Positive Intra-Group Externalities in Facility Location

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ABSTRACT

We study facility location games with multiple groups in one dimension where an agent's utility is not only decided by the distance from the facility but also by their group members. The positive effect of the interactions within a group is captured by *positive intra-group externalities*. Our goal is to design a mechanism that is non-manipulable and respects unanimity while (approximately) optimizing an objective function. We consider three types of manipulation, misreporting only the location, misreporting only the group membership, and misreporting both, under two social objectives, the social utility and the minimum utility. For both objectives, we achieve nearly tight bounds by either designing new mechanisms or extending the existing mechanisms in terms of the first two types of manipulation. As to the negative result, we show that strategyproofness and unanimity are incompatible when each agent can misreport both the location and the group membership, which is independent of the objective functions.

KEYWORDS

Facility location games; Algorithmic mechanism design; Intra-group externalities; Social choice

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1 INTRODUCTION

Ubiquitous economic and social interpersonal situations involve the concept of group and witness the interaction between members within the same group [5, 17]. Consider an urban planner that is planning to build a plaza to facilitate three surrounding neighborhoods where a larger portion of the commodity price will be refunded if there is a higher share of successful recommendations of the commodity within the neighborhood. To maximize the overall well-being of residents in the three neighborhoods, the planner needs to maximize the sum of each resident's benefit received from the commodity itself as well as the added benefit of sharing via intra-neighbor messaging. Moreover, sometimes instead of optimizing the overall welfare of agents, the planner also needs to consider maximizing the minimum benefit among all residences in terms of the egalitarian welfare. In addition, the levels of internal interactions may vary between groups in a collective scenario. Consider the different roles that musicians play in the symphony. Solo and ensemble sections stay united in harmony with various musical effects: only the soloists' own playing is audible to them while in the ensemble, both the player and others in the same section are audible, with the cooperation degree depending on which instrument they are playing. For instance, members in Percussion play distinct percussions individually at the same time and collaborate less than in Strings, where members play together with the aim of establishing diatonic harmony. To have a better radio reception, the musicians in the ensemble want to make the position of the microphone achieve an optimal balance corresponding to the type of instrument between themselves and the other section members.

A simple way to model the above scenarios could be assuming that there are some groups of agents and each group has an intra-group externality, i.e., agents have positive effects on their group members where the level of the effect is decided by a group externality coefficient. The social designer needs to maximize the overall welfare or the underprivileged agent's welfare. Then there comes a natural question:

How should a planner make a decision, given the locations as well as group memberships reported by the agents in the spirit of maximizing the utilitarian or egalitarian welfare, meanwhile eliciting truthful reporting?

The above question shares crucial components with a widely-researched setting in algorithmic game theory: facility location games. In this problem, each agent can be regarded as a point along a real interval, and the goal is typically to select the location of the facility to serve those agents. In facility location games, Li et al. [11] first took the interaction between individual agents into consideration. Zhou [18] considered the setting where an agent can decide whether to participate in a group activity, and those who join the activity will have interaction with other group participants, while the utility of others depends only on the distance to the facility. However, one can see that such a binary division for agents is limited since there are many scenarios that contain multiple groups with potentially different coefficients of intra-group positive interaction, such as the plaza positioning with more than two neighborhoods and the microphone placement with diverse musical sections in a symphony. Moreover, many common group division criteria such as race, age, and political stance require us to consider multiple groups. Hence, a more general setting is required to capture more real-life applications.

1.1 Our Contribution

We study the facility location games with multiple groups where the utility of each agent is not only decided by the distance from the facility but also by the agents in the same group. We consider two

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utility objectives and three types of misreport. Table 1 summarizes our results in this paper. Specifically,

- Results for maximizing the social utility:
  - When we restrict the power of misreport to location only, we extend the mechanism (GI-M) proposed by Zhou [18] from two groups to m groups and prove the approximation ratio of $\Omega(m)$. Then we propose a new mechanism (GA-M) with an approximation ratio of 3, where the new mechanism can be viewed as putting the facility at the weighted group median, and the weight is decided by both the group size and the group externalities. Here we highlight our work for the final mechanism (RGA-M), which improves the upper bound significantly (from 3 to $\sqrt{3} \approx 1.732$). Based on GA-M, RGA-M leverages regularization, a well-known methodology in many other fields such as machine learning, to avoid the situation that when agents are concentrated at both boundaries of the interval and there is no significant difference between those two parts, GA-M may locate the facility at one of the ends. We also give a lower bound of 1.5 in this setting.
  - When we restrict the power of misreport to group membership only, we present a unanimous, strategyproof mechanism (Middle-M) which puts the facility at the middle point between the leftmost agent location and the rightmost agent location and exhibits an approximation ratio of 2. Through an analysis of a particular generic profile set, we also prove a lower bound of 2, implying that Middle-M is the best possible.
  - When we do not restrict the power of misreport, we prove that strategyproofness and unanimity are incompatible by constructing five interlinked profile sets in an intricate way to make a contraction.
- Results for maximizing the minimum utility:
  - When we restrict the power of misreport to location only, we first extend the mechanism (Marginal-M) proposed by Zhou [18] from two groups to m groups and prove an approximation ratio of 2. We highlight our work for the proposed mechanism (Ternary-M), which puts emphasis on the agent with the highest possibility to achieve minimum utility and leverages some fixed points to avoid cognitive bias. Ternary-M has a precise case division and improves the approximation ratio to $\frac{\sqrt{6}}{3} + 1 \approx 1.816$. We also give a lower bound of $\frac{5}{3} \approx 1.667$, implying that Ternary-M is nearly optimal.
  - When we restrict the power of misreport to group membership only, we reuse Middle-M and prove an approximation ratio of 2. We also give the lower bound of 2, implying Middle-M is also the best possible in this setting.
  - When we do not restrict the power of misreport, the negative result shown in the social utility also works here since it is independent of objectives.

In short, due to the nature of the *utility* function, the upper bound of 2 is reasonably easy to be guaranteed by simply locating the facility at the middle point of the interval for both objectives. However, studying beyond 2-approximation faces more difficulties. We overcome the obstacles by 1) proposing sophisticated mechanisms with approximation ratio better than 2 (misreporting only the location) and 2) proving tight lower bounds of 2 (misreporting only the group membership) to show no better mechanism can be guaranteed. We emphasize that both directions are challenging, which requires introducing complex situation analysis to design mechanisms, employing the distinct profile set constructions to tighten the bounds, and proposing fundamental lemmata to help bridge the gap between *group deviations* and *individual deviation* in proofs.

