Impact of Tie-Breaking on the Manipulability of Elections

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ABSTRACT

In this paper, we quantify the impact of manipulation using the price of anarchy measurement and study the impact of the lexicographic and the random candidate tie-breaking rules. We show that neither dominates the other in terms of mitigating the impact of manipulation. Specifically, we show that the random candidate tie-breaking rule lowers the impact of manipulation in plurality elections whereas the lexicographic tie-breaking rule lowers the impact of manipulation in elections determined by majority judgment.

KEYWORDS

Plurality, Majority Judgment, Voting, Manipulation, Minimal Dishonesty, Truth-Bias, Price of Anarchy

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1 INTRODUCTION

Arrow's impossibility theorem [3] and its many descendants (see e.g. [4, 20]) tell us that to choose a voting rule is to trade off some desirable rationality properties against others because they are not mutually attainable. The Gibbard-Satterthwaite [17, 24], Gardenfors [16] and related theorems tell us that every election rule that one would consider to be reasonable is manipulable by a strategic voter.

To understand the impact of manipulations, Farquharson proposed the use of voting games in order to understand outcomes when voters behave strategically [15]. Since then, much work has been done to understand the impact of strategic behavior. In one stream, researchers aim to characterize the outcomes of voting games (see e.g., [22],[13], [21]) to understand what implementations of voting rules look like. In another stream, researchers use the computational complexity of manipulation to either design mechanisms that are difficult to manipulate or to design mechanisms where we are able to predict likely outcomes when agents are strategic [1, 10, 12, 14].

In this paper, we focus on measuring the impact of manipulation using the price of anarchy (PoA) [23]. We are not the first to analyze voting games using PoA (see e.g, [2, 11, 18]). However, our work differs from as others in that we primarily use the PoA to identify the impact of making small changes to a decision rule; specifically, we use PoA to understand the impact of tie-breaking rules and we show that (1) the tie-breaking rule can drastically change the outcome of an election and (2) whether that change is positive varies from voting rule to voting rule.

Another distinction of our work is the type of Nash equilibrium refinement we use. For instance, in [11, 18], the *dynamic* price of anarchy is used to evaluate plurality, veto, and Borda elections. The dynamic price of anarchy considers only equilibria obtained from best-response dynamics from truthful voting; this means that if a single individual is unable to change the election, then the election instance is not manipulable. It is straightforward to show that with an impartial culture, that the probability that a single voter can change the outcome of a plurality, veto, or Borda election rapidly decays to zero and therefore we view this equilibrium as overly restrictive. Instead, we rely on the minimal dishonesty [5] refinement which indicates that voters prefer to be more honest if the outcome remains unchanged.

Our Contributions: We use the PoA to analyze plurality and majority judgment [7] elections with respect to lexicographic and random candidate tie-breaking rules. We show that a tie-breaking rule may impact manipulation in different voting rules in significantly different ways.

For plurality elections, we show that while the PoA for lexciographic tie-breaking is large (Theorems 3.2 and 3.5), random candidate tie-breaking erodes any protections from manipulation and that arbitrarily poor outcomes can be obtained (Theorems 4.1 and 4.2). In contrast, for majority judgment elections, we show that randomized tie-breaking is less prone to manipulation by a large margin (Theorems 3.6, 3.7, 4.3, and 4.5, and Corollary 4.4).

These results emphasize the importance of understanding relatively small changes to a voting rule and demonstrates that the price of anarchy can be used to identify these differences. Thus, we propose that the price of anarchy be one of the metrics by which a voting rule is accessed.

2 NOTATION AND DEFINITIONS

We consider an election with a set $C = \{c_1, ..., c_m\}$ of candidates and a set $V = \{v_1, ..., v_n\}$ voters.

Plurality is a position-based voting rule that relies on ordinal preferences. Each voter $v \in V$ has a transitive, ordinal preference list described by $\pi_v : c_{\sigma_1} \succ_{\pi_v} c_{\sigma_2} \succ_{\pi_v} ... \succ_{\pi_v} c_{\sigma_m}$ where $c_{\sigma_i} \succ_{\pi_v} c_{\sigma_j}$ indicates that v prefers c_{σ_i} to c_{σ_j} for all $i < j$ with respect to π_v. For brevity, we will often denote π_v with the vector $(c_{\sigma_1}, c_{\sigma_2}, ..., c_{\sigma_m})$. The position of candidate $c_i \in C$ in the preference list π_v is $pos(c_i, \pi_v) = |\{c \in C : c_i \succ_{\pi_v} c\}| + 1$, i.e., given $\pi_v = (c_{\sigma_1}, c_{\sigma_2}, ..., c_{\sigma_m})$, $pos(c_{\sigma_i}, \pi_v) = i$. The full preference profile is denoted by $\Pi = \{\pi_v\}_{v \in V}$. The score of candidate c with respect to Π is $S(c, \Pi) = |\{v \in V : pos(c, \pi_v) = 1\}|$, i.e., $S(c, \Pi)$ is the number

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of first place votes that candidate c receives. The winning candidate in a plurality election is selected from $M(\Pi) = \arg\max_{c \in C} S(c, \Pi)$, i.e., from the set of candidates that receive the most first place votes.

Majority judgment [7] is a score-based rule that relies on cardinal preferences. Each voter $v \in V$ has a sincere valuation of candidate c given by $\pi_v(c) \in \{1, ..., u\}$ for some $u \in \mathbb{Z}_{>1}$ and voter v prefers c_i to c_j if and only if $\pi_v(c_i) > \pi_v(c_j)$. A voter v may have the same valuation for different candidates indicating indifference, i.e., $\pi_v(c_i) = \pi_v(c_j)$ does not necessarily imply $i = j$. For brevity, we often represent π_v with the vector $(\pi_v(c_1), \pi_v(c_2), ..., \pi_v(c_m))$. We refer to the collection of scoring valuations as the preference profile $\Pi = \{\pi_v\}_{v \in V}$. The score for candidate c is then given by $S(c, \Pi) = median(\cup_{v \in V}\{\pi_v(c)\})$, i.e., the median score for candidate c. In the event $|V|$ is even, and there are two median scores, we follow the trend in [7–9, 19] of using the lower median score. As a result if $S(c, \Pi) = w$, then a strict majority of voters place a value of at least w on candidate c. We remark that our results can be extended to other methods of taking the median, but the exact bound will change slightly depending on how the median is selected. As before, the winning candidate in majority judgment election is selected from from $M(\Pi) = \arg\max_{c \in C} S(c, \Pi)$.

For both types of elections, ties are possible. In this paper, we consider two different tie-breaking rules.

The **lexicographic** tie-breaking rules include a publicly known sorted list of candidates $\mathcal{L} : c_{\sigma_1} \succ_{\mathcal{L}} c_{\sigma_2} \succ_{\mathcal{L}} ... \succ_{\mathcal{L}} c_{\sigma_m}$ and the winning candidate is the first candidate appearing in the list with the highest score. We use $r(\Pi)$ to denote this candidate. Formally, the winner of the election given Π is $r(\Pi) = \arg\min_{c \in M(\Pi)} pos(c, \mathcal{L})$.

The **random candidate** tie-breaking rule selects the winner uniformly at random from the set of candidates with the highest score. Formally, $r(\Pi)$ is the uniform distribution over $M(\Pi)$.

2.1 The Voting Game

Unfortunately, voters do not necessarily reveal their true preferences and the result of an election may change due to manipulation.

For the remainder of the paper, we consider the **voting game**, where voter v has a sincere preference list/function π_v over the set of candidates C. If ties are broken lexicographically with respect to the publicly known list $\mathcal{L}$ then, the winning candidate is $r(\Pi) = \arg\min_{c \in M(\Pi)} pos(c, \mathcal{L})$. If ties are broken randomly, then the outcome $r(\Pi)$ selected uniformly at random from $M(\Pi)$. However, voter v is not necessarily sincere and may submit a different $\bar{\pi}_v$ over the set of candidates resulting in a possibly different outcome $r(\bar{\Pi})$. The submitted $\bar{\Pi}$ is a Nash equilibrium if no voter can obtain a better outcome by deviating from their submitted strategy.

