Complexity of Controlling Nearly Single-Peaked Elections Revisited

Extended Abstract

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ABSTRACT

In this paper, we investigate the complexity of Constructive Control by Adding/Deleting Votes (CCAV/CCDV) for r-approval, Condorcet, Maximin and Copeland$^\alpha$ in k-axes and k-candidate partition single-peaked elections. In general, we prove that CCAV and CCDV for most of the voting correspondences mentioned above are NP-hard even when k is a very small constant. Exceptions are CCAV and CCDV for Condorcet and CCAV for r-approval in k-axes single-peaked elections, which we show to be fixed-parameter tractable with respect to k. In addition, we give a polynomial-time algorithm for recognizing 2-axes elections, resolving an open question.

KEYWORDS

election control; parameterized complexity; nearly single-peaked

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1 PRELIMINARIES

An *election* is a tuple $\mathcal{E} = (C, \Pi_\mathcal{V})$, where C is a set of *candidates* and $\Pi_\mathcal{V}$ a multiset of *votes*, each of which is defined as a linear order over C. For a vote π and a candidate c, let $\pi(c)$ denote the position of c in π. In particular, the first-ranked candidate has position 1, the second-ranked candidate has position 2, and so forth. We use $N(c, c')$ to denote the number of votes ranking c above c'. For two candidates c and c', we say c *beats* c' if $N(c, c') > N(c', c)$, and c *ties* c' if $N(c, c') = N(c', c)$. For $C \subseteq C$ and a vote $\pi \in \Pi_\mathcal{V}$, π^C is π restricted to C, i.e., for $c, c' \in C$, $\pi(c) < \pi(c')$ implies $\pi^C(c) < \pi^C(c')$. Let $\Pi^C_\mathcal{V} = \{\pi^C \mid \pi \in \Pi_\mathcal{V}\}$. Hence, $(C, \Pi^C_\mathcal{V})$ is the election $(C, \Pi_\mathcal{V})$ restricted to C.

An election $(C, \Pi_\mathcal{V})$ is *single-peaked* if there is a linear order $\lhd$ of C, called an *axis*, such that for every vote $\pi \in \Pi_\mathcal{V}$ and every three candidates $a, b, c \in C$ with $a \lhd b \lhd c$ or $c \lhd b \lhd a$, $\pi(c) < \pi(b)$ implies $\pi(b) < \pi(a)$. An election $(C, \Pi_\mathcal{V})$ is *k-axes single-peaked* if there are k axes $\lhd_1, \ldots, \lhd_k$ such that every $\pi \in \Pi_\mathcal{V}$ is single-peaked with respect to at least one of these axes. In addition, $(C, \Pi_\mathcal{V})$ is *k-candidate partition (CP) single-peaked* if there is a k-partition $(C_1, \ldots, C_k)$ of C such that $(C_i, \Pi^{C_i}_\mathcal{V})$, $1 \leq i \leq k$, is single-peaked.

A *voting correspondence* φ is a function that maps an election $\mathcal{E} = (C, \Pi_\mathcal{V})$ to a non-empty subset $\varphi(\mathcal{E})$ of C. We call the elements in $\varphi(\mathcal{E})$ the *winners* of $\mathcal{E}$ with respect to φ. In this paper, we study the following voting correspondences [6, 11, 14].

r-Approval Each vote approves exactly the top-r ranked candidates. Winners are those with the most approvals. We study only the case where r is a constant.

Borda The Borda score of a candidate $c \in C$ is defined as $\sum_{c' \in C \setminus \{c\}} N(c, c')$. Winners are the ones with the highest Borda score.

Copeland$^\alpha$ ($0 \leq \alpha \leq 1$) For a candidate c, let $B(c)$ (resp. $T(c)$) be the set of candidates beaten by c (resp. tie with c). The Copeland$^\alpha$ score of c is $|B(c)| + \alpha \cdot |T(c)|$. Copeland$^\alpha$ winners are those with the highest Copeland$^\alpha$ score.

Maximin The Maximin score of a candidate c is defined as $\min_{c' \in C \setminus \{c\}} N(c, c')$. Maximin winners are the ones with the highest Maximin score.

A candidate is the *Condorcet winner* if it beats all other candidates [25]. We slightly abuse the term Condorcet by considering it as the following voting correspondence: if the Condorcet winner exists, it is the unique winner; otherwise, all candidates win.