Here we also show the results proposed in Zhou [18] (Table 2). We can observe that most of our results are better than theirs (with smaller upper bounds and larger lower bounds) though we consider a more general setting.

Objective	Restrict the Power of Misreport to: Location only	Group only	Both
Social Utility	UB: 1.732† (RGA-M) LB: 1.5	UB: 2 (MIDDLE-M) LB: 2	∞
Min Utility	UB: 1.816$^\clubsuit$ (TERNARY-M) LB: 1.667$^\diamond$	UB: 2 (MIDDLE-M) LB: 2	∞

Table 1: A summary of our results. The exact bound of $\dagger$: $\sqrt{3}$, $\clubsuit$: $\frac{\sqrt{6}}{3}+1$, $\diamond$: $\frac{5}{3}$. RGA-M, MIDDLE-M, and TERNARY-M are the abbreviations of the Regularized Group-Accumulating Mechanism, the Middle-Point Mechanism, and the Ternary Mechanism respectively.

Objective	Restrict the Power of Misreport to: Location only	Group only	Both
Social Utility	UB: 3 LB: 1.2	UB: 2 LB: 2	$\Omega(n^2)$
Min Utility	UB: 2 LB: 1.5	UB: 2 LB: 1.5	∞

Table 2: A summary of results in Zhou [18].

1.2 Related Work

Procaccia and Tennenholtz [14] first studied approximate mechanism design in facility location games, where all agents strive for securing the facility in the closest proximity to themselves. Li et al. [11] studied facility location games with externalities where agents have effects on each other. However, as they did not consider groups, they only consider the location misreporting. Moreover, most of their results are negative so it is still a big challenge to design mechanisms with group externalities. Zhou [18] studied facility location games with group externalities where there is one group activity with particular internal connections accessible to agents, and two types of agents are separated based on whether or not they participate in the group activity, which could be regarded as a special case of multiple groups. Our model can be seen as the classic facility location game with a novel agent preference structure. There are some other works with different agent preferences, such as dual preferences [6, 21], and fractional preferences [9].

There are also some works studying facility location games with group notions. Filos-Ratsikas and Voudouris [8] and Filos-Ratsikas et al. [7] introduced a setting in which the facility location is selected as part of a distributed process: first, agents within groups (or districts) decide on a representative location and then the mechanism, oblivious to the actual locations of the agents, decides on a location from the set of representatives. Zhou et al. [20] studied mechanism design in facility location games with group-fair objectives. Zhou et al. [19] studied facility location games with altruistic agents, in which one of the agent cost functions is defined as the sum of the distance in a group, which can be viewed as the special case where all group externalities are equal to 1. On the other hand, they considered overlapping groups where each agent can belong to multiple groups and studied both the classical and the group-fair social objectives. Hence, our work and theirs do not degenerate to each other.

More works in facility location games can be found in a recent survey [4]. Moreover, many other fields in both economy and computer science study externalities, such as cake cutting [3, 12], fair allocation [10, 13], and matching [15, 16].

1.3 Paper Organization

The remainder of the paper is organized as follows. We first define the formulation of the problem in Section 2. Then, we study the case for the objective of maximizing the social utility and maximizing the minimum utility in Section 3 and Section 4 respectively. Finally, we conclude our work and discuss the open questions in Section 5. Due to the space limit, some proofs are omitted.

2 PRELIMINARIES

Let $N = \{1, 2, \cdots, n\}$ be a set of agents where all agents are located on a normalized closed interval $I = [0, 1]$. Each agent $i \in N$ has a location x_i and belongs to a group $g_i \in [m]$. A collection of agents with $g_i = j$ is denoted as G_j, so we have $\bigcup_{j\in[m]} G_j = N$ and $G_{j1} \cap G_{j2} = \emptyset$ for all $j1 \neq j2$. Let $med(S)$ be the agent with the median location in the set $S \subseteq N$, tie-breaking by selecting the leftmost. We denote the profile of agent i as $r_i = (x_i, g_i)$ and denote the profile set as $\mathbf{r} = \{r_1, \cdots, r_n\}$. A mechanism is a function f which maps profile set $\mathbf{r}$ to a facility location $y \in I$. We take $d(a, b) = |a - b|$ to represent the distance between a and b.

The externality factor within the group G_j is denoted by $\alpha_j \in [0, 1]$. To perfectly model the benefits that an individual derives from the facility and the interaction within a group, we utilize *utility* function. The cost function corresponds to the loss incurred by group members, which cannot model our setting precisely.

We take an agent i's individual value to be $v(y, x_i) = 1 - d(y, x_i)$. The utility of agent i is defined as

$$u_i(y, \mathbf{r}) = v(y, x_i) + \alpha_{g_i} \sum_{k\in G_{g_i};k\neq i} v(y, x_k),$$

implying that the utility of an agent will be decided by not only using the facility (the first term), but also interacting with their group members (the second term).

Each agent is rational and tries to maximize their utility. Our goal is to design mechanisms that are non-manipulable and respect unanimity while (approximately) optimizing an objective function. Here we use ex-post strategyproofness to represent non-manipulability, which has been widely considered in mechanism design [2] and facility location games [1].

Definition 1. *A mechanism f is **ex-post strategyproof** if an agent can never benefit by reporting a false profile, given the true profiles of the other agents. More formally, given any profile set $\mathbf{r} = \{r_1, ..., r_n\}$ and any profile set $\mathbf{r}' = \{r'_1, ..., r'_n\}$ reported by n rational agents. We have $u_i(f(\mathbf{r}), \mathbf{r}) \geq u_i(f(r'_i, \mathbf{r}_{-i}), \mathbf{r})$ where $\mathbf{r}_{-i}$ is a collection of true profiles of n agents except agent i.*

To simplify the description, we use **strategyproof (SP)** instead of **ex-post strategyproof** in the remaining part of this paper. We can further discuss the three cases shown below.

(1) Restrict the power of misreport to locations only.
(2) Restrict the power of misreport to group memberships only.
(3) Do not restrict the power of misreport.