In the event that ties are broken randomly, then the outcome $r(\bar{\Pi})$ may be determined by a set of candidates. As such, how a voter evaluates $r(\bar{\Pi})$ depends on how the voter evaluates risk, e.g., if a voter v has the preference list (c_1, c_2, c_3), it is unclear if v prefers $r(\bar{\Pi}) = (c_1, c_3)$ or $r(\bar{\Pi}') = (c_2)$; e.g., if v's utility of c_i is proportional to $pos(c_i, \pi_v)$, then v prefers $r(\bar{\Pi}') = c_2$ if v is risk-averse and prefers $r(\bar{\Pi}) = (c_1, c_3)$ if v is risk-prone. In this paper, we make no assumptions about about how voters evaluate risk – all our results hold as long as voters are rational, i.e., v prefers $r(\bar{\Pi})$ to $r(\bar{\Pi}')$ if $r(\bar{\Pi})$ has at least as high of a probability of receiving a candidate as least as good as c (with respect to π_v) for all $c \in C$. This weak assumption or rationality is consistent with standard dominance relations in the literature used to evaluate ties, e.g., Kelly, Gardenfors, and Leximin.

It is straightforward to show that both plurality and majority judgment based elections have Nash equilibria. In fact, is is well-known the set of equilibria for these games is dense; as we show in Proposition 2.1, for any candidate c, there is a Nash equilibrium where c wins regardless of the sincere preferences.

Proposition 2.1. *Let Π be an arbitrarily sincere preference profile for m candidates and $n \geq 3$ voters and let c be an arbitrary candidate. Then for both plurality and majority judgment, there is a Nash equilibrium $\bar{\Pi}$ where candidate c wins the election.*

Proof. Without loss of generality, we show the result for $c = c_1$.

For plurality, let $\bar{\pi}_v = (c_1, ..., c_m)$ so that $S(c, \bar{\Pi}) = n$ and $S(c_i, \bar{\Pi}) = 0$ for all other candidates and for all $v = 1, ..., n$. Trivially, $r(\bar{\Pi}) = c_1$ wins the election with respect to $\bar{\Pi}$. Further, $\bar{\Pi}$ is a Nash equilibrium: if v changes their preferences to $\bar{\pi}'_v$ resulting in the new profile $\bar{\Pi}' = [\bar{\Pi}_{-v}, \bar{\pi}'_v]$, then $S(c_i, \bar{\Pi}')$ is within 1 of $S(c_i, \bar{\Pi})$ since v contributes at most one point to a candidate. As a result, $S(c_1, \bar{\Pi}') \geq n - 1 \geq 2 > 1 \geq S(c_i, \bar{\Pi}')$ for all $i \neq 1$. As a result, candidate c_1 still wins the election and voter v cannot change the outcome by deviating from $\bar{\pi}_v$. Thus $\bar{\Pi}$ is a Nash equilibrium.

For majority judgment, let $\bar{\pi}_v(c_1) = u$ and let $\bar{\pi}_v(c_i) = 1$ for all $i \neq 1$ and for all $v \in V$. As a result, $S(c_1, \bar{\Pi}) = u$ and $S(c_i, \bar{\Pi}) = 1$ for all $i \neq 1$ and $r(\bar{\Pi}) = c_1$ wins the election with respect to $\bar{\Pi}$. Suppose voter v instead submits $\bar{\pi}'_v$ resulting in the new profile $\bar{\Pi}' = [\bar{\Pi}_{-v}, \bar{\pi}'_v]$. Since $n \geq 3$, this deviation does not cause the median to shift and $S(c_1, \bar{\Pi}') = u$ and $S(c_i, \bar{\Pi}') = 1$ for all $i \neq 1$. As a result, v cannot change the outcome and $\bar{\Pi}$ is a Nash equilibrium. □

2.2 Minimal Dishonesty Refinement

To eliminate the spurious equilibria of Proposition 2.1, we make use of the **minimally dishonest** Nash equilibrium [6]. A voter is **minimally dishonest** if submitting a more sincere preference list $\bar{\pi}'$ results in a strictly worse outcome; equivalently, a voter lies as little as possible to receive their best possible outcome. Formally:

Definition 2.2. Given the sincere preference profile Π, a submitted profile $\bar{\Pi}$, and a function $|\cdot|$ that measures the distance between two preference lists/functions, the voter v is *minimally dishonest* with respect to $|\cdot|$ if $|\bar{\pi}'_v - \pi_v| < |\bar{\pi}_v - \pi_v|$ implies that v prefers the outcome $r(\bar{\Pi})$ to $r([\bar{\Pi}_{-v}, \bar{\pi}'_v])$. A Nash equilibrium $\bar{\Pi}$ is a minimally dishonest Nash equilibrium with respect to $|\cdot|$ if all voters are minimally dishonest.

We remark that in this setting, and in any setting where there are a finite number of possible actions from a voter that the minimal dishonesty is equivalent to charging an arbitrarily small cost to a voter that is proportional to the size of their lie – proportional to $|\bar{\pi}_v - \pi_v|$. Thus, in this setting, the minimal dishonest refinement is equivalent to *truth-bias with ϵ-distorted costs* [21] (see [6, Appendix D] for the formal distinction between the two concepts).

Measuring Distance in Plurality Elections: There are two standard methods for measuring the distance between two ordinal lists. The Spearman's footrule distance measures the total positional difference between two lists, while the Kendall's tau distance counts the number of pairwise disparities between the two lists. The distances are defined as follows:

$$F(\pi, \bar{\pi}) = \sum_{i=1}^{m} |pos_{\pi}(c_i) - pos_{\bar{\pi}}(c_i)| \quad \text{(Spearman's Footrule Distance)}$$

$$K(\pi, \bar{\pi}) = \Big|\big\{(c_i, c_j) : pos(c_i, \pi) < pos(c_j, \pi) \ but \ pos(c_i, \bar{\pi}) > pos(c_j, \bar{\pi})\big\}\Big| \quad \text{(Kendall's Tau Distance)}$$

Different metrics can result in different sets of minimally dishonest Nash equilibria (see e.g., [21]), but our main results will hold identically for both metrics.

Measuring Dishonesty in Majority Judgment Elections: In this setting, voter v submit the score vector $\bar{\pi}_v \in \{1, ..., u\}^m$. Standard methods for measuring distance between two vectors include both the ℓ_1 and ℓ_2 norm. Our results only rely on changes to a single component of $\bar{\pi}_v$, i.e., our results hold for any metric where $|\bar{\pi}_v - \bar{\pi}'_v| < |\bar{\pi}_v + e_i - \bar{\pi}'_v|$ when $\bar{\pi}_{vi} \geq \bar{\pi}'_{vi}$. We refer to such metrics as **component-wise norms**.

The *minimal dishonesty refinement removes many absurd Nash equilibria.* For instance, the equilibrium given in Proposition 2.1 is not a minimally dishonest Nash equilibrium since no voter receives a benefit from lying; each agent's unique minimally dishonest best response is to be truthful.

2.3 Price of Anarchy (PoA)

Arrow's impossibility theorem [3] and its many descendants (see e.g. [4, 20]) tell us that every reasonable voting game is manipulable. In this paper, we use the **price of anarchy** (PoA) [23] to measure the impact of manipulation. The PoA is an indicator of a voting rule's ability to provided the promised solution.

In our setting, voting rules are cast as maximizers of a "social utility function" S, i.e., in a plurality election, the "quality" of a candidate is given by the number of first place votes, and in a majority judgment election, the "quality" of a candidate is their median score. The PoA indicates the worst-case ratio between the social utility obtained when voters are honest and the social utility obtained voters are strategic and minimally dishonest. Formally:

Definition 2.3. Let $\mathbf{\Pi}$ be the set of possible preference profiles for a voting rule r that selects a candidate that maximizes a score $S(c, \Pi)$ with respect to the preferences $\Pi \in \mathbf{\Pi}$. Let $\overline{NE}(\Pi)$ denote the set of minimally dishonest Nash equilibria with respect to the sincere Π. The **Price of Anarchy** (PoA) of the voting rule r is

$$\max_{\Pi \in \mathbf{\Pi}} \max_{\bar{\Pi} \in \overline{NE}(\Pi)} \frac{E[S_{r(\Pi)} c(\Pi)]}{E[S_{r(\Pi)} c(\bar{\Pi})]}$$

i.e., the PoA for the voting mechanism r is the worst-case ratio between the (expected[1]) social utility when voters are sincere and the expected social utility obtained when voters are strategic and minimally dishonest. The expectation here is needed due to the possible randomness in tie-breaking.