For a voting correspondence φ, we study the following problems.

Constructive Control by Adding Votes (CCAV)	
Given:	An election $(C, \Pi_\mathcal{V})$, a distinguished candidate $p \in C$, a multiset $\Pi_\mathcal{W}$ of votes, and a positive integer ℓ.
Question:	Is there $\Pi_W \subseteq \Pi_\mathcal{W}$ such that $|\Pi_W| \leq \ell$ and p wins $(C, \Pi_\mathcal{V} \cup \Pi_W)$ with respect to φ?

Constructive Control by Deleting Votes (CCDV)	
Given:	An election $(C, \Pi_\mathcal{V})$, a distinguished candidate $p \in C$, and a positive integer ℓ.
Question:	Is there $\Pi_V \subseteq \Pi_\mathcal{V}$ such that $|\Pi_V| \leq \ell$ and p wins the election $(C, \Pi_\mathcal{V} \setminus \Pi_V)$ with respect to φ?

We study CCAV and CCDV in k-CP/axes elections. This means that for CCAV, $(C, \Pi_\mathcal{V} \cup \Pi_\mathcal{W})$ in the input is a k-CP/axes election, and for CCDV, $(C, \Pi_\mathcal{V})$ in the input is a k-CP/axes election.

2 OUR CONTRIBUTION

The complexity of CCAV and CCDV in general elections was initially studied by Bartholdi III, Tovey, and Trick [1]. Since then, the

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	CCAV				CCDV			
	SP	$(k \geq 2)$-axes	$(k \geq 2)$-CP	general	SP	$(k \geq 2)$-axes	$(k \geq 2)$-CP	general
r-Approval	P [12]	**FPT**	$k = 2$: P [24] $k \geq 3, r \geq 4$: **NP-h**	$r \leq 3$: P [15] $r \geq 4$: NP-h [15]	P [12]	$r \leq 2$: P [15] $r \geq 3$: Open	$r \leq 2$: P [15] $r \geq 3$: **NP-h**	$r \leq 2$: P [15] $r \geq 3$: NP-h [15]
Borda	NP-h [21]			NP-h [19]	NP-h [21]			NP-h [17]
Condorcet	P [3]	**FPT**	$k = 2$: Open $k \geq 3$: **NP-h**	NP-h [1]	P [3]	**FPT**	$k = 2$: Open $k \geq 3$: **NP-h**	NP-h [1]
Copeland$^{\alpha \in [0,1)}$	Open	**NP-h**	NP-h [22]	NP-h [11]	Open	**NP-h**	NP-h [22]	NP-h [11]
Copeland^1	P [3]	**NP-h**		NP-h [11]	P [3]	**NP-h**		NP-h [11]
Maximin	P [3]	**NP-h**		NP-h [10]	P [3]	**NP-h**		NP-h [10]

Table 1: Complexity of CCAV and CCDV. Here, "P" stands for "polynomial-time solvable", "NP-h" for "NP-hard", and "SP" for "single-peaked". Our results are in boldface. FPT results are with respect to k. The FPT results for Condorcet hold only when k axes of the given election are given, while the FPT result for r-approval holds even without knowing the k-axes in advance.

complexity of CCAV and CCDV for a number of voting correspondences has been investigated (see [13] for a survey). It is known that in general elections CCAV and CCDV for Borda, Condorcet, Maximin and Copeland$^\alpha$ are NP-hard [1, 10, 11, 16]. Lin [15] derived dichotomy results for r-approval with respect to the values of r: CCAV is NP-hard iff $r \geq 4$, and CCDV is NP-hard iff $r \geq 3$. In contrast, when restricted to single-peaked elections, CCAV and CCDV for all aforementioned voting correspondences, except Borda and Copeland$^\alpha$ where $0 \leq \alpha < 1$, are polynomial-time solvable [3, 12]. CCAV and CCDV for Borda in single-peaked elections were recently shown to be NP-hard by Yang [21]. To the best of our knowledge, the complexity of CCAV and CCDV for Copeland$^\alpha$ for $0 \leq \alpha < 1$ still remains open so far.