Definition 2. *A mechanism f is **unanimous** if for every profile $\mathbf{r}$ with location $x_1 = x_2 = ... = x_n = x$, $f(\mathbf{r}) = x$.*

We consider two utility-oriented objectives, maximizing the social utility, which is defined as

$$su(y, \mathbf{r}) = \sum_{i\in N} u_i(y, \mathbf{r}),$$

and maximizing the minimum utility, which is defined as

$$mu(y, \mathbf{r}) = \min_{i\in N} u_i(y, \mathbf{r}).$$

We measure the performance of f by approximation ratio.

Definition 3. *We say a mechanism f has an **approximation ratio** of ρ for a certain objective if there exists a number ρ such that for any profile set $\mathbf{r}$, the objective value achieved by the optimal location is within ρ times the objective value achieved by f.*

When there are two groups with $\alpha_1 = 0$ and $\alpha_2 = \alpha$, our setting can be viewed as the setting where each agent is either in the group or alone, which coincides with Zhou [18]. We will use this observation to derive one of our negative results.

Corollary 1. *Neither putting the facility at the median agent location nor putting the facility at the leftmost agent location is strategyproof.*

Moreover, there are two useful lemmata to assist in proving most of lower bounds in Section 3 and Section 4.

Lemma 1. *If a mechanism is strategyproof, the agents with $\alpha = 0$ at the same location cannot benefit even if they misreport their locations or group memberships simultaneously.*

Proof. We denote the set of agents with $\alpha = 0$ at the same location as $S \subseteq N$ with true profiles $r_1, ...r_s$. Let $\mathbf{r}_{-S}$ denote a collection of true profiles of n agents except for agents in S. Consider a series of profile sets $\mathbf{r}^i (0 \leq i \leq s)$ where $s-i$ agents with $r_{i+1}, ..., r_s$ in S misreport their locations or group memberships simultaneously. We have

$$\mathbf{r}^i = \{r_1, ..., r_i, r'_{i+1}, ..., r'_s\} \cup \mathbf{r}_{-S},$$
$$\mathbf{r}^{i-1} = \{r_1, ..., r_{i-1}, r'_i, ..., r'_s\} \cup \mathbf{r}_{-S}.$$

Therefore $\mathbf{r}^{i-1}$ could be regarded as the agent i in $\mathbf{r}^i$ misreporting the profile to r'_i. From the definition of strategyproofness, we have $u_i(f(\mathbf{r}^i), \mathbf{r}^i) \geq u_i(f(\mathbf{r}^{i-1}), \mathbf{r}^i)$. By integrating all the inequalities achieved by $i \in [0, s]$, we further have

$$u_s(f(\mathbf{r}^s), \mathbf{r}^s) \geq u_s(f(\mathbf{r}^{s-1}), \mathbf{r}^s) = u_{s-1}(f(\mathbf{r}^{s-1}), \mathbf{r}^{s-1})$$
$$\geq u_{s-1}(f(\mathbf{r}^{s-2}), \mathbf{r}^{s-1}) = \ldots = u_1(f(\mathbf{r}^0), \mathbf{r}^0).$$

Combining with $u_s(f(\mathbf{r}^s),\mathbf{r}^s) = \ldots = u_1(f(\mathbf{r}^s),\mathbf{r}^s)$ and $u_s(f(\mathbf{r}^0),\mathbf{r}^0) = \ldots = u_1(f(\mathbf{r}^0),\mathbf{r}^0)$, we have $u_i(f(\mathbf{r}^s),\mathbf{r}^s) \geq u_i(f(\mathbf{r}^0),\mathbf{r}^0)$ for each $i \in [0,s]$, which completes the proof. □

Lemma 2. *If a mechanism is strategyproof, the agents in the same group with $\alpha \geq 0.5$ cannot benefit even if they*

(1) *misreport their locations without crossing their group median location simultaneously, or*
(2) *are at the same location and misreport group memberships simultaneously*

Proof. We denote the set of agents within the same group G_j with $\alpha_j \geq 0.5$ as $S \subseteq G_j$ with true profiles $r_1, \ldots r_s$. Let $\mathbf{r}_{-s}$ denote a collection of true profiles of n agents except for agents in S. Then we consider a series of profile sets which is defined similarly to Lemma 1, where $\mathbf{r}^{i-1}$ could be regarded as the agent i in $\mathbf{r}^i$ misreporting profile to r_i'. Firstly, we observe that all agents in G_j can obtain their maximum utility if the facility is located at the group median location $x_{med(G_j)}$. Since both conditions included in Lemma 2 imply that the group median location will not move during the misreporting, our goal is to prove $d(f(\mathbf{r}^s), x_{med(G_j)}) \leq d(f(\mathbf{r}^0), x_{med(G_j)})$. Guaranteed by strategyproofness we first have $u_i(f(\mathbf{r}^i),\mathbf{r}^i) \geq u_i(f(\mathbf{r}^{i-1}),\mathbf{r}^i)$, implying $d(f(\mathbf{r}^i), x_{med(G_j)}) \leq d(f(\mathbf{r}^{i-1}), x_{med(G_j)})$. By integrating all the inequalities achieved by $i \in [0,s]$, we finally have $d(f(\mathbf{r}^s), x_{med(G_j)}) \leq d(f(\mathbf{r}^0), x_{med(G_j)})$, which completes the proof. □

In the following sections, we first investigate maximizing the social utility, and then focus on maximizing the minimum utility.

3 SOCIAL UTILITY

In this section, we first study the setting where each agent can only misreport their location, then we look at the setting where each agent can only misreport their group membership. Finally, we derive a negative result when each agent can misreport both.

3.1 Misreport only the location

In this subsection, we adopt the mechanism proposed by Zhou [18] in a direct way to our setting, and analyze its weakness. To overcome that weakness, we propose a new mechanism with a significantly lower approximation ratio. However, that new mechanism is also flawed. Finally, we refine that new mechanism which gives the nearly best possible performance compared with the lower bound.

First, let us revisit the mechanism proposed by Zhou [18]. As we mentioned, their setting could be regarded as two groups with group externalities $\alpha_1 = 0$ and $\alpha_2 = \alpha$, respectively. They put the facility at $x_{med(G_2)}$ if $\alpha|G_2|^2 - \alpha|G_2| + |G_2| \geq |G_1|$; otherwise, they put the facility at $x_{med(G_1)}$. A simple way to adapt it to our setting is taking $\alpha_j|G_j|^2 - \alpha_j|G_j| + |G_j|$ as the parameter of each group j, and put the facility at the median agent of the group with the largest parameter.