The PoA reports a voting rule's ability to deliver its promised results. For instance, if the PoA of a plurality election is 8/7, then it guarantees that the winner of an election (when agents are strategic) would receive at least 7/8ths as many votes as the sincere winner, i.e., manipulation would have a relatively small impact on the outcome whereas a PoA of ∞ would indicate that it is possible for someone to win the election despite being no one's sincerely most preferred candidate. Like the computational complexity of manipulation [10], it is one of many metrics we should use to evaluate the quality of an election mechanism.

Normalized PoA: The PoA as introduced, is not a fine enough measure to discriminate among scenarios with an unbounded PoA. For example, suppose that the sincere winner receives 40% of the votes in one plurality election and receives 20% of the votes in another. Suppose in both elections there is a strategic equilibrium in which the winner is no voter's top choice. The PoA is equally bad – infinity – in both elections, yet we might wish it to be measured as worse in the first election.

Another potential shortcoming of the PoA is that its value can be greatly altered by changing a score's scale. For instance, in a plurality election it is possible for a winning candidate in the strategic voting game to be no voter's first choice. The PoA can be $1/0 = \infty$. Create a new decision mechanism, plurality+, where a candidate receives 2 points for being most favored by a voter, and 1 point from the voter otherwise. Plurality and plurality+ always yield the same outcome, yet they have significantly different prices of anarchy. With respect to plurality+, every candidate receives between n and $2n$ votes and the PoA is at most 2.

To address both of these issues, we also normalize the PoA so that its value is at most m for all voting mechanisms. We apply an affine transformation to the scoring function such that each candidate receives at least 1 and at most m points. While PoA without normalization gives a pure measure of how much a voting rule can be manipulated, we believe that normalization is appropriate for comparing prices of anarchy of different mechanisms. For example, plurality and plurality+ have the same PoA after normalization.

3 LEXICOGRAPHIC TIE-BREAKING

We begin by studying the PoA for both plurality and majority judgment using a lexicographic tie-breaking rule. We show that both rules are heavily impacted by manipulation when using lexicographic tie-breaking.

3.1 Plurality

We first show that it is possible for someone to win a plurality election despite not receiving any first place votes implying the PoA for plurality is ∞.

Lemma 3.1. *Suppose ties in a plurality election are broken lexicographically. There exists a* Π *and a minimally dishonest Nash equilibrium (with respect to the Spearman's footrule and Kendall's tau distance)* $\bar{\Pi}$ *where* $S(r(\bar{\Pi}), \Pi) = 0$ *– where the winning candidate would receive no votes with respect to the sincere* Π.

Proof. Let $|V| = 3k-1$ for an arbitrary $k \in \mathbb{Z}_{\geq 2}$. Suppose the tie-breaking list $\mathcal{L}$ is such that $c \succ_{\mathcal{L}} c_2$ for all $c \neq c_2$ – c_2 can only win the election if c_2 uniquely receives the most votes. We describe 3 sets of voter preferences, V_1, V_2, and V_3 where $|V_1| = k-2, |V_2| = k+1$ and $|V_3| = k$:

[1]When voters are sincere, the utility is deterministic even when the winning candidate is selected randomly since every candidate in $M(\Pi)$ has the same sincere score.

$$
\begin{array}{lll}
v \in V_1: & \pi_v = (c_1, ..., c_m), & \bar{\pi}_v = \pi_v \text{ (honest)} \\
v \in V_2: & \pi_v = (c_1, ..., c_m), & \bar{\pi}_v = (c_2, c_1, c_3, c_4, ..., c_m) \\
v \in V_3: & \pi_v = (c_3, c_4, ..., c_m, c_1, c_2), & \bar{\pi}_v = \pi_v \text{ (honest)}
\end{array}
$$

With respect to the sincere preferences, candidate c_1 receives $S(c_1, \Pi) = 2k - 1$ points, candidate c_3 receives $S(c_3, \Pi) = k$ points and all other candidates receive 0 points implying $r(\Pi) = c_1$ would win the election if everyone was sincere.

However, with respect to the submitted preferences, candidate c_1 receives $S(c_1, \bar{\Pi}) = k - 2$ points, candidate c_2 receives $S(c_2, \bar{\Pi}) = k + 1$ points, candidate c_3 receives $S(c_3, \bar{\Pi}) = k$ points, and all other candidates receive 0 points implying that $r(\bar{\Pi}) = c_2$ wins the election with respect to the submitted $\bar{\Pi}$. We now show that each voter is submitting a minimally dishonest best response.

First, consider voter $v \in V_1$. The voter is honest and therefore minimally dishonest. Further, voter v is submitting a best response: Suppose voter v submits $\bar{\pi}'$ resulting in the new profile $\bar{\Pi}' = [\bar{\Pi}_{-v}, \bar{\pi}'_v]$. If voter v submits any list with c_1 first, then $S(c_i, \bar{\Pi}) = S(c_i, \bar{\Pi}')$ and the result of the election does not change. Next, suppose v lists c_i first where $i \neq 1$. Then $S(c_1, \bar{\Pi}') = S(c_1, \bar{\Pi}) - 1 = k - 3$, $S(c_i, \bar{\Pi}') = S(c_i, \bar{\Pi}) + 1$ and $S(c_j, \bar{\Pi}') = S(c_j, \bar{\Pi})$ for $j \notin \{1, i\}$. If $i = 3$, then $S(c_2, \bar{\Pi}') = S(c_3, \bar{\Pi}')$ resulting in $r(\bar{\Pi}') = c_3$ winning the election due to the tie-breaking rule, a strictly worse outcome for voter v. If $i \notin \{1, 3\}$, $S(c_2, \bar{\Pi}') \geq k + 1$ and $S(c_j, \bar{\Pi}') \leq k$ for all $j \neq 2$ resulting in the same outcome ($r(\bar{\Pi}') = c_2$). In all cases, voter v cannot alter their preferences to receive a better outcome implying v is providing a minimally dishonest best response.

Next, consider $v \in V_2$. We first show that v is minimally dishonest. The only preference list that is more sincere than $\bar{\pi}_v$, with respect to either the Spearman's footrule distance or the Kendall's tau distance, is the honest $\bar{\pi}'_v = \pi_v$. If voter v is honest, resulting in the new profile $\bar{\Pi}' = [\bar{\Pi}_{-v}, \bar{\pi}'_v]$, then $S(c_1, \bar{\Pi}') = S(c_1, \bar{\Pi}) + 1 = k - 1$, $S(c_2, \bar{\Pi}') = S(c_2, \bar{\Pi}) - 1 = k$, and $S(c_j, \bar{\Pi}') = S(c_j, \bar{\Pi})$ for $j \notin \{1, 2\}$. In particular, $S(c_3, \bar{\Pi}') = S(c_2, \bar{\Pi}') = k$ causing $r(\bar{\Pi}') = c_3$ to win due to the tie-breaking rule, a strictly worse outcome for v implying that v is minimally dishonest. The argument that $v \in V_2$ is providing a best response is identical to the argument for $v \in V_1$; by altering their preferences, v can only cause c_3 or c_2 to win, neither of which are better than the current outcome c_2. Thus, $v \in V_2$ is providing a minimally dishonest best response.

Finally, consider $v \in V_3$. Voter v is honest and therefore minimally dishonest. Voter v cannot change the election result; Suppose v submits $\bar{\pi}'_v$ resulting in the profile $\bar{\Pi}' = [\bar{\Pi}_{-v}, \bar{\pi}'_v]$. Since v currently assigns 1 point to candidate c_3, v can cause c_3's score to decrease by 1 and the score of exactly one other candidate to increase 1. As a result, regardless of $\bar{\pi}'_v$, $S(c_2, \bar{\Pi}') \geq S(c_2, \bar{\Pi}) = k + 1$, $S(c_3, \bar{\Pi}') \leq S(c_3, \bar{\Pi}) = k$, and $S(c_j, \bar{\Pi}') \leq S(c_j, \bar{\Pi}) + 1 = 1$ for $j \notin \{2, 3\}$. As a result $r(\bar{\Pi}') = c_2$ and voter v is providing a minimally dishonest best response which completes the proof of the lemma. □

Theorem 3.2. *The PoA for plurality voting with lexicographic tie-breaking is ∞ for both the Spearman's footrule distance and the Kendall's tau distance.*

Proof. The sincere preferences in Lemma 3.1 is such that if everyone was honest then the sincere winner would receive $S(r(\Pi), \Pi) = 2k - 1$ votes while the minimally dishonest Nash equilibrium is such that $S(r(\bar{\Pi}), \Pi) = 0$ resulting in a PoA of ∞. □

3.1.1 Normalized PoA for Plurality. The result of Theorem 3.2 suggests that arbitrarily poor outcomes can be obtained, which is true in the sense that someone no one sincerely prefers could win the election. However, the proof itself suggests that for this to occur, there cannot be a unanimous winner when individuals are sincere.

