Proportional Fairness in Obnoxious Facility Location

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ABSTRACT

We consider the obnoxious facility location problem (in which agents prefer the facility location to be far from them) and propose a hierarchy of distance-based proportional fairness concepts for the problem. These fairness axioms ensure that groups of agents at the same location are guaranteed to be a distance from the facility proportional to their group size. We consider deterministic and randomized mechanisms, and compute tight bounds on the price of proportional fairness. In the deterministic setting, we show that our proportional fairness axioms are incompatible with strategyproofness, and prove asymptotically tight *epsilon*-price of anarchy and stability bounds for proportionally fair welfare-optimal mechanisms. In the randomized setting, we identify proportionally fair and strategyproof mechanisms that give an expected welfare within a constant factor of the optimal welfare. Finally, we prove existence results for two extensions to our model.

KEYWORDS

Facility location; Fairness; Social choice; Approximate Equilibria

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1 INTRODUCTION

In the *obnoxious facility location problem (OFLP)*, some undesirable facility such as a garbage dump or an oil refinery is to be located on a unit interval (i.e. the domain of locations is $[0, 1]$), and the agents along the interval wish to be as far from the facility as possible [14, 15, 19, 23]. In this problem, agents have single-dipped preferences, contrasting with the single-peaked preferences of agents in the classic facility location problem (in which agents prefer to be located as close as possible to the facility).

The obnoxious facility location problem models many real-world facility placements which negatively impact nearby agents, such as a prison or a power plant [18]. Aside from the geographic placement of an obnoxious facility, the OFLP can also be applied to various collective decision making problems. For instance, when agents are averse to their worst possible social outcomes (represented by their locations), the problem captures issues where a decision needs to be made on a social policy or a budget composition. When a socially sensitive or a politically undesirable policy needs to be implemented, the placements of such a policy in the space of outcomes may need to take equity considerations.

It is known that placing the facility at one of the interval endpoints maximizes the sum of agent distances [16], but such a solution may not be 'proportionally fair' for the agents. To build intuition, consider the instance depicted in Figure 1. The optimal utilitarian solution (which maximizes the sum of agent distances) places the facility at 0, disproportionately disadvantaging the agents at 0.1 who are located only 0.1 distance from the facility. A facility location of 0.45 results in both groups of agents having the same distance from the facility, and would be considered to be more 'fair' in the egalitarian sense. However, it is not proportionally fair: despite having over twice as many agents, the group of agents at 0.8 have the same distance from the facility as the group of agents at 0.1. A proportionally fair solution places the facility at 0.3, and results in the distance between a group of agents and the facility being proportional to the size of the group.

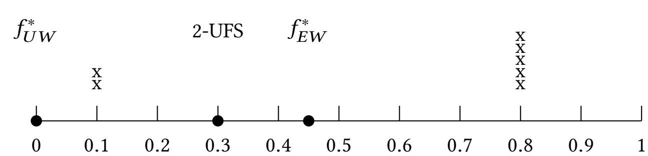

Figure 1: OFLP with agent location profile $(0.1, 0.1, 0.8, 0.8, 0.8, 0.8)$ **represented by** x**. The facility locations (represented by •) correspond to a utilitarian outcome,** $f^*_{UW} = 0$**; a proportionally fair outcome,** 2-UFS = 0.3**; and an egalitarian outcome,** $f^*_{EW} = 0.45$**.**

In this work, we pursue notions of *proportional fairness* as a central concern for the problem. Specifically, we formulate a hierarchy of proportional fairness axioms which guarantee that each group of agents at the same location are a distance from the facility proportional to the relative size of the group. While proportional fairness axioms have been formulated and studied in the classic facility location problem [4], they have not yet been applied to the OFLP. Our

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Table 1: Price of fairness and welfare approximation results.

		Price of Fairness		Best Known
		2-IFS	2-UFS	2-UFS SP Approx.
Det.	UW	2 (Thm. 4.2)	2 (Thm. 4.3)	**Incompatible** (Prop. 4.7)
	EW	1 (Prop. 4.4)	$n-1$ (Thm. 4.5)	
Rand.	UW	12/11 (Cor. 5.9)	1.094. . . (Cor. 5.12)	1.5 (Thm. 5.4)
	EW	1 (Prop. 4.4)	1 (Cor. 5.6)	1 (Prop. 5.5)

paper provides a comprehensive overview of proportionally fair solutions for the obnoxious facility location problem, examining the interplay between proportional fairness and utilitarian/egalitarian welfare, and investigating concerns of agent strategic behaviour in both the deterministic and randomized settings.

Contributions.

- We formalize (approximate) proportional fairness concepts such as 2-Individual Fair Share (2-IFS) and 2-Unanimous Fair Share (2-UFS) in the context of the obnoxious facility location problem. Several of the definitions are natural adaptations of axioms from fair division and participatory budgeting.
- We find tight bounds on the price of 2-IFS and 2-UFS fairness for the objectives of egalitarian and utilitarian welfare, in both the deterministic and randomized settings.
- We prove that our proportional fairness axioms are incompatible with strategyproofness in the deterministic setting, and give strategyproof randomized mechanisms that satisfy these proportional fairness axioms in expectation and either have a constant approximation ratio for utilitarian welfare or are optimal for egalitarian welfare.
- For the deterministic mechanisms that maximize utilitarian welfare under the constraints of 2-IFS and 2-UFS, we prove that a pure ϵ-Nash equilibrium always exists. We then find asymptotically tight linear bounds on the corresponding ϵ-price of anarchy, as well as asymptotically tight constant bounds on the corresponding ϵ-price of stability.
- Finally, we give two possible extensions of our model: the fairness axiom of 2-Proportional Fairness (2-PF), which is stronger than 2-UFS as it captures proportional fairness concerns for groups of agents near but not necessarily at the same location, and the hybrid model, which also includes 'classic' agents who instead want to be near the facility. We prove existence results for both extensions.

Table 1 summarizes some of our results. Results lacking proofs are proven in the appendix.