Group-Imposing Mechanism (GI-M). *Given any profile set $\mathbf{r}$, put the facility at $y = x_{med(G_{j^*})}$ where $j^* = \arg\max_{j \in [m]}\{\alpha_j|G_j|^2 - \alpha_j|G_j| + |G_j|\}$, tie-breaking by selecting the smallest j.*

Here we show how *bad* GI-M is rather than showing its exact approximation ratio, but this is sufficient to show how far apart GI-M is from Group-Accumulating Mechanism (GA-M) and Regularized Group-Accumulating Mechanism (RGA-M) to be proposed.

Proposition 1. *Group-Imposing Mechanism has an approximation ratio of at least $\Omega(m)$ when group memberships are public.*

One may wonder whether another tie-breaking rule can improve GI-M. In fact, we can slightly decrease the externality factors' values of those groups whose agents are at 1 to get the same result, which is independent of tie-breaking rules. Moreover, a constant approximation ratio might be guaranteed if there is always a *powerful* group, i.e., a group covering a majority of agents with a larger externality factor.

Although GI-M cannot guarantee a constant approximation ratio, we can also leverage the idea from it. As we know, social utility is a kind of utilitarian welfare function that focuses on the overall welfare. It would be better to put the facility somewhere in the middle. Thus, we cannot treat the parameters of each group independently. Instead, we can look at those groups from the left to the right, and accumulate these parameters until they reach a certain value.

Group-Accumulating Mechanism (GA-M). *Given any profile set $\mathbf{r}$, without loss of generality we assume that $x_{med(G_1)} \leq x_{med(G_2)} \leq \cdots \leq x_{med(G_m)}$. Put the facility at $y = x_{med(G_{j^*})}$ such that j^* is the smallest index that satisfies*

$$\sum_{j \in [j^*]} (\alpha_j|G_j|^2 - \alpha_j|G_j| + |G_j|) \geq \frac{1}{2} \sum_{j \in [m]} (\alpha_j|G_j|^2 - \alpha_j|G_j| + |G_j|).$$

Before showing the approximation ratio, we first propose a lemma.

Lemma 3. *Given any profile set $\mathbf{r}$, the social utility achieved by y is equivalent to $\sum_{i \in N}(1 + \alpha(|G_{g_i}| - 1))v(y, x_i)$.*

Proposition 2. *Group-Accumulating Mechanism is unanimous, strategyproof, and has an approximation ratio of 3 when group memberships are public.*

Compared with GI-M, GA-M achieves a constant approximation ratio of 3. However, GA-M is also flawed since putting the facility at $\frac{1}{2}$ can guarantee an approximation ratio of 2 (but it does not satisfy unanimity). So far, the outputs of our mechanisms are restricted to the agent locations. It will significantly help improve the approximate ratio if we can leverage the non-agent locations, especially when the agents are concentrated at both boundaries of the interval, and there is no significant difference between those two parts, where putting the facility at one of the ends will incur a lousy performance easily. Taking into account the above situation, and inspired by the regularization term, a well-known technique to prevent overfitting in machine learning and many other fields, we introduce a weighted fixed point to prevent the mechanism from outputting an extreme solution when the above situation occurs.

Regularized Group-Accumulating Mechanism (RGA-M). *Given any profile set $\mathbf{r}$, without loss of generality we assume that $x_{med(G_1)} \leq x_{med(G_2)} \leq \cdots \leq x_{med(G_m)}$. Let $C(j) = |G_j|(1 + \alpha_j(|G_j| - 1))$ and $K(j) = C(1) + \cdots + C(j)$. We define $j^{mid} =$*

$\arg\max_{j\in[m]}\{x_{med(G_j)} \leq \frac{1}{2}\}$. *Put the facility at*

$$\begin{cases} x_{med(G_{j^*})} \text{ where } j^*= \arg\min_{j\in[m]} \{K(j) \geq \frac{1}{2}((1+\lambda)K(m))\} \\ ---if\, K(j^{mid}) > \frac{1}{2}((1+\lambda)K(m)); \\ \frac{1}{2} --if\, \frac{1}{2}((1-\lambda)K(m)) \leq K(j^{mid}) \leq \frac{1}{2}((1+\lambda)K(m)); \\ x_{med(G_{j^*})} \text{ where } j^*= \arg\min_{j\in[m]} \{K(j) \geq \frac{1}{2}((1-\lambda)K(m))\} \\ ---otherwise. \end{cases}$$

Here we briefly discuss how λ affects the output of RGA-M. If $\lambda = 0$, RGA-M is the same as GA-M. When $\lambda > 1$, RGA-M always outputs the facility at $\frac{1}{2}$, implying we need to restrict λ to $[0, 1)$ to satisfy unanimity. Different from GA-M, if y is on the left of $\frac{1}{2}$, implying the majority of agents are on the left since $K(j^*)$ is at least $\frac{1}{2}((1+\lambda)K(m))$, which is larger than $\frac{1}{2}K(m)$. The case where y is on the right of $\frac{1}{2}$ is similar, implying the majority of agents are on the right. Moreover, RGA-M outputs $\frac{1}{2}$ when the agents on both sides are more balanced.

Theorem 1. *Regularized Group-Accumulating Mechanism is unanimous, strategyproof when $\lambda \in [0, 1)$, and has an approximation ratio of $\sqrt{3} \approx 1.732$ for maximizing the social utility when group memberships are public, by setting $\lambda = 2\sqrt{3} - 3 \approx 0.464$.*

Proof. When all agents are at the same location, $K(j^{mid})$ is equal to either $K(m) > \frac{1}{2}((1+\lambda)K(m))$ or $0 < \frac{1}{2}((1-\lambda)K(m))$, RGA-M outputs the facility at the agent location, satisfying unanimity.

For each agent in G_{j^*}, the only way to change the facility location is to misreport their location to the opposite side of $x_{med(G_{j^*})}$, which eventually leads the facility to be farther away from both their true location and the original group median. Their utility decreases after misreporting therefore the agent in G_{j^*} has no incentive to misreport the location. As for each agent in $G_j (j \neq j^*)$, misreporting to make their group j^* is the only way to change the facility location, which puts this agent into a situation similar to the case of G_{j^*}, i.e., moving the facility to be farther away from both their location and their original group median.