To capture a finer measure of the impact of manipulation, we consider the normalized PoA where candidate c receives $\frac{m}{n}$ points from a voter that lists c as their favorite candidate and $\frac{1}{n}$ from all other voters so that each candidate receives between m and 1 point. While the outcome can still be poor, we show that the normalized PoA is at most $\frac{2m+1}{3}$, implying that if the winner with respect to an equilibrium $\bar{\Pi}$ sincerely receives zero votes (a normalized score of 1, then the sincere winner can have a normalized score of at most $\frac{2m+1}{3}$ implying the sincere winner would receive at most $\frac{n(2m+1)}{3m} = \frac{2n}{3} + \frac{n}{3m} \leq \frac{7n}{9}$ votes for elections with $m \geq 3$ candidates.

Prior to establishing a result, we give a partial characterization of minimally dishonest Nash equilibria. Specifically, if the PoA is more than one, then the winning candidate wins by at most one vote and every voter will vote either for their favorite candidate or the candidate that wins the election.

Lemma 3.3. *If $\bar{\Pi}$ is a minimally dishonest Nash equilibrium (with respect to Spearman's footrule or Kendall's tau distance) for a plurality election with lexicographic tie-breaking where $r(\bar{\Pi}) \neq r(\Pi)$, then there is some candidate $c \neq r(\bar{\Pi})$ such that $S(c, \bar{\Pi}) \geq S(r(\bar{\Pi}), \bar{\Pi}) - 1$, i.e., $r(\bar{\Pi})$ wins by at most one vote.*

Proof. For contradiction, suppose that $S(r(\bar{\Pi}), \bar{\Pi}) \geq r(c, \bar{\Pi}) - 2$ for all $c \neq r(\bar{\Pi})$. Since $r(\bar{\Pi}) \neq r(\Pi)$, there must exist a voter v with a most preferred candidate $c \neq r(\bar{\Pi})$ where $pos(c, \pi_v) = 1$ but $pos(c, \bar{\pi}_v) > 1$ since otherwise $S(c', \Pi) \leq S(c', \bar{\Pi})$ for all $c' \in C$ implying $r(\bar{\Pi})$ does not win the election, a contradiction.

Suppose voter v instead submits the honest π_v resulting in the $\bar{\Pi}' = [\bar{\Pi}_{-v}, \pi_v]$; π_v is more honest with respect to both distances. Let c be such that $pos(c, \pi_v) = 1$. Since only c receives a point from v with respect to π_v, only c's score increases while the score of all other candidates decreases or stays the same. As a result,

$$
\begin{aligned}
S(c, \bar{\Pi}') &= S(c, \bar{\Pi}) + 1 \leq S(r(\bar{\Pi}), \bar{\Pi}) - 1 \\
S(r(\bar{\Pi}), \bar{\Pi}') &\geq S(r(\bar{\Pi}), \bar{\Pi}) - 1 \\
S(c', \bar{\Pi}') &\leq S(c', \bar{\Pi}) \leq S(r(\bar{\Pi}), \bar{\Pi}) - 1 \; \forall c' \notin \{c, r(\bar{\Pi})\}
\end{aligned}
$$

implying either $M(\bar{\Pi}) = \{r(\bar{\Pi})\}$ ($r(\bar{\Pi})$ still wins the election), or $M(\bar{\Pi}) = \{c, r(\bar{\Pi})\}$ (there is a tie between c and $r(\bar{\Pi})$ with respect to $\bar{\Pi}'$). If $r(\bar{\Pi}') = r(\bar{\Pi})$, then we contradict minimal dishonesty since v is more honest. If $r(\bar{\Pi}') = c$, then we contradict the equilibrium property since, by definition, $c \succ_{\pi_v} r(\bar{\Pi})$ and v can submit π_v to obtain a better outcome. In both cases, $\bar{\Pi}$ is not a minimally dishonest Nash equilibrium, a contradiction. □

Lemma 3.4. *Let c be such that $pos(c, \pi_v) = 1$. If $\bar{\Pi}$ is a minimally dishonest Nash equilibrium (with respect to Spearman's footrule or Kendall's tau distance) for a plurality election with lexicographic tie-breaking, then either (1) $pos(c, \bar{\pi}_v) = 1$ or (2) $pos(r(\bar{\Pi}), \bar{\pi}_v) = 1$, i.e., v votes for either their favorite candidate or the winning candidate.*

Proof. Let $\mathcal{L}$ be the list used for lexicographic tie-breaking and suppose without loss of generality that $r(\bar{\Pi}) = c_2$ wins the election. For contradiction, suppose $c_i \notin \{c_2, c\}$ is such that $pos(c_j, \bar{\pi}_v) = 1$.

Suppose instead v submits the honest π_v resulting in $\bar{\Pi}' = [\bar{\Pi}_{-v}, \pi_v]$. As in the proof of Lemma 3.3, only the score of c_j and c change and, as a result, $M(\bar{\Pi}') \subseteq M(\bar{\Pi}) \setminus \{c_j\} \cup \{c\}$. Since $r(\bar{\Pi}) = c_2$, the tie-breaking rule is such that $c_2 \succ_{\mathcal{L}} c'$ for all $c' \in M(\bar{\Pi})$ and therefore $c_2 \succ_{\mathcal{L}} c'$ for all $c' \in M(\bar{\Pi}') \setminus \{c\}$. As a result, $r(\bar{\Pi}') \in \{c_2, c\}$. As in Lemma 3.3, in both cases, the voter v has not submitted a minimally dishonest best response, a contradiction. □

With this partial characterization, we show that the normalized PoA is $\frac{2m+1}{3}$.

Theorem 3.5. *The normalized PoA for plurality voting with lexicographic tie-breaking is $\frac{2m+1}{3}$ for both the Spearman's footrule distance and the Kendall's tau distance.*

Proof. For the preferences given in Lemma 3.1, the normalized score of c_1 is $S(c_1, \Pi) = (2k-1) \cdot \frac{m}{n} + k \cdot \frac{1}{n} = \frac{(2m+1)\cdot k - m}{3k-1}$ and the normalized score of c_2 is $S(c_2, \Pi) = 1$. The proof of Lemma 3.1 shows there is a minimally dishonest Nash equilibrium where c_2 wins implying the normalized PoA is $\frac{(2m+1)\cdot k-m}{3k-1} \to \frac{2m+1}{3}$ as $k \to \infty$. It remains to show the PoA is at most $\frac{2m+1}{3}$.

Consider any Π and a corresponding minimally dishonest Nash equilibrium $\bar{\Pi}$. The PoA is 1 when when $r(\Pi) = r(\bar{\Pi})$. Next we consider, without loss of generality, when $c_1 = r(\Pi) \neq r(\bar{\Pi}) = c_2$ and break the problem into two cases. In the first case, c_1 has one of the top two scores (possibly tied) with respect to the submitted $\bar{\Pi}$. In the second case, c_1 does not have a top two score and there is a third candidate c_3 that has a top two score.

Suppose first that c_1 has a top two score. By Lemma 3.3, $S(c_1, \bar{\Pi}) \geq S(c_2, \bar{\Pi}) - 1$. Let $A_1 := \{v \in V : pos(c_1, \pi_v) = 1\}$ be the set of voters that sincerely most prefer c_1 and let $\bar{A}_2 := \{v \in V : pos(c_2, \bar{\pi}_v) = 1\}$ be the set of voters that report that they most prefer c_2. We claim $A_1 \cap \bar{A}_2 = \emptyset$. For contradiction, suppose that $v \in A_1 \cap \bar{A}_2$ and suppose v submits the honest π_v resulting in $\bar{\Pi}' = [\bar{\Pi}_{-v}, \pi_v]$. As a result, $S(c_1, \bar{\Pi}') = S(c_1, \bar{\Pi}) + 1 \geq S(c_2, \bar{\Pi}) = S(c_2, \bar{\Pi}') + 1$. Further, since c_1 has a top two score with respect to $\bar{\Pi}$, $S(c_1, \bar{\Pi}') = S(c_1, \bar{\Pi}) + 1 \geq S(c_i, \bar{\Pi}) + 1$ for all $i \notin \{1, 2\}$ implying $r(\bar{\Pi}') = c_1$, contradicting that $\bar{\Pi}$ is an equilibrium. Therefore $A_1 \cap \bar{A}_2 = \emptyset$.

