-THREE-3SAT): Given a boolean formula φ that contains only clauses with three positive literals, does there exist a satisfying assignment such that each clause has exactly one true literal?

Now we describe the reduction. Let $\ell_1, \ldots, \ell_n$ be the variables, $C_1, \ldots, C_m$ the clauses of φ, and r_i the number of distinct clauses ℓ_i appears in. For every variable ℓ_i that appears in some clause, we introduce three *variable players*, a_i, b_i, and c_i, and for every clause C_k, we add three *clause players*, D_k, E_k, and F_k.

The valuation functions are defined as follows:

Variable players of type 1, a_i, $1 \leq i \leq n$, value every other variable player except for b_i and c_i with $-3m-1$. They value b_i with $3r_i$, c_i with 0, all clause players D_k, E_k, and F_k for which $\ell_i \in C_k$ with -1, and all remaining players with 0. Variable players of type 2, b_i, $1 \leq i \leq n$, value c_i with $3r_i$, all clause players D_k, E_k, and F_k for which $\ell_i \in C_k$ with 1, and all remaining players with 0. Variable players of type 3, c_i, $1 \leq i \leq n$, value every clause player with -1 and all remaining players with 0.

Clause players of type 1, D_k, $1 \leq k \leq m$, value E_k with -10, F_k with 15, and all remaining players with 0. Clause players of type 2, E_k, $1 \leq k \leq m$, value D_k with -20, F_k with 21, all a_i for which $\ell_i \in C_k$ with 20, and all remaining players with 0. Clause players of type 3, F_k, $1 \leq k \leq m$, value D_k with 10, E_k with 20, and all remaining players with 0.

We compute the min-max thresholds before showing the required equivalence. The min-max threshold of type-1 variable players a_i is 0, of type-2 variable players b_i is $3r_i$, of type-3 variable players c_i is 0, of type-1 clause players D_k is 5, of type-2 clause players E_k is 20 (consider the coalition structure where D_k and F_k are together and all remaining players are in single coalitions), and of type-3 clause players F_k is 20. We show that the given formula is satisfied by an assignment with exactly one true literal per clause if and only if there is a coalition structure satisfying min-max fairness.

From left to right: Suppose there is a satisfying assignment τ with the above property. Denote by $\ell_1, \ldots, \ell_o$ all variables which are true under τ and by $\ell_{o+1}, \ldots, \ell_n$ all remaining variables. Denote by $T(\ell_i)$ the set of clauses that become true under τ via ℓ_i and let $\mathscr{C}(\ell_i) = \{D_k, E_k, F_k \mid C_k \in T(\ell_i)\}$. Consider the following coalition structure: $\pi = \{\mathscr{C}(\ell_1) \cup \{a_1, b_1\}, \{c_1\}, \ldots, \mathscr{C}(\ell_o) \cup \{a_o, b_o\}, \{c_o\}, \{a_{o+1}, b_{o+1}, c_{o+1}\}, \ldots, \{a_n, b_n, c_n\}\}$. In words, for each clause satisfied by some literal, we put into one coalition the corresponding type-1 and type-2 variable players and all three corresponding clause players. In this case, the corresponding type-3 variable player stays alone. If a variable satisfies no clause, then the corresponding variable players are in a coalition that only consists of them. Since each clause is satisfied by exactly one literal, no player is in multiple coalitions simultaneously.

We compute every player's value. Type-1 clause players D_k are in the same coalition as E_k and F_k. Therefore, they have a value of 5. Similarly, type-3 clause players F_k have a value of 30. For variables ℓ_i that are true under τ, since type-1 variable players a_i never share coalitions with other variable players except for b_i and all clause players corresponding to clauses that contain ℓ_i, these a_i have a value of $-3r_i + 3r_i = 0$. For the same reason, type-2 variable players b_i have a value of $3r_i$ and type-2 clause players E_k have a value of $-20 + 21 + 20 = 21$. Type-3 variable players c_i are alone and, hence, have value 0. For variables ℓ_j that are false under τ, the coalition consisting of all variable players corresponding to such a variable gives the corresponding type-1 and type-2 variable players, a_j and b_j, a value of $3r_j$ and the corresponding type-3 variable player c_j a value of 0. Overall, every player achieves her min-max threshold, so coalition structure π satisfies min-max fairness.