Now, we show the approximation ratio of RGA-M. Given any profile set $\mathbf{r}$, we denote the optimal solution for maximizing the social utility as y^* and the facility location output by mechanism RGA-M as y. Without loss of generality, we assume that $y \leq y^*$. We first move all the agents in (y, y^*) to y^* since the approximation ratio will not decrease. Let K_1 be the set of the agents within $[y^*, 1]$, K_2 be the set of agents within $[0, y]$ after the movement. Let d denote the distance between y and y^*. We further divide the scenario into two cases based on the relative locations of y and $\frac{1}{2}$.

Case 1: We first consider the case where $y > \frac{1}{2}$, as shown in Figure 1.

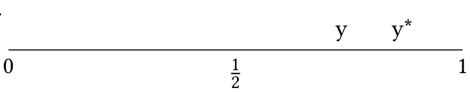

Figure 1: $y > \frac{1}{2}$**.**

Combined with Lemma 3, we have

$$su(y^*,\mathbf{r}) - su(y,\mathbf{r}) = (\sum_{i\in K_1}(1+\alpha_{g_i}(|G_{g_i}|-1)) - \sum_{i\in K_2}(1+\alpha_{g_i}(|G_{g_i}|-1)))d$$
$$\leq (\sum_{j^*+1\leq j\leq m} C(j) + \frac{1}{2}\sum_{1\leq j\leq j^*} C(j))d - \frac{1}{2}\sum_{1\leq j\leq j^*} C(j)d.$$

From the definition of RGA-M, we further have

$$su(y^*,\mathbf{r}) - su(y,\mathbf{r}) \leq \sum_{j^*+1\leq j\leq m} C(j)d$$
$$= (K(m) - K(j^*))d \leq \frac{1}{2}((1+\lambda)K(m))d.$$

Hence, we have the approximation ratio

$$\rho = \frac{su(y^*,\mathbf{r})}{su(y,\mathbf{r})} \leq 1 + \frac{((1+\lambda)K(m))d}{2su(y,\mathbf{r})}.$$

When there are half of agents in $G_j (j \in [1, j^*])$ at 0, half of agents in $G_j (j \in [1, j^*])$ at 1, and all of the agents in $G_j (j \in [j^*+1, m])$ at 1, the minimum $su(y, \mathbf{r})$ is achieved,

$$su(y,\mathbf{r}) \geq \frac{1}{2}\sum_{1\leq j\leq j*} C(j)(1-y) + \frac{1}{2}\sum_{1\leq j\leq j*} C(j)y + \sum_{j*+1\leq i\leq m} C(j)y$$
$$= \frac{1}{2}\sum_{1\leq i\leq j*} C(j) + \sum_{j*+1\leq i\leq m} C(j)y.$$

Since $y \geq \frac{1}{2}$ and $d \leq \frac{1}{2}$, we have the approximation ratio

$$\rho \leq 1 + \frac{((1+\lambda)K(m))d}{2(\frac{1}{2}\sum_{1\leq j\leq j^*} C(j) + \sum_{j^*+1\leq j\leq m} C(j)y)}$$
$$\leq 1 + \frac{\frac{1}{2}((1+\lambda)K(m))}{\sum_{1\leq j\leq j^*} C(j) + \sum_{j^*+1\leq j\leq m} C(j)} \leq 1 + \frac{1+\lambda}{2}.$$

Case 2: Now we consider the second case where $y \leq \frac{1}{2}$, as shown in Figure 2.

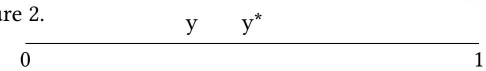

Figure 2: $y \leq \frac{1}{2}$**.**

Similar to the first case, we have

$$su(y^*,\mathbf{r}) - su(y,\mathbf{r}) \leq \sum_{j^*+1\leq j\leq m} C(j)d$$
$$= (K(m) - K(j^*))d \leq \frac{1}{2}((1-\lambda)K(m))d,$$

and the approximation ratio

$$\rho = \frac{su(y^*,\mathbf{r})}{su(y^*,\mathbf{r})} \leq 1 + \frac{((1-\lambda)K(m))d}{2su(y,\mathbf{r})}.$$

When there are half of agents in $G_j (j \in [1, j^*])$ at 0, half of agents in $G_j (j \in [1, j^*])$ at 1, and all of the agents in $G_j (j \in [j^*+1, m])$ at 1, the minimum $su(y, \mathbf{r})$ is achieved,

$$su(y,\mathbf{r}) \geq \frac{1}{2}\sum_{1\leq j\leq j*} C(j)(1-y) + \frac{1}{2}\sum_{1\leq j\leq j*} C(j)y + \sum_{j*+1\leq i\leq m} C(j)y$$
$$= \frac{1}{2}\sum_{1\leq i\leq j*} C(j) + \sum_{j*+1\leq i\leq m} C(j)y.$$

Since $y \geq 0$ and $d \leq 1$, we have the approximation ratio

$$\rho \leq 1 + \frac{((1-\lambda)K(m))d}{2(\frac{1}{2}\sum_{1\leq j\leq j^*} C(j) + \sum_{j^*+1\leq j\leq m} C(j)y)}$$
$$\leq 1 + \frac{(1-\lambda)K(m)}{\sum_{1\leq j\leq j^*} C(j)} \leq 1 + \frac{2-2\lambda}{1+\lambda}.$$

Finally, we have the approximation ratio

$$\rho \leq \max\{1 + \frac{1+\lambda}{2}, 1 + \frac{2-2\lambda}{1+\lambda}\},$$

which is at most $\sqrt{3} \approx 1.732$ by setting $\lambda = 2\sqrt{3} - 3$. □

Then we give the lower bound of this setting.

Theorem 2. *Any deterministic unanimous, strategyproof mechanism has an approximation ratio of at least* 1.5 *for maximizing the social utility when group memberships are public.*

3.2 Misreport only the group membership

In this subsection, we put the facility at the middle point between the location of the leftmost agent and the location of the rightmost agent.

Middle-Point Mechanism (Middle-M). *Put the facility at* $y = \frac{1}{2}(\min_{i\in N}\{x_i\} + \max_{i\in N}\{x_i\})$.