Combining $A_1 \cap \bar{A}_2 = \emptyset$ with Lemma 3.4 implies that for all $v \in A_1$ truthfully reveals that $pos(c_1, \bar{\pi}_v) = pos(c_1, \pi_v)$ and $S(c_1, \bar{\Pi}) = |A_1|$. By definition of $\bar{A}_2$, $S(c_2, \bar{\Pi}) = |\bar{A}_2|$ and $|\bar{A}_2| \geq |A_1|$ since $r(\bar{\Pi}) = c_2$. Since $A_1 \cap \bar{A}_2 = \emptyset$, $|A_1| + |\bar{A}_2| \leq |V| = n$ and $|A_1| \leq n/2$. Finally, observe that $S(c_1, \Pi) = |A_1| \cdot \frac{m}{n} + (n - |A_1|)$ and $S(c_2, \Pi) \geq 1$. Combining everything, the normalized PoA in this case is at most

$$\frac{|A_1| \cdot \frac{m}{n} + (n - |A_1|) \cdot \frac{1}{n}}{1} \leq \frac{m+1}{2} < \frac{2m+1}{3}.$$

Next, suppose that that c_3 has a top two score but c_1 does not. We will generate several inequalities, apply weights to them, and combine them to yield an upper bound on the PoA. As in the previous case, let $A_i := \{v \in V : pos(c_i, \pi_v) = 1\}$ and let $\bar{A}_i := \{v \in V : pos(c_i, \bar{\pi}_v) = 1\}$. With this notation $S(c_i, \Pi) = |A_i|$ and $S(c_i, \bar{\Pi}) = |\bar{A}_i|$.

Unlike the previous case, $A_1 \cap \bar{A}_2$ is not necessarily empty. By Lemma 3.4, for $v \in A_1$, $v \in \bar{A}_1 \cup \bar{A}_2$ since $r(\bar{\Pi}) = c_2$ implying $|A_1| = |\bar{A}_1| + |A_1 \cap \bar{A}_2|$. Since there are n voters, $n \geq |\bar{A}_1| + |\bar{A}_2| + |\bar{A}_3|$. Combining both expression yields

$$2 \cdot \left(|A_1| - |A_1 \cap \bar{A}_2| + |\bar{A}_2| + |\bar{A}_3|\right) \leq 2 \cdot n$$

where the use of the multiplier 2 will become apparent later.

Next, since c_3 is in second place and by Lemma 3.3, c_3's reported score is within 1 of c_2's reported score, $|\bar{A}_3| \geq |\bar{A}_2| - 1$ yielding

$$|\bar{A}_2| - |\bar{A}_3| \leq 1.$$

Next, by definition, $|\bar{A}_2| \geq |A_1 \cap \bar{A}_2|$ yielding

$$3 \cdot (|A_1 \cap \bar{A}_2| - |\bar{A}_2|) \leq 3 \cdot 0.$$

Next, since c_3 is a top two scorer and c_1 is not, $|\bar{A}_3| \geq |\bar{A}_1| + 1 = |A_1| - |A_1 \cap \bar{A}_2| + 1$ yielding

$$|A_1| - |A_1 \cap \bar{A}_2| - |\bar{A}_3| \leq -1.$$

Adding together the four inequalities yields $3 \cdot |A_1| \leq 2n$ implying $|A_1| \leq \frac{2n}{3}$. Following identically to the first case, the normalized PoA is at most

$$\frac{|A_1| \cdot \frac{m}{n} + (n - |A_1|) \cdot \frac{1}{n}}{1} \leq \frac{2m+1}{3}.$$

□

3.2 Majority Judgment

Next, we study the impact of manipulation on elections determined by majority judgment. Since the score of candidate c satisfies $1 \leq S(c, \Pi) \leq u$, the PoA is trivially at most u. We begin by showing that this trivial bound is nearly tight – like plurality elections, majority judgment can be significantly impacted by manipulation.

Theorem 3.6. *The PoA for majority judgment with lexicographic tie-breaking is $u - 1$ for any component-wise norm.*

Proof. We begin by showing an upper bound of $u - 1$. We first claim that if $\pi_v(c) = u$, then $\bar{\pi}_v(c) = u$ at every minimally dishonest Nash equilibrium. Suppose for contradiction, this is not the case. Then for any component-wise norm, v can be more honest by submitting $\bar{\pi}'_v(c) = \pi_v(c) = u$ and $\bar{\pi}'_v(c') = \bar{\pi}(c')$ for all $c' \neq c$ resulting in the new profile $\bar{\Pi}' = [\bar{\Pi}_{-v}, \bar{\pi}'_v]$. Since a candidate's score is determined by their median score, $S(c', \bar{\Pi}') = S(c', \bar{\Pi})$ for $c' \neq c$ and $S(c, \bar{\Pi}') \geq S(c, \bar{\Pi})$. As a result, the set of candidates with the highest median score is $M(\bar{\Pi}') \subseteq M(\bar{\Pi}) \cup \{c\}$. As shown in the proof of Lemma 3.3, for lexicographic tie-breaking rules, $r(\bar{\Pi}') \in \{r(\bar{\Pi})\} \cup \{c\}$. Since $\pi_v(c) = u$, v (weakly) prefers c to $r(\bar{\Pi})$ and c obtains at least as good of a result by submitting the more honest $\bar{\pi}'_v$, contradicting that v is minimally dishonest. As a result, $\pi_v(c) = u$, then $\bar{\pi}_v(c) = u$.

Through a nearly identical argument, we claim if $\pi_v(c) = 1$, then $\bar{\pi}_v(c) = 1$. If not, v can submit the more honest $\bar{\pi}'_v$ where $\bar{\pi}'_v(c) = \pi_v(c) = 1$ and $\bar{\pi}'_v(c') = \bar{\pi}_v(c')$ for all other c' resulting in $\bar{\Pi}' = [\bar{\Pi}_{-v}, \bar{\pi}'_v]$. Following identically to the previous case, either $M(\bar{\Pi}') = M(\bar{\Pi})$ or $M(\bar{\Pi}') = M(\bar{\Pi}) \setminus \{c\}$. If $r(\bar{\Pi}') = r(\bar{\Pi})$, then v is more honest while obtaining the same outcome, a contradiction. If $r(\bar{\Pi}') \neq r(\bar{\Pi})$, then $r(\bar{\Pi}) = c$ since the outcome of a lexicographic tie-breaking can only change if $M(\cdot)$ changes. Voter v (weakly) prefers all other candidates to $c = r(\bar{\Pi})$ and therefore obtains at least as good an outcome, once again contradicting minimal dishonest. As a result, if $\pi_v(c) = 1$, then $\bar{\pi}_v(c) = 1$.

We now prove a PoA of $u-1$. First, suppose there is a candidate $r(\Pi) = c$ where $S(c, \Pi) = u$, i.e., where a majority of voters sincerely assign a utility of u to c. By the first claim, these voters are truthful about c and $S(c, \bar{\Pi}) = u$. Symmetrically, if there is a candidate c' where $S(c', \Pi) = 1$, then the second claim implies $S(c', \bar{\Pi}) = 1$ and such a voter cannot defeat c even when voters are strategic. Therefore the equilibrium $\bar{\Pi}$ is such that $S(r(\bar{\Pi}), \pi) \geq 2$ yielding a PoA of at most $\frac{u}{2}$.

Next, suppose $S(r(\Pi), \Pi) \leq u-1$. For every candidate, $S(c, \Pi) \geq 1$ yielding a trivial PoA of $u-1 \geq u/2$. In the following example, we show the bound of $u-1$ is tight.