Related Work. The papers most relevant to our research are those that treat the facility as obnoxious: agents prefer the facility to be as far from them as possible. Similar to the classical facility location problem, early operations research on the OFLP apply an optimization approach to compute solutions; a review of these approaches is given by Church and Drezner [18]. There are several papers on the obnoxious facility location problem that apply a mechanism design approach, assuming agents' location are private information. This was initiated by Cheng et al. [15, 16], who define an agent's utility as its distance from the facility, and design strategyproof mechanisms which approximate the optimal utilitarian welfare on the path and network metrics. Other recent examples of related papers include [14, 19, 23, 36]. These papers do not pose or study the fairness concepts that we explore in this paper.

Notions of fairness in various collective decision problems have been widely explored over the last few decades [26, 28, 31]. Fairness objectives specifically relevant to the facility location problem include maximum cost/egalitarian welfare (see, e.g. [30, 34]) and maximum total/average group cost [37]. Rather than optimize/approximate fairness objectives, we focus on solutions enforcing proportional fairness axioms, in which groups of agents with similar or identical preferences/locations) have a minimum utility guarantee relative to the group size. The axioms of proportional fairness that we present stem from several related areas of social choice. Individual Fair Share (IFS) is closely related to the axiom of proportionality proposed by Steinhaus [32], and appears in participatory budgeting along with Unanimous Fair Share (UFS) [3, 9]. Asll of our proportional fairness axioms have been studied in the classical facility location problem by Aziz et al. [4].

In our paper, we also analyse the loss of efficiency, defined as the price of fairness, of implementing the proportional fairness axioms that we have proposed. The price of fairness has been studied for some variations of the facility location problem, such as when there is a lexicographic minimax objective [11], or when facilities have preferences over subsets of agents, and the fairness is observed from the facilities' perspectives [35]. Many recent results on price of fairness have also been found in various social choice contexts, such as fair division and probabilistic social choice [6–8, 12].

As strategyproofness is impossible in our deterministic setting, we present results on the existence of pure Nash equilibria, and the prices of anarchy and stability. Similar models where such results are proven include a variation of the Hotelling-Downs model [21], and two-stage facility location games where both facilities and clients act strategically [25]. In the classic facility location problem, Aziz et al. [4] characterize the pure Nash equilibria of strictly monotonic facility location mechanisms satisfying UFS and show that the resulting equilibrium facility location is also guaranteed to satisfy UFS. For certain mechanisms in our setting, a pure Nash equilibrium may not exist, so we prove the existence of the approximate *pure ϵ-Nash equilibrium*. Examples of papers applying this notion to other settings include [17, 27].

The second half of our paper focuses on the randomized setting to overcome the incompatibility with strategyproofness. The use of randomized mechanisms to overcome impossibility results is prevalent in many social choice contexts (see, e.g., [2, 10]). Additionally, Aziz et al. [5] use a randomized approach in the classic facility location problem to achieve stronger notions of proportional fairness, providing a unique characterization of universally anonymous and universally truthful mechanisms satisfying an axiom called Strong Proportionality. The use of randomized mechanisms also results in better approximation ratio/price of fairness bounds. This is common in many variants of the facility location problem, such as when agents have fractional or optional preferences [13, 22], or in the hybrid facility location model [19].

2 MODEL

Let $N = \{1, \ldots, n\}$ be a set of agents, and let $X := [0, 1]$ be the domain of locations.[1] Agent i's location is denoted by $x_i \in X$; the profile of agent locations is denoted by $x = (x_1, \ldots, x_n) \in X^n$. We also assume the agent locations are ordered such that $x_1 \leq \cdots \leq x_n$. A *deterministic mechanism* is a mapping $f : X^n \to X$ from a location profile $\hat{x} \in X^n$ to a facility location $y \in X$. Given a facility location $y \in X$, agent i's utility[2] is equal to its distance from the facility $u(y, x_i) := |y - x_i|$. We are interested in maximizing the objectives of *Utilitarian Welfare* (UW), defined for a facility location y and location profile x as the sum of agent utilities $\sum_i u(y, x_i)$, and *Egalitarian Welfare* (EW), defined as the minimum agent utility $\min_i u(y, x_i)$.

Note that the preferences in OFLP can be viewed as *single-dipped*, contrasting with the *single-peaked preferences* of the classical facility location problem (FLP). The underlying model of both FLP and OFLP is the same except that the agents' preferences have a different structure. Unless specified otherwise, we will state results for the obnoxious facility location problem (OFLP).

3 PROPORTIONAL FAIRNESS AXIOMS

In this section, we introduce proportional fairness axioms for the obnoxious facility location problem.

3.1 Individual Fair Share

We first present an adaptation of Individual Fair Share (IFS), the weakest of our proportional fairness axioms (as studied by Aziz et al. [4] in the context of the classic facility location problem). IFS provides a minimum distance guarantee between each agent and the facility, requiring that each agent has at least $\frac{1}{n}$ utility. By placing two agents at $\frac{1}{4}$ and $\frac{3}{4}$, it is easy to see that an IFS solution may not exist. As a result, we turn to approximations of IFS.

Definition 3.1 (α-Individual Fair Share (IFS)). Given a profile of locations x, a facility location y satisfies *α-Individual Fair Share (α-IFS)* if

$$u(y, x_i) \geq \frac{1}{\alpha n} \quad \forall i \in N.$$

We find that the lowest value of α such that an α–IFS solution always exists is $\alpha = 2$. Intuitively, with $\alpha = 2$, each agent has an open interval of radius $\frac{1}{2n}$ around its location. The sum of interval lengths is 1, meaning there will always be a 2-IFS solution. For any $\alpha < 2$, an α–IFS solution may not always exist as the sum of interval lengths will exceed 1.

Proposition 3.2. *The lowest value of α for which an α-IFS solution always exists is $\alpha = 2$.*

A polynomial time 2-IFS mechanism (which we denote as f^*_{2IFS}) that maximizes the utilitarian welfare simply iterates through the endpoints of the intervals which satisfy the constraint and outputs the optimal facility location, breaking ties in favour of the leftmost optimal location.