From right to left: Suppose there is some coalition structure π satisfying min-max fairness. Let F_k be the type-3 clause player for some clause C_k. Since $v_{F_k}(\pi(F_k)) \geq 20$, $E_k \in \pi(F_k)$. Because $F_k \in$

$\pi(F_k)$ and $v_{D_k}(\pi(D_k)) \geq 5$, $D_k \in \pi(F_k)$. Thus $\{D_k, E_k, F_k\} \subseteq \pi(F_k)$. Because $v_{E_k}(\pi(F_k)) \geq 20$, some variable player a_i corresponding to a literal occurring in clause C_k has to be in $\pi(F_k)$; otherwise, $v_{E_k}(\pi(F_k)) = 1$. If variable players different from b_i or c_i are in $\pi(F_k)$, no such variable player can satisfy the min-max threshold. Since variable player a_i joins $\{D_k, E_k, F_k\}$, $b_i \in \pi(F_k)$. Because $v_{b_i}(\pi(F_k)) \geq 3r_i$, all clause players that are valued positively by b_i are in $\pi(F_k)$. Otherwise, $c_i \in \pi(F_k)$, but then $u_{c_i}(\pi(F_k)) < 0$ because clause players are in $\pi(F_k)$. If clause players that are valued 0 by a_i are in the same coalition, other variable players have to be in that coalition as well, but then min-max fairness is not satisfied. Therefore, every $\pi(F_k)$ contains exactly one a_i. Then we can construct a satisfying assignment with the one-in-three property by making all variables ℓ_i true where a_i is in a coalition with clause players. □

4.2 Grand-Coalition & Max-Min Fairness

In this section, we can consider grand-coalition and max-min fairness at the same time because of the following result:

THEOREM 6. *In additively separable hedonic games, for every $i \in N$ we have*

$$\text{MaxMin}_i = \text{GC}_i.$$

PROOF. It remains to show $\max\{0, v_i(N)\} = \text{GC}_i \geq \text{MaxMin}_i$ because of Theorem 3.

If $v_i(N) < 0$, suppose there is some π such that $\min_{C\in\pi} v_i(C \cup \{i\}) > 0$. Then $v_i(C \cup \{i\}) > 0$ for every coalition $C \in \pi$. This, however, implies $v_i(N) = \sum_{C\in\pi} v_i(C \cup \{i\}) > 0$.

If $v_i(N) \geq 0$, suppose there is some π such that $\min_{C\in\pi} v_i(C \cup \{i\}) > v_i(N)$. Then $v_i(C \cup \{i\}) > v_i(N)$ for every coalition $C \in \pi$, which would imply $v_i(N) = \sum_{C\in\pi} v_i(C \cup \{i\}) > v_i(N)$. □

Define the threshold and existence problems for grand-coalition and max-min fairness analogously to MIN-MAX-THRESHOLD and MIN-MAX-EXIST. Since computing the value of the grand coalition is easy in additively separable hedonic games, we have

COROLLARY 7. MAX-MIN-THRESHOLD *and* GRAND-COALITION-THRESHOLD *are in* P.

However, checking whether there exists a grand-coalition fair or max-min fair coalition structure is hard.

THEOREM 7. *The problems* GRAND-COALITION-EXIST *and* MAX-MIN-EXIST *are* NP*-complete.*

PROOF. Membership in NP follows from guessing and checking. Checking works in polynomial time because of Corollary 7. We reduce from PARTITION. Let $b_1, \dots, b_n$ be a PARTITION instance with $\sum_{i=1}^{n} b_i = 2L$.