Suppose our tie-breaking rule $\mathcal{L}$ is such that $c_1 \succ_{\mathcal{L}} c$ for all $c \neq c_1$. Consider three disjoint sets of voters V_1, V_2, and V_3 with the following sincere and submitted preferences.

$$\begin{aligned}
v \in V_1: \quad & \pi_v(c_1) = \bar{\pi}_v(c_1) = 1, \\
& \pi_v(c_i) = \bar{\pi}_v(c_i) = u \text{ for } i \neq 1 \text{ (honest)} \\
v \in V_2: \quad & \pi_v(c_1) = \bar{\pi}_v(c_1) = 1, \\
& \pi_v(c_i) = \bar{\pi}_v(c_i) = 1 \text{ for } i \neq 1 \text{ (honest)} \\
v = V_3: \quad & \pi_v(c_1) = \bar{\pi}_v(c_1) = u, \\
& \pi_v(c_i) = u-1 \text{ but } \bar{\pi}_v(c_i) = 1 \text{ for } i \neq 1
\end{aligned}$$

where $|V_1| = k, |V_2| = k$ and $|V_3| = 1$. With respect to Π, some candidate $c_i \neq c_1$ wins the election since $S(c_i, \Pi) = u-1$ for $i \neq 1$ and $S(c_1, \Pi) = 1$. However, with respect to the submitted preferences, $S(c_j, \bar{\Pi}) = 1$ for all j and $r(\bar{\Pi}) = c_1$ since $c_1 \succ_{\mathcal{L}} c_i$ for all $i \neq 1$. Thus, the PoA is $u-1$ is all agents are minimally dishonest.

Voters in V_1 and V_2 are honest. Voters in V_1 cannot change the outcome of the election at all and therefore are minimally dishonest. Voters in V_2 can change the election by increasing their reported utility for $c_i \neq c_1$, however, these voters are indifferent between all candidates and therefore providing a best response.

Finally, voter $v \in V_3$ can only become more honest by increasing their reported valuation for $c_i \neq c_1$. However, doing so raises the median score of c_i causing c_i to win the election, a strictly worse outcome. As a result, v is providing a minimally dishonest best response. Thus, the PoA of majority judgment is $u-1$. □

3.3 Normalized PoA for Majority judgment

If voter v's score for candidate c is x, then v's normalized score for c is $\frac{m-1}{u-1}(x-1)+1$. After the transformation, the maximum score for any candidate is m and the minimum score for any candidate is 1. The bound for the PoA is obtained by updating Theorem 3.6.

Theorem 3.7. *The normalized PoA for majority judgment with lexicographic tie-breaking is* $\frac{(u-2)m+1}{u-1}$ *for any component-wise norm.*

Proof. The proof follows identically to Theorem 3.6. For the upper bound, we broke the problem in two cases: In the first case, the largest score a candidate can receive is m while the 2nd lowest score a candidate can receive is $\frac{m-1}{u-1}+1$ yielding a ratio of

$$\frac{m}{\frac{m-1}{u-1}+1} = \frac{u(m-1)}{m+u}.$$

In the second case, the largest score a candidate can receive is $\frac{m-1}{u-1}(u-2)+1$ while the lowest score a candidate can receive is 1 yielding a ratio of

$$\frac{m-1}{u-1}(u-2)+1 = \frac{(u-2)m+1}{u-1}.$$

To see that the second bound is larger, it suffices to show

$$\begin{aligned}
&\frac{u(m-1)}{m+u} \leq \frac{(u-2)m}{u-1} \\
\Leftrightarrow &(u-1)^2 \leq (u-2)(u+m) \\
\Leftrightarrow &(u-1)^2 \leq (u-2)(u+m) \\
\Leftrightarrow &1 \leq (u-2)m
\end{aligned}$$

which is true for all $u \geq 3$. For $u = 2$, both bounds evaluate to 1. Thus, the normalized PoA is at most $\frac{(u-2)m+1}{u-1}$. The instance provided in the proof of Theorem 3.6 yields the second bound demonstrating that the bound is tight. □

3.4 Discussion of Lexicographic Rules

Both plurality and majority judgment have near worst-case prices of anarchy and are significantly impacted by manipulation. As discussed in Section 2.3, we recommend using the normalized PoA when comparing two dissimilar voting rules.

The normalized PoA of majority judgment is $\frac{(u-2)m+1}{u-1} \to m$ as $u \to \infty$, i.e., as voters are allowed to provide more granular rating of candidates, the PoA can approach the worst-case possible value of m. The normalized PoA of plurality on the other hand is $\frac{2m+1}{3}$, and therefore arbitrarily bad outcomes are avoided. But neither rule is able to offer meaningful guarantees that a solution will be close to the intended outcome.

However, the PoA of majority judgment can be artificially lowered, e.g., as mentioned in the proof of Theorem 3.7, the normalized PoA becomes 1 when $u = 2$. Further, it is straightforward to show the normalized PoA of majority judgment is strictly less than the normalized PoA if and only if $u < 4$. This lower PoA is obtained by unnaturally removing choice from voters, i.e., if we force voters to all report identical values for candidates, then it appears like manipulation has no impact on the outcome because all candidates look the same. Instead, in the next section, we look for more natural way to lower the impact of manipulation.

4 RANDOM CANDIDATE TIE-BREAKING

In this section, we consider plurality and majority judgment elections where ties are broken uniformly at random. Formally, the outcome $r(\bar{\Pi})$ is the uniform distribution over $M(\bar{\Pi})$, the set of candidates with the largest score. As defined in Section 2.3, we examine the expected PoA, i.e., the PoA for a risk-neutral society. Our results hold for all rational voters; we make no additional assumptions on how voters evaluate distributions of candidates.

For plurality, we show that that random candidate tie-breaking actually increases the PoA – the worst-case impact of manipulation. In contrast, we show that the random candidate tie-breaking decreases the PoA for majority judgment.

4.1 Plurality

In this section, we show that the random candidate tie-breaking rule negatively impacts the PoA for plurality elections.

Theorem 4.1. *The PoA for plurality voting with the random candidate tie-breaking rule is ∞ for both the Spearman's footrule distance and the Kendall's tau distance.*

The proof of Theorem 4.1 follows identically to Theorem 3.2; in the proof of Theorem 3.2, the tie-breaking rule is never invoked.

Unlike plurality elections with lexicographic tie-breaking, we show that a candidate with no sincere votes can win even when all voters sincerely agree on which candidate should win the election. This implies that the normalized PoA is m – which is the worst any election mechanism can be; in the setting of plurality elections, the random candidate tie-breaking rule completely erodes any protections against manipulation.

Theorem 4.2. *The normalized PoA for plurality voting with the random candidate tie-breaking rule is m for both the Spearman's footrule distance and the Kendall's tau distance.*

Proof. By definition, the normalized PoA is at most m for any voting rule and it remains to provide a lower bound. We consider two sets of voters:

$$\begin{aligned} v \in V_1: \quad & \pi_v = (c_m, c_1, c_2, c_3, ..., c_{m-1}), \\ & \bar{\pi}_v = (c_1, c_m, c_2, c_3, ..., c_{m-1}) \\ v \in V_2: \quad & \pi_v = (c_m, c_{m-1}, c_{m-2}, c_{m-3}, ..., c_1), \\ & \bar{\pi}_v = (c_{m-1}, c_m, c_{m-2}, c_{m-3}, ..., c_1) \end{aligned}$$

where $|V_1| = |V_2| = k$. If voters are sincere, then $r(\Pi) = c_m$ and the normalized score for c_m is $S(c_m, \Pi) = n \cdot \frac{m}{n} = m$ and the normalized score for all other candidates is 1.

With respect to $\bar{\Pi}$, candidate c_1 and c_{m-1} tie and $r(\bar{\Pi})$ is the uniform distribution over $M(\bar{\Pi}) = \{c_1, c_{m-1}\}$. If voter $v \in V_1$ reports $\bar{\pi}'_v$ such that $pos(c_j, \bar{\pi}'_v) = 1$ for $j \neq 1$ resulting in $\bar{\Pi}' = [\bar{\Pi}_{-v}, \bar{\pi}'_v]$, then $M(\bar{\Pi}') = \{c_{m-1}\}$ and v obtains a worse outcome. Thus v is providing a best response. Further v is minimally dishonest since the only preference list more honest (with respect to either distance) than $\bar{\pi}_v$ is π_v, which causes c_{m-1} to win the election. Thus, $v \in V_1$ is providing a minimally dishonest best response. Symmetrically, so is $v \in V_2$ and $\bar{\Pi}$ is a minimally dishonest Nash equilibrium with a PoA of m. □

Finally, we note that the proof for Theorem 3.5 remains valid if there is a unique winner even when ties are broken randomly. Thus, given a unique winner, the normalized PoA is at most $\frac{2m+1}{3}$.