3.2 Unanimous Fair Share

We now present Unanimous Fair Share (UFS), a strengthening and generalization of IFS to groups of agents at the same location. Informally, if there are k agents at the same location, then UFS requires that the facility is placed at least $\frac{k}{n}$ distance from these agents. Under UFS, agents are not considered to be in the same group if they are very close but not exactly co-located. However, the co-location of agents often naturally arises in practice, such as when multiple citizens live in the same apartment building, or when considering populations of towns. Towards the end of the paper, we propose a stronger proportional fairness axiom which considers agents at near but not necessarily the same location to be part of the same group.

Again, we focus on approximations of UFS as a UFS solution may not exist.

Definition 3.3 (α-Unanimous Fair Share (UFS)). Given a profile of locations $\boldsymbol{x}$, a facility location y satisfies *α-Unanimous Fair Share (α-UFS)* if for any set of agents S with identical location,

$$u(y, x_i) \geq \frac{|S|}{\alpha n} \quad \forall i \in S.$$

Note that α–UFS implies α–IFS. As with α–IFS, we find that the optimal value of α for which an α-UFS solution always exists is $\alpha = 2$. The proof intuition is similar to that of Proposition 3.2, but the intervals vary in size depending on the number of agents in the group.

Proposition 3.4. *The lowest value of α for which an α-UFS solution always exists is $\alpha = 2$.*

Similar to f^*_{2IFS}, a polynomial time mechanism (which we denote as f^*_{2UFS}) that computes the optimal 2-UFS facility location for utilitarian welfare iterates through the endpoints of the intervals satisfying 2-UFS and outputs the optimal facility location, breaking ties in favour of the leftmost optimal location.

4 DETERMINISTIC SETTING

We begin with the deterministic setting, analyzing the price of proportional fairness and agent strategic behaviour. All results stated in this section are for the deterministic setting.

4.1 Price of Fairness

In this section, we analyze the price of fairness for our (approximate) fairness axioms.[3] Informally, the price of fairness measures the loss of efficiency from imposing a certain fairness constraint. We focus on the objectives of utilitarian and egalitarian welfare, defined as the sum of utilities and the minimum agent utility, respectively.

A *fairness property P* is a mapping from an agent location profile $x \in X^n$ to a (possibly empty) set of facility locations $P(x) \in X$. Every facility location $P(x)$ satisfies the fairness property P. The price of fairness for property P is the worst-case ratio between the optimal welfare and the optimal welfare from a facility location satisfying P.

[1] Our results can be naturally extended to any compact interval on $\mathbb{R}$.
[2] This definition is consistent with [16].
[3] The price of fairness can also be interpreted as the approximation ratio for the respective optimal mechanism satisfying the fairness constraint.

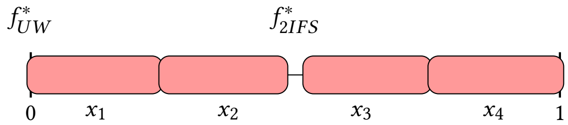

Figure 2: The lower bound instance in the proof of Theorem 4.2 for $n = 4$. f^*_{UW} represents the utilitarian welfare maximizing facility placement, whilst f^*_{2IFS} maximizes utilitarian welfare under the constraints of 2-IFS. The red intervals denote locations that are infeasible under 2-IFS.

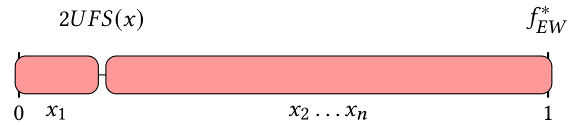

Figure 3: The instance in the proof of Theorem 4.5. f^*_{EW} represents the egalitarian welfare maximizing facility placement, whilst $2UFS(x)$ represents the interval of facility placements satisfying 2-UFS. The red intervals denote locations that are infeasible under 2-UFS.

Definition 4.1 (Price of Fairness for Utilitarian/Egalitarian Welfare). Let $\{f^*_{UW}, f^*_{EW}\}$ be the mechanism that returns the solution maximizing utilitarian/egalitarian welfare. For UW/EW and fairness property P, we define the price of fairness as the worst-case ratio (over all location profiles) between the optimal UW/EW and the optimal UW/EW achieved by a facility location satisfying P:

$$\max_{x\in[0,1]^n} \frac{W(f^*(x), x)}{\max_{y\in P(x)} W(y, x)}.$$

For UW, $f^*(x) := f^*_{UW}(x)$ and $W(y, x) := \sum_i u(y, x_i)$.
For EW, $f^*(x) := f^*_{EW}(x)$ and $W(y, x) := \min_i u(y, x_i)$.

4.1.1 Utilitarian Welfare. The utilitarian welfare (UW) of an instance is a standard measure of efficiency. Finding the price of our proportional fairness axioms for utilitarian welfare quantifies the impact on efficiency when the OFLP system is constrained to be proportionally fair.

We now move to compute the prices of 2-IFS and 2-UFS fairness for utilitarian welfare. Recall that the solution maximizing utilitarian welfare must be either 0 or 1 [16]. To prove the price of fairness lower bounds, we place the agents such that the only feasible 2-IFS/UFS solution lies in the optimal median interval (see, e.g. Figure 2).

Theorem 4.2. *The price of 2-IFS for utilitarian welfare is 2, and this bound is tight.*

Lower Bound Proof. Suppose n is even, and that the agents are located at $\frac{1}{2n} - \epsilon$, $\frac{3}{2n} - 2\epsilon, \dots, \frac{n-1}{2n} - \frac{n}{2}\epsilon$, $\frac{n+1}{2n} + \frac{n}{2}\epsilon, \dots, \frac{2n-3}{2n} + 2\epsilon$, $\frac{2n-1}{2n} + \epsilon$ for some sufficiently small ϵ (see, e.g. Figure 2). Under this symmetric profile, either a facility location of 0 or 1 leads to the maximum utilitarian welfare of $\frac{n}{2}$. The only facility locations satisfying 2-IFS are within the interval $[\frac{1}{2} - \frac{n}{2}\epsilon, \frac{1}{2} + \frac{n}{2}\epsilon]$. Any location in this interval gives the same utilitarian welfare as there are an equal number of agents on both sides, so suppose the facility is at $\frac{1}{2}$. This corresponds to a utilitarian welfare of $\frac{n}{4} + \epsilon n(1 + \frac{n}{2})$. Taking the limit $\epsilon \longrightarrow 0$ gives a ratio of 2. □

Theorem 4.3. *The price of 2-UFS for utilitarian welfare is 2, and this bound is tight.*

As the price of fairness for utilitarian welfare is the same for both proportional fairness axioms, it may be desirable to implement 2-UFS in favour of 2-IFS when loss of utilitarian welfare is the primary concern.