Theorem 3. *Middle-Point Mechanism is unanimous, strategyproof, and has an approximation ratio of* 2 *for maximizing the social utility when locations are public.*

Theorem 4. *Any deterministic unanimous, strategyproof mechanism has an approximation ratio of at least* 2 *for maximizing the social utility when locations are public.*

Proof. Given any unanimous, strategyproof mechanism, consider a profile set $\mathbf{r}$ where all the agents' locations $x_1 = \cdots = x_{\frac{n}{2}} = 0$ and $x_{\frac{n}{2}+1} = \cdots = x_n = 1$ (n is an even number). Suppose that all agents at 0 belong to G_1 and all agents at 1 belong to G_2 where $\alpha_1 = \alpha_2 = 0$. Due to symmetry, we assume that $f(\mathbf{r}) \in [\frac{1}{2}, 1]$. Then we consider another profile set $\mathbf{r}'$ where for all $\frac{n}{2}$ agents at 0, $g_i = 3$, and $\alpha_3 = 1$. Guaranteed by Lemma 1, the utility of each agent i at 0 satisfies $u_i(y, \mathbf{r}) \geq u_i(y, \mathbf{r}')$. Then we further have $f(\mathbf{r}') \geq f(\mathbf{r})$. Finally, we have the approximation ratio

$$\rho = \frac{su(0, \mathbf{r}')}{su(f(\mathbf{r}'), \mathbf{r}')} \geq \frac{\sum_{i\in G_3} |G_{g_3}| v(0, x_i)}{\sum_{i\in G_2} v(\frac{1}{2}, x_i) + \sum_{i\in G_3} |G_{g_3}| v(\frac{1}{2}, x_i)}$$
$$= \frac{(\frac{n}{2})^2}{\frac{n}{2}\frac{1}{2} + (\frac{n}{2})^2\frac{1}{2}} = \frac{2}{\frac{2}{n}+1} \xrightarrow{n \text{ approaches } \infty} 2.$$

□

3.3 Misreport both the location and preference

In this subsection, we will show that strategyproofness and unanimity are incompatible.

Theorem 5. *There does not exist any unanimous and strategyproof mechanism when each agent can misreport both the location and the group membership.*

Proof. Given any unanimous, strategyproof mechanism f, we prove the theorem by analyzing a series of profile sets with the same constant $k (k \geq 2)$, and $\alpha_1 = \alpha_2 = 1$.

Firstly, we consider a profile set $\mathbf{r}^1$ where k agents have $r_i^1 = (0, 1)$, $k+1$ agents have $r_i^1 = (1, 1)$. Note that all agents have the same utility $k(1-y) + (k+1)y = k + y$, where y is the facility location. The mechanism f has to output $f(\mathbf{r}^1) = 1$. Otherwise, all agents at 0 can benefit by misreporting their locations to 1, making the facility move from $[0, 1)$ to 1 guaranteed by unanimity, in contradiction to Lemma 2.

Secondly, consider another profile set $\mathbf{r}^2$ where k agents have $r_i^2 = (0, 1)$, $k-1$ agents have $r_i^2 = (1, 1)$ and two agents have $r_i^2 = (1, 2)$, $\alpha_1 = \alpha_2 = 1$. In this profile set, $f(\mathbf{r}^2)$ should be equal to 1. Otherwise, two agents in G_2 can benefit by misreporting their group memberships to G_1, making the profile set the same as $\mathbf{r}^1$, which makes the facility move from $f(\mathbf{r}^2) \in [0, 1)$ to $f(\mathbf{r}^1) = 1$, in contradiction to Lemma 2.

Continuing with the profile set $\mathbf{r}^2$, we consider a profile set $\mathbf{r}^3$ where $2k-1$ agents have $r_i^3 = (0, 1)$ and two agents have $r_i^3 = (1, 2)$, which is shown in Figure 3. We then have $f(\mathbf{r}^3) = 1$. Otherwise, given the profile set $\mathbf{r}^2$, $k-1$ agents in G_1 at $x = 1$ can benefit by misreporting their locations to 0, which makes the facility move from $f(\mathbf{r}^2) = 1$ to $f(\mathbf{r}^3) \in [0, 1)$, in contradiction to Lemma 2.

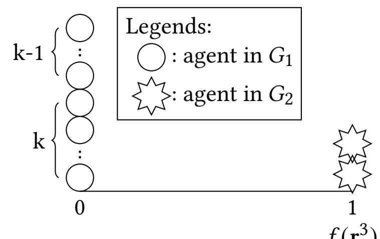

Figure 3: Profile set $\mathbf{r}^3$

Fourthly, consider a profile set $\mathbf{r}^4$ where $2k-1$ agents have $r_i^4 = (0, 2)$, two agents have $r_i^4 = (1, 2)$. We have $f(\mathbf{r}^4) = 1$. Otherwise, given the profile set $\mathbf{r}^3$, all agents at $x = 0$ can benefit by misreporting their group memberships to G_2, which makes the facility move from $f(\mathbf{r}^3) = 1$ to $f(\mathbf{r}^4) \in [0, 1)$, in contradiction to Lemma 2.

Lastly, consider a profile set $\mathbf{r}^5$ where $2k+1$ agents have $r_i^5 = (0, 2)$. We have $f(\mathbf{r}^5) = 1$. Otherwise, given the profile set $\mathbf{r}^4$, two agents at 1 can benefit by misreporting their locations to 0, which makes the facility move from $f(\mathbf{r}^4) = 1$ to $f(\mathbf{r}^5) \in [0, 1)$, in contradiction to Lemma 2.

Finally, we have the scenario where all agent locations are 0 while the facility location is 1, which contradicts the definition of unanimity. Therefore, we finish the proof that when misreporting both locations and group memberships is allowed, there does not exist any mechanism with a bounded approximation ratio that satisfies both strategyproofness and unanimity simultaneously. □

Remark 1. *Intuitively, one may consider the dictatorship mechanism which empowers an agent to decide where to locate the facility could satisfy strategyproofness. However, though ordinarily a dictator's misreporting does not increase their utility and fulfill strategyproofness, we demonstrate that agents other than the dictator can profit by misreporting, leading to the conclusion that the dictatorship mechanism is not an exception to our setting. Consider the following profile set where one agent has the profile* $(x_1, 1)$ *with* $a_1 = 1$ *and two agents have the profile* $(x_2, 2)$*. Assume that we empower the agent* *in G_1 to decide the facility location. Then it is easy to derive that two agents in G_2 can move the facility from x_1 to x_2 by misreporting their group memberships to G_1, in contradiction to mechanism strategyproofness.*

While strategyproofness and unanimity cannot be satisfied simultaneously, the problem will be easy if only one property is desired. If we only design a strategyproof mechanism, putting facility at $\frac{1}{2}$ can guarantee a 2-approximation ratio. If we only design a unanimous mechanism, we can just put the facility at the optimal location in every profile set.