We construct the following additively separable hedonic game: $N = \{1, \dots, n, n+1, n+2, n+3, n+4, n+5\}$ with $v_{n+1}(n+2) = -1$, $v_{n+2}(n+1) = -1$, $v_{n+3}(n+1) = v_{n+3}(n+2) = v_{n+4}(n+1) = v_{n+4}(n+2) = L$, $v_{n+3}(i) = v_{n+4}(i) = -b_i$, $v_i(n+3) = v_i(n+4) = 1$, $v_i(n+5) = -1$ for every i, $1 \leq i \leq n$. All other values are 0. The threshold is 1 for player i, $1 \leq i \leq n$, and 0 for the remaining players. We show that there is a partition if and only if there is a coalition structure satisfying grand-coalition (and thus, equivalently in ASHGs, max-min) fairness.

From left to right: Suppose (A, B) is a partition. Then the coalition structure $\{\{n+1, n+3\} \cup A, \{n+2, n+4\} \cup B, \{n+5\}\}$ satisfies grand-coalition fairness: Player $n+1$ is not with player $n+2$. Players $n+3$ and $n+4$ receive 0 value. Each i, $1 \leq i \leq n$, gets a value of 1. Player $n+5$ gets 0.

From right to left: Suppose π satisfies grand-coalition fairness. Then players $n+1$ and $n+2$ are not in the same coalition. Since each i, $1 \leq i \leq n$, needs to get at least a value of 1, every such player is with players $n+3$ or $n+4$. Let C_k contain all i, $1 \leq i \leq n$, that are with player $n+k$, $k \in \{3,4\}$. Then $C_3 \cap C_4 = \emptyset$; otherwise, $n+3$ and $n+4$ are in the same coalition with nonnegative values, which means that $n+1$ and $n+2$ are in the same coalition as well. We also have $C_3 \neq \emptyset$ (otherwise, all i, $1 \leq i \leq n$, are in C_4: $v_{n+4}(\pi(n+4)) \leq -2L + L < 0$). Since $v_{n+3}(\pi(n+3)) \geq 0$, and players $n+1$ and $n+2$ are the only players valued positively by $n+3$, players $n+1$ or $n+2$ are in $\pi(n+3)$. Both of them cannot be in $\pi(n+3)$ because then player $n+4$ would have a negative value. Therefore, $L \geq \sum_{i\in C_k} b_i$, $k \in \{3,4\}$. Since $\sum_{i\in C_3} b_i + \sum_{i\in C_4} b_j = 2L$, we have $\sum_{i\in C_3} b_i = L = \sum_{i\in C_4} b_j$. □

5. PRICE OF FAIRNESS

Now we study the price of fairness in additively separable hedonic games. Informally, the price of fairness captures the loss in social welfare of a worst (best) coalition structure that satisfies some fairness criterion. We denote by $\text{SW}_G(\pi)$ the utilitarian social welfare of coalition structure π in an additively separable hedonic game $G = (N, v)$, that is, $\text{SW}_G(\pi) = \sum_{i\in N} v_i(\pi(i))$. We omit G when it is clear from the context.

Definition 4. Let $G = (N, v)$ be an additively separable hedonic game and let π^* denote a coalition structure maximizing utilitarian social welfare. Define the *maximum price of min-max fairness* by

$$\text{Max-PoMMF}(G) = \max_{\pi\in\Pi(N), \pi \text{ is min-max fair}} \frac{\text{SW}(\pi^*)}{\text{SW}(\pi)}$$

if there is some min-max fair $\pi \in \Pi(N)$ and $\text{SW}(\pi) > 0$ for all min-max fair $\pi \in \Pi(N)$; by $\text{Max-PoMMF}(G) = 1$ if $\text{SW}(\pi^*) = 0$ and $\text{SW}(\pi) = 0$ for some min-max fair $\pi \in \Pi(N)$; and by setting $\text{Max-PoMMF}(G) = +\infty$ otherwise.

Define the *minimum price of min-max fairness* (Min-PoMMF) analogously.

Note that we have $\text{SW}(\pi^*) \geq 0$ and $\text{SW}(\pi) \geq 0$, where π^* maximizes utilitarian social welfare and π is min-max fair.