4.1.2 Egalitarian Welfare. The egalitarian welfare (EW) is an alternate measure of fairness frequently observed in the literature, focusing on the worst off agent. Our price of fairness analysis gives an insight into the tradeoff between egalitarian welfare/maximin fairness and proportional fairness in the OFLP.

Our first result is that the price of 2-IFS is 1, meaning that a mechanism that maximizes egalitarian welfare is guaranteed to satisfy 2-IFS. This follows from Proposition 3.2, which states that a 2-IFS solution (in which every agent obtains at least $\frac{1}{2n}$ utility) always exists.

Proposition 4.4. *The price of 2-IFS for egalitarian welfare is* 1.

On the other hand, we find that the price of 2-UFS is noticeably worse, taking a linear factor of $n - 1$. The intuition behind this is that a coalition of $n - 1$ agents at one point can ensure that the facility is distant from their location (and closer to the remaining agent's location) by a 'factor' of $n - 1$ (see, e.g. Figure 3).

Theorem 4.5. *The price of 2-UFS for egalitarian welfare is* $n - 1$.

Proof. We first prove that the lower bound is $n - 1$. It suffices to consider $n \geq 3$. Consider the location profile with 1 agent at $\frac{1}{2n} - \epsilon$ and $n - 1$ agents at $\frac{n+1}{2n} + \epsilon$ for sufficiently small $\epsilon > 0$, (see, e.g. Figure 3). The optimal solution places the facility at 1 resulting in an egalitarian welfare of $\frac{n-1}{2n} - \epsilon$. The only 2-UFS solutions are in the interval $[\frac{1}{n} - \epsilon, \frac{1}{n} + \epsilon]$, and the solution of $\frac{1}{n} + \epsilon$ results in an egalitarian welfare of $\frac{1}{2n} + 2\epsilon$. As $\epsilon \longrightarrow 0$, the ratio approaches $n - 1$.

We now prove that the upper bound is $n - 1$. Firstly, it clearly suffices to consider location profiles where groups contain at most $n - 1$ agents. Suppose there exists such an x where $\min_i u(f^*_{EW}(x), x_i) \geq \frac{n-1}{2n}$, i.e. there is a solution where every agent has at least $\frac{n-1}{2n}$ utility. Then this also satisfies 2-UFS and results in an egalitarian ratio of 1. Therefore the maximum ratio must have $\min_i u(f^*_{EW}(x), x_i) < \frac{n-1}{2n}$. Due to 2-UFS, we also have $\max_{y\in 2UFS(x)} \min_i u(y, x_i) \geq \frac{1}{2n}$. The theorem statement follows from dividing these two terms. □

4.2 Incompatibility with Strategyproofness

In mechanism design, the normative property of *strategyproofness* is often sought as it disincentivizes agents from misreporting their true location.

Definition 4.6 (Strategyproofness). A (deterministic) mechanism f is strategyproof if for every agent $i \in N$, we have for every x_i, x'_i and $\hat{x}_{-i}$,

$$u(f(x_i, \hat{x}_{-i}), x_i) \geq u(f(x'_i, \hat{x}_{-i}), x_i).$$

We say that a randomized mechanism is *strategyproof in expectation* if no agent can improve its expected utility by misreporting its own location.

We note that no strategyproof and deterministic mechanism can achieve any approximation of IFS (and therefore also UFS).

Proposition 4.7. *There exists no deterministic and strategyproof mechanism that achieves any approximation of IFS.*

Proof. From the characterization by Feigenbaum and Sethuraman [20], it can be seen that for any deterministic and strategyproof mechanism, there exists a location profile where the facility is placed at an agent's location. Such a mechanism does not satisfy any approximation of IFS. □

Since strategyproofness is incompatible with our fairness axioms, we are interested in the performance of proportionally fair mechanisms in our model when accounting for agent strategic behaviour. Such performance can be quantified by the price of anarchy, and the price of stability.

4.3 ϵ-Price of Anarchy and ϵ-Price of Stability

In this section, we compute the loss of efficiency by agents misreporting their location (in a pure Nash equilibrium of reports) under the mechanisms f^*_{2IFS} and f^*_{2UFS}. Recall these are the mechanisms which maximize utilitarian welfare under the constraints of 2-IFS and 2-UFS, respectively. This efficiency loss can be quantified in the 'worst-case' sense, by the *price of anarchy* [24, 29], or in the 'best case' sense, by the *price of stability* [1].

However, for f^*_{2IFS} and f^*_{2UFS}, we show that a pure Nash equilibrium may not necessarily exist, and hence the price of anarchy is not well-defined.

Proposition 4.8. *A pure Nash equilibrium may not exist for f^*_{2IFS} or f^*_{2UFS}.*

Proof Sketch. Consider the location profile $x = (\frac{1}{4} - \epsilon, \frac{3}{4} + \epsilon)$, where $f^*_{2IFS} = \frac{1}{2} - \epsilon$. The agent at $\frac{1}{4} - \epsilon$ can, for all $\epsilon > 0$, shift its location to the right to improve its utility, but if it reports exactly $\frac{1}{4}$, then it will lose utility as the facility moves to location 0. □

As a result, we turn to proving existence of the approximate notion of pure ϵ-Nash equilibria, and computing the corresponding notions of ϵ-price of anarchy and ϵ-price of stability.