Yang and Guo [22–24] studied CCAV and CCDV in elections with single-peaked width k and k-peaked elections. Generally, an election has *single-peaked width* k if the candidates can be divided into groups, each of size at most k, such that every vote ranks all candidates in each group consecutively and, moreover, considering each group as a single candidate results in a single-peaked election. An election is *k-peaked* if there is an axis $\lhd$ such that for every vote π there is a k-partition of $\lhd$ such that π restricted to each component of the partition is single-peaked. Yang and Guo [22] proved that CCAV and CCDV for Copeland$^\alpha$, where $0 \leq \alpha < 1$, in elections with single-peaked width k are NP-hard for every $k \geq 2$. Erdélyi, Lackner, and Pfandler [8] proved that every election with single-peaked width k is a k'-CP election for some $k' \leq k$. It then follows that CCAV and CCDV for Copeland$^\alpha$, where $0 \leq \alpha < 1$, are NP-hard in k-CP elections for every $k \geq 2$. For Copeland^1 and Maximin, Yang and Guo [22] proved that CCAV and CCDV in elections with single-peaked k is polynomial-time solvable if $k = 2$, but become NP-hard for every $k \geq 3$. Then, from the relation between nearly single-peaked elections studied in [8], it follows that CCAV and CCDV for Copeland^1 and Maximin are NP-hard in k-CP elections for every $k \geq 3$. We complete the final gap by showing that CCAV and CCDV for Copeland^1 and Maximin remain NP-hard in 2-CP elections. For Condorcet, Yang and Guo [22] proved that CCAV and CCDV are fixed-parameter tractable (FPT) with respect to the single-peaked width. In contrast, we show that the problems are NP-hard in k-CP elections for every $k \geq 3$. Concerning k-peaked elections, Yang and Guo [23] obtained the following results: for Condorcet, Maximin and Copeland$^\alpha$, where $0 \leq \alpha \leq 1$, CCAV is NP-hard in 3-peaked elections and CCDV is NP-hard in 4-peaked elections [1]. As k-CP elections are a special case of k-peaked elections, our study fills several gaps left in [23] and shows NP-hardness results for even special cases of 2-peaked elections.

Yang and Guo [24] also derived dichotomy results for CCAV and CCDV for r-approval in k-peaked elections, with respect to the values of k and r. Particularly, they showed that CCAV for r-approval in 2-peaked elections is polynomial-time solvable if r is a constant, but becomes NP-hard if r is not a constant. As 2-CP elections are 2-peaked elections, their polynomial-time algorithm applies to CCAV for r-approval in 2-CP elections for all constants r. In addition, they proved that CCAV for r-approval in k-peaked elections for $k \geq 3$ and $r \geq 4$ is NP-hard. We strengthen this result by showing that the problem is NP-hard in k-CP elections for every $k \geq 3$ and $r \geq 4$. Moreover, Yang and Guo proved that CCDV for r-approval in 2-peaked elections is NP-hard iff $r \geq 3$. We strengthen their result by showing that CCDV for r-approval remains NP-hard in k-CP elections for every $r \geq 3$ and $k \geq 2$.

In addition, we study CCAV and CCDV in k-axes elections. We prove that CCAV for r-approval and Condorcet, and CCDV for Condorcet are FPT with respect to k. However, CCAV and CCDV for Maximin and Copeland$^\alpha$, $0 \leq \alpha \leq 1$, turn out to be NP-hard for every $k \geq 2$ and $0 \leq \alpha \leq 1$.

Table 1 summarizes our results and some related previous results.

Finally, we study the complexity of determining whether an election is a k-axes election. It is known that for $k = 1$, the problem is polynomial-time solvable [2, 7, 9]. Erdélyi, Lackner, and Pfandler [8] proved that the problem is NP-hard for every $k \geq 3$. We complement these results by showing that determining whether an election is a 2-axes election is polynomial-time solvable, filling the last complexity gap of the problem with respect to k.

Theorem 2.1. *Determining whether an election is a 2-axes election is polynomial-time solvable.*

Many other problems pertaining to voting in nearly single-peaked elections have also been studied in the literature, see, e.g., [4, 5, 20, 23, 26] and references therein for further details. Moreover, voting problems in other restricted elections such as single-crossing elections have also been investigated recently, see, e.g., [18].

[1] Precisely, they achieved W[1]-hardness results with respect to the solution size.

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