4.2 Majority Judgment

In contrast, we show that the random candidate tie-breaking rule improves the PoA for majority judgment.

Theorem 4.3. *The PoA for majority judgment with the random candidate tie-breaking rule is* $\max\{\frac{um-m}{u+m-2}, \frac{u-1}{2}\}$ *for any component-wise norm.*

Proof. Following the first part of Theorem 3.6, we show that if $\pi_v(c) = u$, then $\bar{\pi}_v(c) = u$ at a minimally dishonest Nash equilibrium: If not, then for any component-wise norm, v can be more honest by submitting $\bar{\pi}'_v(c) = \pi_v(c) = u$ and $\bar{\pi}'_v(c') = \bar{\pi}_v(c)$ for all $c' \in C$ resulting in the new profile $\bar{\Pi}' = [\bar{\Pi}_{-v}, \bar{\pi}'_v]$. As in Theorem 3.6, the set of candidates with the highest median score is $M(\bar{\Pi}') \subseteq M(\bar{\Pi}) \cup \{c\}$. If $M(\bar{\Pi}') = M(\bar{\Pi})$ then v obtains the same outcome, contradicting minimal dishonesty.

Next, suppose $c \in M(\bar{\Pi}')$ but $c \notin M(\bar{\Pi})$. Given $C' \subseteq C$, let $pref(w, \pi_v, C') = \{c \in C' : \pi_v(c) \geq w\}$ for all $w \in \{1, ..., u\}$ be the set of candidates in C' that v sincerely values with weight at least w. With this notation, the probability that v receives an outcome with weight at least w is $\frac{|pref(w,\pi_v,M(\bar{\Pi}'))|}{|M(\bar{\Pi}')|}$. In particular, since $\pi_v(c) = u$, $c \in pref(w, \pi_v, M(\bar{\Pi}'))$ and

$$\begin{aligned} \frac{|pref(w, \pi_v, M(\bar{\Pi}'))|}{|M(\bar{\Pi}')|} &= \frac{|pref(w, \pi_v, M(\bar{\Pi})| + 1}{|M(\bar{\Pi})| + 1} \\ &\geq \frac{|pref(w, \pi_v, M(\bar{\Pi})|}{|M(\bar{\Pi})|} \end{aligned}$$

for all $w \in \{1, ..., u\}$, i.e., voter v weakly prefers $r(\bar{\Pi}')$ to $r(\bar{\Pi})$, contradicting that v is minimally dishonest. Thus, $\pi_v(c) = u$ implies $\bar{\pi}_v(c) = u$.

Symmetrically, $\pi_v(c) = 1$, then $\bar{\pi}_v(c) = 1$ since v can potentially remove a candidate with weight 1 from $M(\bar{\Pi})$.

Next, we break the problem into two cases. The bound of $\frac{um-m}{u+m-2}$ occurs when there is a tie and the bound $\frac{u-1}{2}$ occurs when there is a unique winner.

Case 1: We consider the Nash equilibrium $\bar{\Pi}$ where $M(\bar{\Pi})$ is a singleton – when there is not a tie – and show the PoA is $\frac{u-1}{2}$.

First, consider a candidate c where $S(c, \Pi) = 1$. By the earlier claim $\bar{\pi}_v(c) = 1$ for all v where $\pi_v(c) = u$ and $S(c, \bar{\Pi}) = 1$. Since all candidates receive at least one point, and since there is not a tie with respect to $\bar{\Pi}$, $c \neq r(\bar{\Pi})$ and $S(r(\bar{\Pi}), \Pi) \geq 2$. Next, suppose there is a candidate c where $S(c, \Pi) = u$. Then by the earlier claim, $\bar{\pi}_v(c) = u$ for all v where $\pi_v(c) = u$ and $S(c, \bar{\Pi}) = u$ implying $c = M(\bar{\Pi})$ and the PoA is 1. Thus, if the PoA is more than 1, then $S(c, \Pi) \leq u - 1$ for all $c \in C$.

Combining the two statements, we obtain a bound on the PoA of $\frac{u-1}{2}$. We now demonstrate that this bound is tight. We consider 3 sets of voters with the following preferences:

$$\begin{aligned} v \in V_1: \quad & \pi_v(c_1) = \bar{\pi}_v(c_1) = u - 1, \\ & \pi_v(c_2) = \bar{\pi}_v(c_2) = u \text{ (honest)} \\ v \in V_2: \quad & \pi_v(c_1) = \bar{\pi}_v(c_1) = u - 1, \\ & \pi_v(c_2) = \bar{\pi}_v(c_2) = 1 \text{ (honest)} \\ v = V_3: \quad & \pi_v(c_1) = \bar{\pi}_v(c_1) = 1, \\ & \pi_v(c_2) = 2 \text{ but } \bar{\pi}_v(c_2) = u \end{aligned}$$

where $|V_1| = k$, $|V_2| = k$ and $|V_3| = 1$ and where $\pi_v(c_i) = \bar{\pi}_v(c_i) = 1$ for all $i \geq 3$. If voters are honest then $r(\bar{\Pi}) = c_1$ since $S(c_1, \Pi) = u - 1$, $S(c_2, \Pi) = 2$ and $S(c_i, \Pi) = 1$ for all $i \geq 3$.

However, with respect to the submitted preferences, $S(c_1, \bar{\Pi}) = u - 1$, $S(c_2, \bar{\Pi}) = u$ and $S(c_i, \bar{\Pi}) = 1$ for all $i \geq 3$ and $r(\bar{\Pi}) = c_2$. If $\bar{\Pi}$ is minimally dishonest Nash equilibrium, then we yield the desired bound of $\frac{u-1}{2}$. For $v \in V_1$, voter v receives their preferred outcome and is honest. For $v \in V_2$, voter v cannot change the score of any candidate and is honest. For $v = V_3$, voter v is receiving their most preferred outcome and the only set of preferences that is more honest (with respect to any component-wise norm) assigns c_2 a lower score causing c_3 to tie or win the election, both of which are worse for v. As a result, all voters are providing a minimally dishonest best response and the PoA is $\frac{u-1}{2}$ when there is a unique winner.

Case 2: We considers $\bar{\Pi}$ where $|M(\bar{\Pi})| = m' \geq 2$.

First, suppose there is a candidate $c \in M(\Pi)$ such that $S(c,\Pi) = u$ implying $E[S(r(\Pi),\Pi)] = u$. By the earlier claim, $S(c,\bar{\Pi}) = u$ and $c \in M(\bar{\Pi})$. Similarly, $S(c',\Pi) \geq 2$ for all $c' \in M(\bar{\Pi})$ since our earlier claim also implies $S(c'',\bar{\Pi}) = 1$ for all c'' where $S(c'',\Pi) = 1$ and since $S(c,\bar{\Pi}) = u$. As a result, $E[S(r(\bar{\Pi}),\Pi)] \geq u \cdot \frac{1}{m'} + 2 \cdot \frac{m'-1}{m'} \geq \frac{u+2m-2}{m}$ yielding an expected PoA of

$$\frac{u}{\frac{u+2m-2}{m}} = \frac{um}{u+2m-2} \leq \frac{um-m}{u+2m-2-m} = \frac{um-m}{u+m-2}$$

Alternatively, suppose $E[S(r(\Pi),\Pi)] = x \leq u-1$. If $S(c,\Pi) \geq 2$ for all $c \in M(\bar{\Pi})$, then we obtain an upper bound of $\frac{x}{2} \leq \frac{u-1}{2}$ matching the bound when there are no ties. Alternatively, suppose there is a $c \in M(\bar{\Pi})$ such that $S(c,\Pi) = 1$. By our earlier claim, this implies that $S(c,\bar{\Pi}) = 1$. Since 1 is the minimum score, $S(c',\bar{\Pi}) = 1$ for all c' (everyone ties) yielding a PoA of at most

$$\frac{x}{x\frac{1}{m} + \frac{m-1}{m}} \leq \frac{u-1}{\frac{u+m-2}{m}} = \frac{um-m}{u+m-2}.$$