Definition 4.9 (Tardos and Vazirani [33]). A *pure ϵ-Nash equilibrium* is a profile of reported agent locations $x' = (x'_1, \dots, x'_n)$ such that no single agent can improve its own utility (with respect to its true location) by strictly more than ϵ by changing its reported location. A pure Nash equilibrium is a pure ϵ-Nash equilibrium where $\epsilon = 0$.

To prove the following theorems, we divide the space of agent location profiles into several subcases, and for each subcase, we describe a pure ϵ-Nash equilibrium.

Theorem 4.10. *For any $\epsilon > 0$, a pure ϵ-Nash equilibrium always exists for f^*_{2IFS}.*

Theorem 4.11. *For any $\epsilon > 0$, a pure ϵ-Nash equilibrium always exists for f^*_{2UFS}.*

In real-world settings, the value of ϵ could represent a discretization of the domain, or the smallest distance of which an agent can change their reported location.

For a mechanism f, the ϵ-price of anarchy (resp. stability) is defined as the worst-case ratio (over all location profiles x) between the utilitarian welfare corresponding to all agents reporting truthfully and the *minimum* (resp. *maximum*) utilitarian welfare corresponding to agents reporting in a pure ϵ-Nash equilibrium.

Definition 4.12. Given f and x, define the set of pure ϵ-Nash equilibria location profiles as ϵ-$Equil(f, x)$. The ϵ-price of anarchy for utilitarian welfare is defined as:

$$\epsilon\text{-}PoA(f) := \max_{x \in X^n} \frac{\sum_i u(f(x), x_i)}{\min_{x' \in \epsilon\text{-}Equil(f,x)} \sum_i u(f(x'), x_i)}.$$

The ϵ-price of stability for utilitarian welfare is defined as:

$$\epsilon\text{-}PoS(f) := \max_{x \in X^n} \frac{\sum_i u(f(x), x_i)}{\max_{x' \in \epsilon\text{-}Equil(f,x)} \sum_i u(f(x'), x_i)}.$$

We now proceed to find ϵ-price of anarchy bounds for utilitarian welfare. The same proof arguments can be applied to find identical bounds for both f^*_{2IFS} and f^*_{2UFS}.

Theorem 4.13. *For any $\epsilon \in (0, \frac{1}{n})$, the ϵ-price of anarchy for f^*_{2IFS} and f^*_{2UFS} of utilitarian welfare is at least $\frac{2n-1+n\epsilon}{1-n\epsilon}$. The price of anarchy is unbounded for $\epsilon \geq \frac{1}{n}$.*

Proof Sketch. When all agents are located at $\frac{1}{2n} - \frac{\epsilon}{2}$, the location profile $x' = (1, \dots, 1)$ is a pure ϵ-Nash equilibrium, which leads to the stated ϵ-price of anarchy lower bound. □

Theorem 4.14. *For any $\epsilon \in (0, \frac{1}{2n})$, the ϵ-price of anarchy for f^*_{2IFS} and f^*_{2UFS} of utilitarian welfare is at most $\frac{2n}{1-2n\epsilon}$.*

Proof. Under a pure ϵ-Nash equilibrium, each agent must have at least $\frac{1}{2n} - \epsilon$ utility. This is because an agent can achieve at least $\frac{1}{2n}$ utility by reporting its true location. Therefore the utilitarian welfare under a pure ϵ-Nash equilibrium must be at least $\frac{1}{2} - n\epsilon$. Now the utilitarian welfare under any instance is at most n, from all agents being located at 0 and the facility being placed at 1. The theorem statement follows from dividing these terms. □

As $\epsilon \to 0$, we see that the ϵ-price of anarchy of f^*_{2IFS} and f^*_{2UFS} is linear and thus the mechanisms perform quite poorly in the worst-case equilibria. In contrast, we prove asymptotically tight constant bounds on the ϵ-price of stability. To prove the lower bound, we give a location profile where the optimal f^*_{2IFS} and f^*_{2UFS} facility placements are near an interval endpoint, but the only ϵ-Nash equilibria facility placement is in the middle of the agents.

Theorem 4.15. *For f^*_{2IFS} and f^*_{2UFS}, if $\epsilon \in (0, \frac{1}{2n})$, the ϵ-price of stability is at least*

$$\frac{4n^2 - 4n + 4 + 8n\epsilon}{2n^2 + n + 2 + (2n^3 + 4n^2 - 8n)\epsilon}.$$

This expression approaches 2 as $\epsilon \to 0$ and $n \to \infty$.

Proof Sketch. Suppose we have even n, and that the (true) agent location profile is $x = (0, \frac{3}{2n} - 2\delta, \frac{5}{2n} - 3\delta, \ldots, \frac{n-1}{2n} - \frac{n}{2}\delta, \frac{n+1}{2n} + \frac{n}{2}\delta, \frac{n+3}{2n} + (\frac{n}{2} - 1)\delta, \ldots, \frac{2n-1}{2n} + \delta)$, where $\delta > \epsilon$ and $\delta - \epsilon$ is sufficiently small. Here, f^*_{2IFS} and f^*_{2UFS} place the facility at $\frac{1}{2n}$, which results in a UW of $\frac{n^2-n+1}{2n} + \delta$. Under a pure ϵ-Nash equilibrium, the facility can only be placed in the interval $[\frac{1}{2} - \frac{n}{2}\delta, \frac{1}{2} + \frac{n}{2}\delta]$. The utilitarian welfare corresponding to the equilibrium facility placement of $\frac{1}{2} - \frac{n}{2}\delta$ is $\frac{2n^2+n+2}{8n} + (\frac{n^2}{4} + \frac{n}{2} - 1)\delta$. Dividing the welfares and taking the limit $\delta \to \epsilon$ gives the lower bound in the theorem statement. □

We next prove that the ϵ-price of stability for f^*_{2IFS} and f^*_{2UFS} has an upper bound of 2. The proof iterates through the ϵ-equilibria in each subcase of the proof of Theorem 4.10, and constructs the location profile that maximizes the ratio between the utilitarian welfare when agents report truthfully, and the utilitarian welfare corresponding to the given ϵ-equilibrium facility placement.

