and k'_v are the number of goods assigned to u and v in allocations A and B^{i-1} respectively. Since $\ln \text{NSW}(B^i) - \ln \text{NSW}(B^{i-1}) = \ln(k'_u - 1) + \ln(k'_v + 1) - (\ln k'_u + \ln k'_v)$ and $\ln \text{NSW}(A') - \ln \text{NSW}(A) = \ln(k_u - 1) + \ln(k_v + 1) - (\ln k_u + \ln k_v)$, the concavity of $\ln(\cdot)$ implies Equation (9). Finally, Equations (8) and (9) give

$$\ln \text{NSW}(A^*) - \ln \text{NSW}(A') \leq \left(1 - \frac{1}{m}\right)(\ln \text{NSW}(A^*) - \ln \text{NSW}(A)),$$

as desired. □

Notice that the proof of Lemma 5.1 works exactly the same way when for each agent i, $v_i(A_i) = f_i(|A_i|)$ for some *concave* function f_i. That is, the valuation of an agent can be an (agent-specific) concave function of the cardinality (i.e., the number of nonzero valued goods owned by the agent). Thus, ALG-BINARY can find a Nash optimal allocation in polynomial time even when the valuation functions of agents are concave in cardinality. This observation is formalized in Corollary 5.2.

COROLLARY 5.2. *Given any fair division instance with concave and binary valuations, a Nash optimal allocation can be computed in polynomial time.*

Remark 2. A well-studied class of valuation functions captured by Corollary 5.2 is that of *budget-additive* valuations [13]. Under this class, the valuation of an agent $i \in [n]$ for a set of goods $G \subseteq [m]$ is given by $v_i(G) := \min\{c_i, \sum_{j \in G} v_{i,j}\}$, where $c_i > 0$ is an (agent-specific) constant, known as the *utility cap*.

Garg et al. [10] recently gave a $(2.404 + \varepsilon)$-approximation algorithm for maximizing Nash social welfare under budget-additive valuations (for any $\varepsilon > 0$). For binary valuations, a budget-additive valuation function turns out to be a special case of the concave-in-cardinality functions mentioned above. Hence, by Corollary 5.2, a Nash optimal allocation can be found in polynomial time when the valuations are binary and budget-additive. It is unclear whether the existing techniques for finding a Nash optimal allocation under binary and additive valuations [8] admit a similar generalization.

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