Finally, to observe that this bound tight, consider the following three disjoint sets of voters:

$$\begin{array}{ll} v \in V_1: & \pi_v(c_i) = \bar{\pi}_v(c_i) = 1 \text{ for all } i \text{ (honest)} \\ v \in V_2: & \pi_v(c_1) = \bar{\pi}_v(c_1) = u, \\ & \pi_v(c_i) = \bar{\pi}_v(c_i) = 1 \text{ for } i \neq 1 \text{ (honest)} \\ v = V_3: & \pi_v(c_1) = u-1 \text{ but } \bar{\pi}_v(c_1) = 1, \\ & \pi_v(c_i) = \bar{\pi}_v(c_i) = u \text{ for } i \neq 1 \end{array}$$

where $|V_1| = |V_2| = k$ and $|V_3| = 1$ With respect to Π, $S(c_1,\Pi) = u-1$ and $S(c_i,\Pi) = 1$ for all $i \neq 1$. For the submitted $\bar{\Pi}$, $S(c,\bar{\Pi}) = 1$ for all c yielding a ratio of $\frac{um-m}{u+m-2}$. As in the previous proofs, $\bar{\Pi}$ is a minimally dishonest Nash equilibrium: For $v \in V_1 \cup V_2$, voter v is honest and cannot change the outcome. For $v = V_3$, for v to be more honest, v must increase the score of c_1, which causes c_1 to be the unique winner, a result that is strictly worse for v. As a result, $\bar{\Pi}$ is a minimally dishonest Nash equilibrium. □

Theorem 4.3 represents a constant factor improvement over lexicographic tie-breaking.

Corollary 4.4. *When there are ties and when $m \leq \frac{u-2}{\alpha-1}$, the PoA for majority judgment with the random candidate tie-breaking rule is α times better than the PoA for majority judgment with lexicographic tie-breaking.*

Proof. First, observe that $\frac{um-m}{u+m-2} = \frac{u-1}{1+\frac{u-2}{m}}$ is increasing with m and therefore $\frac{um-m}{u+m-2} = \frac{u-1}{1+\frac{u-2}{m}} \leq \frac{u-1}{1+\alpha-1} = \frac{u-1}{\alpha}$ – an α times improvement over the lexicographic tie-breaking rule. □

As with the lexicographic case, to compute the normalized PoA, we simply replace every score x with the normalized score $\frac{m-1}{u-1}(x-1)+1$ resulting in the following normalized PoA.

Theorem 4.5. *The normalized PoA for majority judgment with the random candidate tie-breaking rule is* $\max\{\frac{um^2-2m^2+m}{2um-u-3m+2}, \frac{um-2m+1}{u+m-2}\}$ *for any component-wise norm.*

The proof follows identically to Theorem 4.3 with updated scores.

5 DISCUSSION AND CONCLUSION

Section 4 demonstrated that small changes to voting rules can significantly alter the impact of manipulation. For plurality elections, lexicographic tie-breaking lessens the impact of manipulation and randomization erodes all protections from manipulation. In contrast, for majority judgment, the random candidate tie-breaking rule significantly lessens the impact of manipulation (by a factor of α for $m \leq \frac{u-1}{\alpha-1}$).

Further, tie-breaking rules fundamentally changed the relationship between plurality and majority judgment; with respect to lexicographic tie-breaking, plurality appears better than majority judgment at resisting manipulation, whereas for the random candidate tie-breaking rule, majority judgement performs better.

These results indicate that election designers should take care when considering slightly different variants of a voting rule since small changes can have a large impact. Moreover, these results indicate that manipulation is not equivalent in all settings. Further, the price of anarchy is able to discriminate between these types of manipulation and is also able to identify the impact of small changes to voting rules. As such, we propose that price of anarchy be one of the metrics in which a voting rule is accessed.

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REFERENCES

[1] Aditya Anand and Palash Dey. 2021. Distance restricted manipulation in voting. *Theoretical Computer Science* 891 (2021), 149–165.
[2] Barbara M Anthony and Christine Chung. 2020. Equilibria in Doodle polls under three tie-breaking rules. *Theoretical Computer Science* 822 (2020), 61–71.
[3] Kenneth J Arrow. 2012. *Social choice and individual values*. Vol. 12. Yale university press.
[4] Kenneth J. Arrow, Amartya Sen, and Kotaro Suzumura. 2010. *Handbook of Social Choice & Welfare*. Vol. 2. Elsevier.
[5] James P. Bailey. 2017. *The Price of Deception in Social Choice*. Ph.D. Dissertation. Georgia Institute of Technology.
[6] James P. Bailey and Craig A. Tovey. 2018. The Price of Deception in Facility Location. *Seventh International Workshop on Computational Social Choice* (2018).
[7] M.L. Balinski and R. Laraki. 2010. *Majority Judgment: Measuring, Ranking, and Electing*. MIT Press. https://books.google.com/books?id=sSXxXBxuorIC
[8] Michel Balinski and Rida Laraki. 2014. Judge: Don't Vote! *Operations Research* 62, 3 (2014), 483–511.
[9] Michel Balinski and Rida Laraki. 2020. Majority judgment vs. majority rule. *Social Choice and Welfare* 54, 2-3 (2020), 429–461.
[10] John J Bartholdi III, Craig A Tovey, and Michael A Trick. 1989. The computational difficulty of manipulating an election. *Social Choice and Welfare* 6, 3 (1989), 227–241.
[11] Simina Brânzei, Ioannis Caragiannis, Jamie Morgenstern, and Ariel Procaccia. 2013. How bad is selfish voting?. In *Proceedings of the AAAI conference on artificial intelligence*, Vol. 27. 138–144.
[12] Edith Elkind, Umberto Grandi, Francesca Rossi, and Arkadii Slinko. 2020. Cognitive hierarchy and voting manipulation in k-approval voting. *Mathematical Social Sciences* 108 (2020), 193–205.
[13] Edith Elkind, Evangelos Markakis, Svetlana Obraztsova, and Piotr Skowron. 2015. Equilibria of plurality voting: Lazy and truth-biased voters. In *Algorithmic Game Theory: 8th International Symposium, SAGT 2015, Saarbrücken, Germany, September 28-30, 2015. Proceedings 8*. Springer, 110–122.
[14] Edith Elkind, Evangelos Markakis, Svetlana Obraztsova, and Piotr Skowron. 2016. Complexity of finding equilibria of plurality voting under structured preferences. In *AAMAS'16: Proceedings of the 2016 International Conference on Autonomous Agents & Multiagent Systems*. ACM.
[15] Robin Farquharson. 1969. Theory of voting. *(No Title)* (1969).

[16] Peter Gärdenfors. 1976. Manipulation of social choice functions. *Journal of Economic Theory* 13, 2 (1976), 217–228.

[17] Allan Gibbard. 1973. Manipulation of voting schemes: a general result. *Econometrica: Journal of the Econometric Society* (1973), 587–601.

[18] Joshua Kavner and Lirong Xia. 2021. Strategic behavior is bliss: iterative voting improves social welfare. *Advances in Neural Information Processing Systems* 34 (2021), 19021–19032.

[19] Rida Laraki. 2023. Electoral reform: the case for majority judgment. *Constitutional Political Economy* 34, 3 (2023), 346–356.

[20] Herve Moulin. 1991. *Axioms of cooperative decision making.* Cambridge University Press.

[21] Svetlana Obraztsova, Omer Lev, Evangelos Markakis, Zinovi Rabinovich, and Jeffrey S. Rosenschein. 2017. Distant Truth: Bias Under Vote Distortion Costs. In *Proceedings of the 16th Conference on Autonomous Agents and MultiAgent Systems* (São Paulo, Brazil) *(AAMAS '17)*. International Foundation for Autonomous Agents and Multiagent Systems, Richland, SC, 885–892. http://dl.acm.org/citation.cfm?id=3091125.3091250

[22] Svetlana Obraztsova, Evangelos Markakis, and David R. M. Thompson. 2013. Plurality Voting with Truth-Biased Agents. In *Algorithmic Game Theory*, Berthold Vöcking (Ed.). Springer Berlin Heidelberg, Berlin, Heidelberg, 26–37.

[23] Tim Roughgarden. 2005. Selfish Routing and the Price of Anarchy.

[24] Mark Allen Satterthwaite. 1975. Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions. *Journal of economic theory* 10, 2 (1975), 187–217.