Theorem 4.16. *For f^*_{2IFS} and f^*_{2UFS}, taking the limit $\epsilon \to 0$, the ϵ-price of stability is at most 2.*

As we have shown, when maximizing the utilitarian welfare under 2-IFS or 2-UFS, the degradation of efficiency under a ϵ-Nash equilibrium can range from a constant factor to a linear factor. To avoid a pessimistic outcome, we may wish to employ a randomized mechanism, achieving strategyproofness along with 2-IFS or 2-UFS in expectation. We give examples of such mechanisms in the upcoming section.

5 RANDOMIZED MECHANISMS

By using randomized mechanisms, we can achieve a better price of fairness for 2-IFS and 2-UFS, and overcome the incompatibility with strategyproofness. We define a *randomized mechanism* as a probability distribution over deterministic mechanisms, and an agent's utility as its expected distance from the facility.

In the randomized setting, the optimal approximation of IFS and UFS for which a solution always exists is $\alpha = 2$, as seen by setting 1 agent at $\frac{1}{2}$. Our fairness axioms are adapted as follows:

Definition 5.1 (α-Individual Fair Share (IFS) in expectation). A mechanism f satisfies *α-Individual Fair Share in expectation (α-IFS in expectation)* if for any location profile x,

$$\mathbb{E}[u(f(x), x_i)] \geq \frac{1}{\alpha n} \quad \forall i \in N.$$

Definition 5.2 (α-Unanimous Fair Share (UFS) in expectation). A mechanism f satisfies *α-Unanimous Fair Share in expectation (α-UFS in expectation)* if for any location profile x and any set of agents S at the same location,

$$\mathbb{E}[u(f(x), x_i)] \geq \frac{|S|}{\alpha n} \quad \forall i \in S.$$

We first show that when maximizing welfare in the randomized setting, it suffices to consider mechanisms which can only place the facility at 0 or 1.

Lemma 5.3. *Consider an agent location profile x. For every 2-IFS/UFS randomized mechanism that gives strictly positive probability to a facility placement between 0 and 1, there exists a 2-IFS/UFS randomized mechanism that only gives positive support to a facility placement at 0 or 1 that leads to weakly higher expected utility for each agent.*

5.1 Strategyproofness

From Proposition 4.7, we know that in the deterministic setting, strategyproofness is incompatible with our proportional fairness axioms. In the randomized setting, the space of mechanisms is much larger and hence we are able to overcome this impossibility.

We first consider **Mechanism 2** from [16]. Denoting the numbers of agents located in $[0, 1/2]$ and $(1/2, 1]$ by n_1 and n_2 respectively, **Mechanism 2** places the facility at 0 with probability α and at 1 with probability $(1 - \alpha)$, where $\alpha = \frac{2n_1n_2+n_2^2}{n_1^2+n_2^2+4n_1n_2}$. This mechanism is known to be group strategyproof (in expectation) and $\frac{3}{2}$−approximates the utilitarian welfare. We show that this mechanism satisfies 2-UFS (and therefore also 2-IFS).

Theorem 5.4. ***Mechanism 2** satisfies 2-UFS in expectation.*

5.2 Egalitarian Welfare

We now provide some results on egalitarian welfare. Specifically, we give a randomized, strategyproof mechanism which maximizes egalitarian welfare subject to the constraints of 2-IFS and 2-UFS in expectation.

The **Randomized Egalitarian Welfare mechanism** places the facility at 1 if all agents are in $[0, \frac{1}{2}]$, at 0 if all agents are in $(\frac{1}{2}, 1]$, and at 0 or 1 with 0.5 probability otherwise.

By considering cases, it is easy to see that this mechanism is optimal and satisfies ideal normative properties.

Proposition 5.5. ***Randomized Egalitarian Welfare mechanism** is strategyproof in expectation, egalitarian-welfare optimal, and satisfies 2-UFS.*

Proof. We first prove strategyproofness. If all agents are in $[0, \frac{1}{2}]$ or all agents are in $(\frac{1}{2}, 1]$, then each agent has at least $\frac{1}{2}$ expected utility. Any misreport either causes their expected utility to either stay the same or be reduced to $\frac{1}{2}$ from the facility being placed at 0 or 1 with probability $\frac{1}{2}$ each. If there is at least one agent in each interval, then an agent can only affect the outcome if it is the only agent in its interval and it misreports to be in the other interval, but this weakly reduces the agent's expected utility.

We now prove egalitarian welfare optimality and satisfaction of 2-UFS. The cases where all agents are in $[0, \frac{1}{2}]$ and all agents are in $(\frac{1}{2}, 1]$ are trivial, so it remains to examine the case where both intervals have at least one agent. An agent at x_i has $\frac{1}{2}x_i + \frac{1}{2}(1 - x_i) = 1/2$ expected distance from the facility, hence this mechanism satisfies 2-UFS in expectation. By Lemma 5.3, it suffices to only consider mechanisms which can only place the facility at 0 or 1. Suppose that instead of having $\frac{1}{2}$ probability of placing the facility at either endpoint, we place the facility at 1 with $\frac{1}{2} + p$ probability and at 0 with $\frac{1}{2} - p$ probability, where $p \in (0, \frac{1}{2}]$. The expected utility of the rightmost agent is $x_n(\frac{1}{2} - p) + (1 - x_n)(\frac{1}{2} + p) = \frac{1}{2} + p(1 - 2x_n) < \frac{1}{2}$. By a symmetric argument, if the facility was placed at 1 with $\frac{1}{2} - p$ probability and at 0 with $\frac{1}{2} + p$ probability, the expected utility of the leftmost agent would be strictly less than $\frac{1}{2}$. Hence, our mechanism is optimal in this case. □

Remark 1. *As each agent has at least* $1/2$ *expected distance from the facility under the* ***Randomized Egalitarian Welfare mechanism****, this mechanism even satisfies 1-IFS for* $n \geq 2$.