Because the grand coalition maximizes utilitarian welfare under nonnegative valuation functions, the minimum and maximum price of grand-coalition fairness is one. Since this bound is not really informative, we now make some suitable assumptions to strengthen our results. Elkind et al. [22] argue that these notions are only sensible if the set of coalition structures that we consider is large enough. Because of that we only consider min-max fairness, the weakest fairness notion, in order to constrain the set of feasible coalition structures as least as possible. In addition, Elkind et al. [22] focus on Pareto optimality because such coalition structures always exist. Similarly, we restrict our study to symmetric additively separable hedonic games so as to guarantee the existence of min-max fair coalition structures.

Unfortunately, the maximum price of min-max fairness is not bounded by a constant value even for nonnegative valuation functions.

THEOREM 8. *Let $G = (N, v)$ be a symmetric ASHG of n players with $v_i(j) \geq 0$ for every $i, j \in N$. Then*

$$\text{Max-PoMMF}(G) \leq n - 1.$$

In addition, this bound is tight.

PROOF. If $\text{SW}(\pi^*) = 0$, then $\text{Max-PoMMF}(G) = 1$. Otherwise, there are $i, j \in N$, $i \neq j$, such that $v_i(j) > 0$. We can upper-bound

$\mathrm{SW}(\pi^*)$ by $\sum_{i\in N} v_i(N)$. By Observation 2, we can lower-bound the value of every player i by $\max_{j\in N} v_i(j)$. Thus

$$\begin{aligned}\text{Max-PoMMF}(G) &\leq \frac{\sum_{i\in N} v_i(N)}{\sum_{i\in N}\max_{j\in N} v_i(j)}\\ &\leq \frac{\sum_{i\in N}(n-1)\max_{j\in N} v_i(j)}{\sum_{i\in N}\max_{j\in N} v_i(j)}\\ &= n-1.\end{aligned}$$

To see that this bound is tight, consider a game with n players, n even. Every player values every other player with $a > 0$. Thus the min-max threshold of every player is a. Therefore, the coalition structure that consists of $n/2$ pairs satisfies min-max fairness and has minimum utilitarian social welfare of na among all min-max fair coalition structures. The coalition structure consisting of the grand coalition that maximizes utilitarian social welfare, however, has a utilitarian social welfare of $n(n-1)a$. □

To obtain a meaningful bound in the above result, we need the existence of a min-max fair coalition structure, which is guaranteed in symmetric ASHGs. We need the following result before we can turn to Min-PoMMF.

THEOREM 9. *Let $G = (N, v)$ be a symmetric ASHG. Then every coalition structure π that maximizes utilitarian social welfare satisfies min-max fairness.*

PROOF. Suppose, for the sake of contradiction, that there is some $i \in N$ with $v_i(\pi(i)) < \mathrm{MinMax}_i$. By a result of Brânzei and Larson [14], we know that every coalition structure maximizing utilitarian social welfare satisfies individual rationality. On the other hand, considering π with respect to MinMax, we have

$$0 \leq v_i(\pi(i)) < \mathrm{MinMax}_i \leq \max_{C_j\in\pi} v_i(C_j \cup \{i\}). \quad (1)$$

Because of the strict inequality $\pi(i) \neq C_\ell$, where C_ℓ is a maximizer of $\max_{C_j\in\pi} v_i(C_j \cup \{i\})$, we have $v_i(\pi(i)) < v_i(C_\ell \cup \{i\})$. Since π is a maximizer of utilitarian social welfare, the social welfare of π should not be lower than the coalition structure where i joins C_ℓ:

$$\begin{aligned}&\sum_{k\in\pi(i)} v_k(\pi(i)) + \sum_{k\in C_\ell} v_k(C_\ell) \geq\\ &\sum_{k\in\pi(i),\, k\neq i} v_k(\pi(i)\setminus\{i\}) + \sum_{k\in C_\ell} v_k(C_\ell) + 2v_i(C_\ell \cup \{i\}).\end{aligned}$$