4 MINIMUM UTILITY

In this section, we first study the setting where each agent can only misreport their location, then we look at the setting where each agent can only misreport group membership. When misreporting both the location and the group membership is allowed, strategyproofness and unanimity, as we discussed in the section on social utility, are also incompatible in this minimum utility objective since those two properties are independent of the objective functions.

4.1 Misreport only the location

In this subsection, we first extend the mechanism proposed in Zhou [18] to the multiple groups setting. We say that two points a, b are on the same side if $a, b \leq \frac{1}{2}$ or $a, b \geq \frac{1}{2}$, and an agent is a marginal agent if their location is closest to $\frac{1}{2}$. Recall that the mechanism in Zhou [18] is mapping all agents in G_2 with $\alpha_2 = \alpha$ to $x_{med(G_2)}$, then putting the facility at the location of the marginal agent if all agents are on the same side; otherwise, putting the facility at $\frac{1}{2}$. Next, we extend the mechanism as follows.

Marginal-Point Mechanism (Marginal-M). *Given any profile set, map all agents in G_j from their own locations to $x_{med(G_j)}$. Put the facility at the location of the marginal agent after mapping if all agents are on the same side; otherwise, putting the facility at $\frac{1}{2}$.*

Proposition 3. *Marginal-Point Mechanism is unanimous, strategyproof, and has an approximation ratio of 2 for maximizing the minimum utility when group memberships are public.*

Since Marginal-Point Mechanism does not leverage the group externalities and simply categorizes cases, it is not surprising that it achieves a large approximation ratio. In addition, we observe that the minimum utility is usually achieved by an agent in a smaller group with a smaller group externality. Hence, we propose the following mechanism.

Ternary Mechanism (Ternary-M). *Given any profile set, map all the agents to the respective ideal locations (either their own location or the location of their group median) where they can obtain their maximum utilities. Consider G_{j^*} where $j^* = \arg\min_{j\in[m]}\{1+\alpha(|G_j|-1)\}$, tie-breaking by selecting the smallest j. Put the facility y at*
1. $x_{med(G_{j^*})}$, if $x_{med(G_{j^*})} \in [\sqrt{6}-2, 3-\sqrt{6}]$;
2. $3-\sqrt{6}$, if $x_{med(G_{j^*})} \in (3-\sqrt{6}, 1]$ and there exists an agent in $[0, 3-\sqrt{6}]$;
3. the agent location closest to $3-\sqrt{6}$, if $x_{med(G_{j^*})} \in (3-\sqrt{6}, 1]$ and there is no agent in $[0, 3-\sqrt{6}]$.

We apply a similar categorization to $x_{med(G_{j^*})} \in [0, \sqrt{6}-2)$ as $x_{med(G_{j^*})} \in (3-\sqrt{6}, 1]$, due to their symmetry about $\frac{1}{2}$.

Theorem 6. *Ternary Mechanism is unanimous, strategyproof, and has an approximation ratio of $\frac{\sqrt{6}}{3}+1 \approx 1.816$ for maximizing the minimum utility when group memberships are public.*

Proof Sketch. If all agents are at the same location, Ternary-M puts the facility at $x_{med(G_{j^*})}$ or the location of the agent with the ideal location closest to $3-\sqrt{6}$ or $\sqrt{6}-2$, which satisfies unanimity. For strategyproofness, we use symmetry-based analysis for the case of $x_{med(G_{j^*})}$ in different intervals. We conclude that in each case, any misreporting eventually leads the facility location to remain unchanged or move farther away from that agent's ideal location (i.e., leads their utility to remain unchanged or decrease), regardless of whether a case transition occurs or not.

Then we show the approximation ratio through a discussion of four subcases divided by the location of y and the agent who achieves the minimum utility. We denote the facility location as y and the optimal facility location as y^*. In the first two subcases $y \in [\sqrt{6}-2, 3-\sqrt{6}]$, without loss of generality we assume that $y^* \leq y$.

Subcase 1-1: $x_{med(G_{j^*})} \in [\sqrt{6}-2, 3-\sqrt{6}]$ and $mu(y, \mathbf{r})$ is achieved by an agent in G_{j^*}. We have the minimum utility achieved by y^* is less than or equal to the utility of every agent in G_{j^*}. Hence, our analysis can focus on group G_{j^*} only. By using the fact that y is the median of G_{j^*}, we show an approximation ratio of $\frac{1-\frac{3-\sqrt{6}}{2}}{1-(3-\sqrt{6})} \approx 1.612$.

Subcase 1-2: $x_{med(G_{j^*})} \in [\sqrt{6}-2, 3-\sqrt{6}]$ and $mu(y, \mathbf{r})$ is achieved by an agent in G_{j^+} ($j^+ \in [m], j^+ \neq j^*$). The minimum utility achieved by y^* is less than or equal to the utility of every agent in G_{j^*} and in G_{j^+}. If y^* is closer to y, we use the utility of an agent in G_{j^+} to amplify $mu(y^*, \mathbf{r})$. The largest approximation ratio is achieved when all agents in G_{j^+} are at 0. If y^* is closer to 0, we use the utility of an agent in G_{j^*} to amplify $mu(y^*, \mathbf{r})$. Then we focus on groups G_{j^*} and G_{j^+}. The largest approximation ratio is achieved when all agents in G_{j^+} are at 0, half of the agents in G_{j^*} are at y^* and half of them are at y. Finally, we have the approximation ratio of $\frac{\frac{\sqrt{6}}{3}}{\sqrt{6}-2} \approx 1.816$.

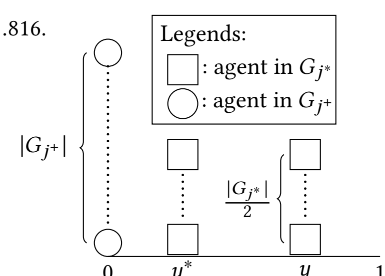

Figure 4: A worst case of Subcase 1-2.

Subcase 2-1: $x_{med(G_{j^*})} \in (3-\sqrt{6}, 1]$ and there exists an agent with an ideal location in $[0, 3-\sqrt{6}]$. If $mu(y, \mathbf{r})$ is achieved by an agent within $[0, y]$, we can use the similar analysis as subcases 1-1 and 1-2 to show a 1.816-approximation ratio. If $mu(y, \mathbf{r})$ is achieved by an agent within $(y, 1]$, the profile set that achieves the largest approximation ratio satisfies that all agents in G_{j^*} are at 1 and $y^* = 1$. Finally we conclude the approximation ratio of $\frac{\sqrt{6}}{3}+1 \approx 1.816$.