In other words, the approximation ratio of this mechanism for egalitarian welfare is 1. Recall that the price of fairness can be interpreted as the approximation ratio of the respective optimal mechanism that satisfies the fairness constraint. This leads us to the following corollary.

Corollary 5.6. *In the randomized setting, the price of fairness of 2-UFS for EW is* 1.

This is in stark contrast to the deterministic setting where the respective price of fairness is $n - 1$.

5.3 2-IFS

We now analyze utilitarian welfare, beginning with the axiom of 2-IFS. Consider the randomized mechanism below which maximizes the utilitarian welfare subject to 2-IFS:

2-IFS Randomized mechanism

- If $\sum_{i=1}^{n} x_i = \frac{n}{2}$, place the facility at 0 with probability $\frac{1}{2}$ and at 1 with probability $\frac{1}{2}$.
- If $\sum_{i=1}^{n} x_i > \frac{n}{2}$,
  - If $x_1 \geq \frac{1}{2n}$, place the facility at 0.
  - If $x_1 < \frac{1}{2n}$, place the facility at 0 with probability $1 - \alpha$, and at 1 with probability α, where $\alpha = \frac{1-2nx_1}{2n(1-2x_1)}$.
- If $\sum_{i=1}^{n} x_i < \frac{n}{2}$,
  - If $x_n \leq 1 - \frac{1}{2n}$, place the facility at 1.
  - If $x_n > 1 - \frac{1}{2n}$, place the facility at 0 with probability $1 - \beta$, and at 1 with probability β, where $\beta = \frac{1-2nx_n}{2n(1-2x_n)}$.

The intuition behind this mechanism is as follows. When $\sum_{i=1}^{n} x_i = \frac{n}{2}$, both facility locations of 0 and 1 are tied in terms of maximizing utilitarian welfare, and by placing the facility at either location with probability $\frac{1}{2}$, we achieve 2-IFS in expectation. When $\sum_{i=1}^{n} x_i > \frac{n}{2}$, the optimal facility location is 0, so the mechanism places the facility there if it does not violate 2-IFS for any agent, else it places the facility at 1 with the minimum probability that ensures 2-IFS is ensured for all agents. The case where $\sum_{i=1}^{n} x_i < \frac{n}{2}$ is symmetric.

Our proof of the mechanism's welfare-optimality is based on its intuition.

Lemma 5.7. ***2-IFS Randomized mechanism*** *is optimal for utilitarian welfare amongst all randomized mechanisms satisfying 2-IFS in expectation.*

We now prove, using an algebraic approach, a tight, constant approximation ratio for this mechanism.

Theorem 5.8. ***2-IFS Randomized mechanism*** *has an approximation ratio for utilitarian welfare of* $\frac{12}{11} \approx 1.091$.

This implies the following price of fairness result for 2-IFS.

Corollary 5.9. *In the randomized setting, the price of fairness of 2-IFS for UW is* $\frac{12}{11} \approx 1.091$.

5.4 2-UFS

We now move to analyze the axiom of 2-UFS in the context of utilitarian welfare. As in the previous subsection, we begin by describing a randomized mechanism which maximizes the utilitarian welfare subject to the 2-UFS constraint:

2-UFS Randomized mechanism

- Order the m **unique** agent locations such that $x_1 < x_2 < \cdots < x_m$.
- Let $S_1, \ldots, S_m$ denote the groups of agents at the m unique agent locations.
- If $\sum_{i=1}^{m} |S_i| x_i = \frac{n}{2}$, place the facility at 0 with probability $\frac{1}{2}$ and at 1 with probability $\frac{1}{2}$.
- If $\sum_{i=1}^{m} |S_i| x_i > \frac{n}{2}$,
  - Let k denote the index of the largest unique agent location satisfying $x_k < \frac{1}{2}$.
  - For i in $\{1, \ldots, k\}$, set $\alpha_i = \frac{|S_i| - 2nx_i}{2n(1-2x_i)}$.
  - Letting $\alpha = \max\{\alpha_1, \ldots, \alpha_k\}$, place the facility at 0 with probability $1 - \alpha$ and at 1 with probability α.
- If $\sum_{i=1}^{m} |S_i| x_i < \frac{n}{2}$,
  - Let k denote the index of the smallest unique agent location satisfying $x_k > \frac{1}{2}$.
  - For i in $\{k, \ldots, m\}$, set $\alpha_i = \frac{|S_i| - 2nx_i}{2n(1-2x_i)}$.
  - Letting $\alpha = \min\{\alpha_k, \ldots, \alpha_m\}$, place the facility at 0 with probability $1 - \alpha$ and at 1 with probability α.

This mechanism is similar to the 2-IFS Randomized mechanism, but we must now iterate through the groups of agents to find the optimal value of α that guarantees 2-UFS for all agents. Specifically, if $\sum_{i=1}^{m} |S_i| x_i > \frac{n}{2}$, then α_i denotes the smallest probability weight on location 1 such that 2-UFS is achieved for S_i. Hence by setting α to be the largest α_i, we achieve 2-UFS for all agents.

Again, our proof of this mechanism's optimality is based on the aforementioned intuition.

Lemma 5.10. ***2-UFS Randomized mechanism*** *is optimal for utilitarian welfare amongst all randomized mechanisms satisfying 2-UFS in expectation.*

Surprisingly, imposing the stronger fairness axiom of 2-UFS as opposed to 2-IFS has a minimal effect on the welfare-optimal mechanism's approximation ratio. Again, the approximation ratio is computed algebraically.

Theorem 5.11. ***2-UFS Randomized mechanism*** *has an approximation ratio of* $\frac{2}{7}(1 + 2\sqrt{2}) \approx 1.09384$.

From Theorem 5.11, we have the following corollary.

Corollary 5.12. *In the randomized setting, the price of fairness of 2-UFS for UW is* $\frac{2}{7}(1 + 2\sqrt{2}) \approx 1.09384$.