Using symmetry, this is equivalent to the contradiction

$$v_i(\pi(i)) \geq v_i(C_\ell \cup \{i\}),$$

which completes the proof. □

Since min-max fairness is strictly stronger than individual rationality, Theorem 9 strengthens the result by Brânzei and Larson [14] that we used in the proof. Note that the above theorem also implies that we have an alternative proof of Corollary 6 (that every symmetric additively separable hedonic game admits a min-max fair coalition structure) that does not depend on the guaranteed existence of Nash-stable coalition structures but on the guaranteed existence of coalition structures maximizing utilitarian social welfare. From Theorem 9 we immediately have the following corollary.

COROLLARY 8. *Let G be a symmetric ASHG. Then*

$$\text{Min-PoMMF}(G) = 1.$$

6. DISCUSSION & CONCLUSION

We have introduced three new notions of fairness in hedonic games and studied the connection with previously studied notions. Our notions themselves form a strict hierarchy: Every max-min fair coalition structure is grand-coalition fair (but not vice versa), and every grand-coalition fair coalition structure is min-max fair (but not vice versa). Although our fairness criteria are inspired from the field of fair division, our results are very different. Bouveret and Lemaître's scale of fairness criteria for additive utility functions [13] says that an envy-free partition of goods satisfies min-max fair share, which in turn implies proportionality, which in turn implies max-min fair share. So our strongest notion of fairness is the weakest notion in fair division of indivisible goods (according to this scale). In addition, in additively separable hedonic games, we have seen that grand-coalition fairness and max-min fairness coincide. This is not the case in fair division (if one equates grand-coalition fairness with proportionality). Also note that Nash stability (or, equivalently, a definition of envy-freeness based on joining) implies min-max fairness but none of the stronger notions. In the setting of indivisible goods, envy-freeness even implies min-max fair share. So it is one of the strongest notions there. We consider these results surprising, as the intution from fair division of indivisible goods is no longer valid in this different context. The main reasons are the already mentioned difference between the number of allowed subsets of a partition and that players can "share" coalitions. This missing intuition is also a reason of why we have checked in detail whether any known stability notions imply one of our fairness notions.

Then we have studied the complexity of computing threshold coalitions and deciding whether an additively separable hedonic game admits a fair coalition structure. Although nearly all of these problems are intractable, our fairness criteria still have some meaning. They give additional motivation to notions of stability, such as Nash stability. Moreover, in a decentralized setting the hardness of a problem can be "distributed" (of course, the intractability cannot disappear). Giving players a yardstick for fairness that only depends on their own preferences reduces the amount of communication that is necessary to check whether a coalition structure is fair. Our complexity results are also comparable to the results by Bouveret and Lemaître [13] and Heinen et al. [27] with the exception that no lower bound is known for deciding whether a max-min fair-share allocation exists, whereas in ASHGs we know that the corresponding problem is NP-complete. Also note that with min-max fairness we have found a notion that is strictly stronger than individual rationality, but is still satisfied by every coalition structure maximizing utilitarian social welfare in symmetric additively separable hedonic games. At last, we have initiated the study of price of fairness in hedonic games. Our results here are unsatisfactory in the sense that either the price is unbounded or not very informative.

Therefore, we consider finding suitable restrictions to players' valuation functions such that the maximum price of min-max fairness is bounded by a nontrivial constant an interesting research question for the future. Interesting future work would also be identifying (other) sufficient conditions that imply the existence of a fair coalition structure, determining the complexity of searching for a min-max fair coalition structure in symmetric additively separable hedonic games, and showing Σ_2^p-hardness of MIN-MAX-EXIST.

Acknowledgments

We thank the anonymous reviewers for their helpful comments. This work was supported in part by DFG grant RO-1202/14-2.

REFERENCES

[1] G. Amanatidis, E. Markakis, A. Nikzad, and A. Saberi. Approximation algorithms for computing maximin share allocations. In *Proceedings of the 42nd International Colloquium on Automata, Languages, and Programming*, pages 39–51. Springer-Verlag *Lecture Notes in Computer Science #9134*, July 2015.

[2] H. Aziz and F. Brandl. Existence of stability in hedonic coalition formation games. In *Proceedings of the 11th International Joint Conference on Autonomous Agents and Multiagent Systems*, pages 763–770. IFAAMAS, June 2012.

[3] H. Aziz, F. Brandt, and P. Harrenstein. Fractional hedonic games. In *Proceedings of the 13th International Joint Conference on Autonomous Agents and Multiagent Systems*, pages 5–12. IFAAMAS, May 2014.

[4] H. Aziz, F. Brandt, and H. Seedig. Computing desirable partitions in additively separable hedonic games. *Artificial Intelligence*, 195:316–334, 2013.

[5] H. Aziz, S. Gaspers, J. Gudmundsson, J. Mestre, and H. Täubig. Welfare maximization in fractional hedonic games. In *Proceedings of the 24th International Joint Conference on Artificial Intelligence*, pages 461–467. AAAI Press/IJCAI, July 2015.

[6] H. Aziz and R. Savani. Hedonic games. In F. Brandt, V. Conitzer, U. Endriss, J. Lang, and A. Procaccia, editors, *Handbook of Computational Social Choice*, chapter 15. Cambridge University Press, 2016. To appear.

[7] C. Ballester. NP-completeness in hedonic games. *Games and Economic Behavior*, 49(1):1–30, 2004.

[8] S. Banerjee, H. Konishi, and T. Sönmez. Core in a simple coalition formation game. *Social Choice and Welfare*, 18(1):135–153, 2001.

[9] V. Bilò, A. Fanelli, M. Flammini, G. Monaco, and L. Moscardelli. Nash stability in fractional hedonic games. In *Proceedings of the 10th International Workshop on Internet & Network Economics*, pages 486–491. Springer-Verlag *Lecture Notes in Computer Science #8877*, Dec. 2014.

[10] V. Bilò, A. Fanelli, M. Flammini, G. Monaco, and L. Moscardelli. On the price of stability of fractional hedonic games. In *Proceedings of the 14th International Joint Conference on Autonomous Agents and Multiagent Systems*, pages 1239–1247. IFAAMAS, May 2015.

[11] A. Bogomolnaia and M. Jackson. The stability of hedonic coalition structures. *Games and Economic Behavior*, 38(2):201–230, 2002.

[12] S. Bouveret, Y. Chevaleyre, and N. Maudet. Fair allocation of indivisible goods. In F. Brandt, V. Conitzer, U. Endriss, J. Lang, and A. Procaccia, editors, *Handbook of Computational Social Choice*, chapter 12. Cambridge University Press, 2016. To appear.

[13] S. Bouveret and M. Lemaître. Characterizing conflicts in fair division of indivisible goods using a scale of criteria. *Journal of Autonomous Agents and Multi-Agent Systems*. To appear. A preliminary version appeared in the proceedings of AAMAS-2014.

[14] S. Brânzei and K. Larson. Coalitional affinity games and the stability gap. In *Proceedings of the 21st International Joint Conference on Artificial Intelligence*, pages 79–84. AAAI Press/IJCAI, 2009.

[15] E. Budish. The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. *Journal of Political Economy*, 119(6):1061–1103, 2011.

[16] I. Caragiannis, C. Kaklamanis, P. Kanellopoulos, and M. Kyropoulou. The efficiency of fair division. *Theory of Computing Systems*, 50(4):589–610, 2012.

[17] K. Cechlárová and A. Romero-Medina. Stability in coalition formation games. *International Journal of Game Theory*, 29(4):487–494, 2001.

[18] Y. Chevaleyre, P. Dunne, U. Endriss, J. Lang, M. Lemaître, N. Maudet, J. Padget, S. Phelps, J. Rodríguez-Aguilar, and P. Sousa. Issues in multiagent resource allocation. *Informatica*, 30(1):3–31, 2006.

[19] S. de Jong, K. Tuyls, and K. Verbeeck. Fairness in multi-agent systems. *The Knowledge Engineering Review*, 23(2):153–180, 2008.