6 EXTENSION 1: PROPORTIONAL FAIRNESS

In our analyses of price of fairness and randomized mechanisms, we have considered 2-IFS and 2-UFS, which give minimum distance guarantees for individual agents and groups of agents at the same location, respectively. One downside of the 2-UFS definition is that agents located near each other but not at the same location are considered to be in separate groups. An axiom which accounts for groups of agents located relatively close to each other is Proportional Fairness (PF), from [4]. As with IFS and UFS, a PF solution may not exist so we define approximate $\alpha-$PF as follows:

Definition 6.1 (α-PF). Given a profile of locations $\boldsymbol{x}$, a facility location y satisfies *α-PF* if for any set of agents S within range $r := \max_{i\in S}\{x_i\} - \min_{i\in S}\{x_i\}$,

$$u(y, x_i) \geq \frac{1}{\alpha}(|S|/(n)) - r \quad \forall i \in S.$$

Note that $\alpha-$PF implies $\alpha-$UFS, and therefore also implies $\alpha-$IFS. However, $\alpha-$UFS does not imply $\alpha-$PF, hence $\alpha-$PF is a stronger notion than $\alpha-$UFS.

LEMMA 6.2. *For $\alpha = 2$, there exists an $\alpha-$UFS facility location y that does not satisfy $\alpha-$PF.*

We now show that a 2-PF solution always exists. The proof uses induction on the number of groups of agents at the same location.

THEOREM 6.3. *A 2-PF solution always exists.*

From Theorem 3.4, we see that 2-PF is the optimal approximation of PF for the obnoxious facility location problem.

7 EXTENSION 2: HYBRID MODEL

In the hybrid model, agents either want to be located close to the facility (as in the FLP), or wish to be located far away from the facility (as in our OFLP model). Such a model has several real-world applications such as the placement of schools or religious places of worship; families with children or religious people would want to live near the facility for convenience, whilst others would want to be far from the facility due to the increased noise and traffic. In our model, we say an agent is type C if it is a classic agent and prefers to be closer to the facility, and an agent is type O if it is an obnoxious agent and prefers to be further away from the facility.[4] We denote the set of classic agents as N_C and the set of obnoxious agents as N_O.

A type C agent has utility $u(y, x_i) = 1 - d(y, x_i)$ and a type O agent has utility $u(y, x_i) = d(y, x_i)$.[5]

When defining IFS and UFS in the hybrid model, we use definitions consistent with [4] and this paper. Our definition of Hybrid-Individual Fair Share (H-IFS) provides an appropriate distance guarantee for each agent.

Definition 7.1 (Hybrid-Individual Fair Share (H-IFS)). Given a profile of locations x, a facility location y satisfies Hybrid-Individual Fair Share (H-IFS) if for all $i \in N_C$,

$$u(y, x_i) \geq \frac{1}{n} \quad \text{or, equivalently,} \quad d(y, x_i) \leq 1 - \frac{1}{n},$$

and for all $i \in N_O$,

$$u(y, x_i) \geq \frac{1}{2n} \quad \text{or, equivalently,} \quad d(y, x_i) \geq \frac{1}{2n}.$$

When defining UFS, we aim to capture proportional fairness guarantees for subsets of agents of the same type at the same location. Consider every subset $S \subseteq N$ of agents at the same location, where $S = S_C \cup S_O$. S_C denotes the agents of S that are of type C, and S_O denotes the agents of S that are of type O.

[4]Our model is based on the model presented by Feigenbaum and Sethuraman [20].
[5]This choice of utility function is adapted from [4, 20]. We refer the reader to those papers for a justification of the utility model.

Definition 7.2 (Hybrid-Unanimous Fair Share (H-UFS)). Given a profile of locations x such that a subset of $S_j \subseteq N$ agents[6] share the same type and location, a facility location y satisfies Hybrid-Unanimous Fair Share (H-UFS) if for all $i \in S_C$,

$$u(y, x_i) \geq \frac{|S_C|}{n} \quad \text{or, equivalently,} \quad d(y, x_i) \leq 1 - \frac{|S_C|}{n},$$

and for all $i \in S_O$,

$$u(y, x_i) \geq \frac{|S_O|}{2n} \quad \text{or, equivalently,} \quad d(y, x_i) \geq \frac{|S_O|}{2n}.$$

Example 7.3. Suppose there are $n - k$ type C agents and k type O agents, all at the same location. The facility needs to be between $\frac{k}{2n}$ and $\frac{k}{n}$ distance from the group.

Although our definitions have a discrepancy in utility functions between the classic and obnoxious agents, we have specified them to be consistent with related literature and to be the optimal bounds such that a solution is guaranteed to exist. Furthermore, existence of a H-UFS solution under our definition implies existence of a solution under a weaker definition where a set S_C of classic agents at the same location instead have a utility guarantee of $\frac{|S_C|}{2n}$.

THEOREM 7.4. *Under the hybrid model, a H-UFS solution always exists.*

8 DISCUSSION

In this paper we have formulated proportional fairness axioms for the obnoxious facility location problem, and given welfare-optimal deterministic and randomized mechanisms satisfying these axioms. In both the deterministic and randomized setting, we prove tight price of fairness bounds for 2-IFS and 2-UFS, for the objectives of utilitarian and egalitarian welfare. These correspond to the approximation ratios of the respective welfare-optimal mechanisms. For the deterministic utilitarian welfare-optimal mechanisms, we prove existence of pure ϵ-Nash equilibria, linear ϵ-price of anarchy bounds, and constant ϵ-price of stability bounds. We also give a randomized, strategyproof mechanism satisfying 2-UFS with a constant utilitarian approximation ratio.

Future directions to this work could stem from our proposed extension of 2-PF, as well as the extension of our proportional fairness axioms to the *hybrid facility location model*. Further research could compute the price of fairness for these two extensions, and the prices of anarchy and stability for the welfare-optimal mechanisms in these settings.

Further extensions to the price of fairness results could involve different objective and utility functions. It is also worth analyzing the Nash equilibria of the randomized utilitarian welfare-optimal mechanisms, as they are not strategyproof in expectation. Although our proportional fairness axioms are incompatible with strategyproofness in the deterministic setting, we may consider weaker notions of strategyproofness which may be compatible with our fairness properties.

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[6]$j \in \{C, O\}$

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