[20] D. Dimitrov, P. Borm, R. Hendrickx, and S. Sung. Simple priorities and core stability in hedonic games. *Social Choice and Welfare*, 26(2):421–433, 2006.

[21] J. Drèze and J. Greenberg. Hedonic coalitions: Optimality and stability. *Econometrica*, 48(4):987–1003, 1980.

[22] E. Elkind, A. Fanelli, and M. Flammini. Price of Pareto optimality in hedonic games. In *Proceedings of the 30th AAAI Conference on Artificial Intelligence*. AAAI Press, Feb. 2016. To appear.

[23] E. Elkind and J. Rothe. Cooperative game theory. In J. Rothe, editor, *Economics and Computation. An Introduction to Algorithmic Game Theory, Computational Social Choice, and Fair Division*, chapter 3, pages 135–193. Springer-Verlag, 2015.

[24] E. Elkind and M. Wooldridge. Hedonic coalition nets. In *Proceedings of the 8th International Joint Conference on Autonomous Agents and Multiagent Systems*, pages 417–424. IFAAMAS, May 2009.

[25] M. Garey and D. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman and Company, 1979.

[26] J. Hajduková. Coalition formation games: A survey. *International Game Theory Review*, 8(4):613–641, 2006.

[27] T. Heinen, N. Nguyen, and J. Rothe. Fairness and rank-weighted utilitarianism in resource allocation. In *Proceedings of the 4th International Conference on Algorithmic Decision Theory*, pages 521–536. Springer-Verlag *Lecture Notes in Artificial Intelligence #9346*, Sept. 2015.

[28] M. Karakaya. Hedonic coalition formation games: A new stability notion. *Mathematical Social Sciences*, 61(3):157–165, 2011.

[29] D. Kurokawa, A. Procaccia, and J. Wang. When can the maximin share guarantee be guaranteed? In *Proceedings of the 30th AAAI Conference on Artificial Intelligence*. AAAI Press, Feb. 2016. To appear.

[30] J. Lang, A. Rey, J. Rothe, H. Schadrack, and L. Schend. Representing and solving hedonic games with ordinal preferences and thresholds. In *Proceedings of the 14th International Joint Conference on Autonomous Agents and Multiagent Systems*, pages 1229–1238. IFAAMAS, May 2015.

[31] J. Lang and J. Rothe. Fair division of indivisible goods. In J. Rothe, editor, *Economics and Computation. An Introduction to Algorithmic Game Theory, Computational*

Social Choice, and Fair Division, chapter 8, pages 493–550. Springer-Verlag, 2015.

[32] T. Morrill. The roommates problem revisited. *Journal of Economic Theory*, 145(5):1739–1756, 2010.

[33] T. Nguyen, M. Roos, and J. Rothe. A survey of approximability and inapproximability results for social welfare optimization in multiagent resource allocation. *Annals of Mathematics and Artificial Intelligence*, 68(1–3):65–90, 2013.

[34] D. Peters. Complexity of hedonic games with dichotomous preferences. In *Proceedings of the 30th AAAI Conference on Artificial Intelligence*. AAAI Press, Feb. 2016. To appear.

[35] D. Peters. Graphical hedonic games of bounded treewidth. In *Proceedings of the 30th AAAI Conference on Artificial Intelligence*. AAAI Press, Feb. 2016. To appear.

[36] D. Peters and E. Elkind. Simple causes of complexity in hedonic games. In *Proceedings of the 24th International Joint Conference on Artificial Intelligence*, pages 617–623. AAAI Press/IJCAI, July 2015.

[37] A. Procaccia and J. Wang. Fair enough: Guaranteeing approximate maximin shares. In *Proceedings of the 15th ACM Conference on Economics and Computation*, pages 675–692. ACM Press, 2014.

[38] M. Wright and Y. Vorobeychik. Mechanism design for team formation. In *Proceedings of the 29th AAAI Conference on Artificial Intelligence*, pages 1050–1056. AAAI Press, Jan. 2015.