Subcase 2-2: $x_{med(G_{j^*})} \in (3-\sqrt{6}, 1]$ and there is no agent with an ideal location in $[0, 3-\sqrt{6}]$. If $y^* \leq y$, we have the optimal solution since all locations on the left of y achieve a minimum utility less than y. If $y^* > y$, we can use the similar analysis as subcase 2-1 to show an approximation ratio of 1.816.

For the case where $x_{med(G_{j^*})} \in [0, \sqrt{6}-2)$, we can use the same analysis to show the approximation ratio. Finally, we complete the proof that Ternary-M has an approximation ratio of 1.816. □

We introduce a lemma before giving the lower bound.

Lemma 4. *Given any unanimous and strategyproof mechanism f, there exists a case profile set $\mathbf{r}$ where k agents have the profile $(x_1, 1)$, k agents have the profile $(x_2, 2)$, $\alpha_1 = \alpha_2 = 1$ and $x_2 - x_1 > \frac{1}{2}$, for which $f(\mathbf{r}) \in [x_1, x_1+\epsilon] \cup [x_2-\epsilon, x_2]$.*

Proof. Firstly, we prove that y should not be outside of the interval $[x_1, x_2]$. If $y > x_2$, consider the case where agents at x_1 misreport their locations to x_2 sequentially. In each misreporting, the facility location output by any unanimous and strategyproof mechanism f cannot be x_2, due to the strategyproofness. Hence, we finally have the scenario where all agents are at x_2 but $y > x_2$, in contradiction to unanimity. A similar analysis also holds for $y < x_1$.

Now we focus on showing that for any given $q \in N^+$, there exists a profile set $\mathbf{r}$ where k agents have the profile $(x_1, 1)$, k agents have the profile $(x_2, 2)$, $\alpha_1 = \alpha_2 = 1$, and $x_2 - x_1 = (\frac{q+3}{2})\epsilon$ $(x_2 > x_1)$, such that all the possible y output by any unanimous and strategyproof mechanism satisfies $y \leq x_1 + \epsilon$ or $y \geq x_2 - \epsilon$.

By utilizing mathematical induction, firstly we construct the base in which $x_2 - x_1 = 2\epsilon$, i.e., $q = 1$. It is obvious that $[x_1, x_1+\epsilon] \cup [x_2-\epsilon, x_2]$ covers the whole interval $[x_1, x_2]$.

Secondly, we show the induction step in terms of two different ranges of y. We first consider the case where $y \in [x_1, x_1+\epsilon]$ and suppose that the lemma holds for the given q $(q \geq 1)$. Combined with the next profile set where k agents have the profile $(x_1, 1)$ and k agents have the profile $(x_2 + \frac{1}{2}\epsilon, 2)$, which could be regarded as k agents originally at x_2 in the previous profile misreporting their locations to $x_2' = x_2 + \frac{1}{2}\epsilon$ sequentially, increasing the distance to $x_2' - x_1 = (\frac{(q+1)+3}{2})\epsilon$. We first observe that when there are less than $\frac{k}{2}$ agents at x_2', the ideal facility location for each agent in G_2 is x_2. Otherwise, the ideal facility location for each agent in G_2 is x_2'. Therefore, we denote y^i as the facility location after i agents in G_2 misreport to $x_2' (1 \leq i \leq k)$ and further have $d(y^i, x_2) \geq d(y^{i-1}, x_2)$ when $i \leq \lceil \frac{k}{2} \rceil$ and $d(y^i, x_2') \geq d(y^{i-1}, x_2')$ when $i > \lceil \frac{k}{2} \rceil$, due to the strategyproofness. From both inequalities we can conclude that $y^i \leq x_2 - \frac{q+3}{2}\epsilon + \epsilon = x_1 + \epsilon$ or $y^i \geq x_2 + \frac{q+1}{2}\epsilon > x_2'$ for all $i \in [k]$. Combined with $y^k \in [x_1, x_2']$, we have $y^k \in [x_1, x_1+\epsilon]$.

If $y \in [x_2-\epsilon, x_2]$, combined with the next profile set where k agents have the profile $(x_1 - \frac{1}{2}\epsilon, 1)$ and k agents have the profile $(x_2, 2)$, which could be regarded as k agents originally at x_1 in the previous profile misreporting their locations to $x_1' = x_1 - \frac{1}{2}\epsilon$ sequentially, increasing the distance to $x_2 - x_1' = (\frac{(q+1)+3}{2})\epsilon$. Then we can use a similar analysis as the case $y \in [x_1, x_1+\epsilon]$ to show the mechanism will output the facility within $[x_2-\epsilon, x_2]$.

Since x_1 and x_2 in the base case are on the opposite side of $\frac{1}{2}$, and either x_1 moves to the left or x_2 moves to the right in every induction step. The termination condition of induction is that one of the two points reaches the boundary after movements, implying that $x_2 - x_1 > \frac{1}{2}$. □

Theorem 7. *Any deterministic unanimous, strategyproof mechanism has an approximation ratio of at least $\frac{5}{3} \approx 1.667$ for maximizing the minimum utility when group memberships are public.*

Proof Skecth. We construct the profile set $\mathbf{r}'$ where $2k+1$ agents have the profile $(x_1, 1)$, k agents have the profile $(x_{2'}, 2)$, $k+1$ agents have the profile $(x_2, 2)$, $x_{2'} = \frac{(2k+1)x_1 + (k+1)x_2}{3k+2}$, $\alpha_1 = \alpha_2 = 1$ and $x_2 - x_1 \geq \frac{1}{2} + \epsilon$. The mechanism has to output the facility within $[0, x_2 - \epsilon)$ to achieve an approximation ratio smaller than $\frac{5}{3}$. Then k agents at $x_{2'}$ can benefit by misreporting locations to x_2, moving the facility location to $[x_2 - \epsilon, x_2]$, in contradiction to Lemma 2. □

4.2 Misreport only the group membership

Theorem 8. *Middle-Point Mechanism is unanimous, strategyproof, and has an approximation ratio of 2 for maximizing the minimum utility when locations are public.*

Theorem 9. *Any deterministic unanimous, strategyproof mechanism has an approximation ratio of at least 2 for maximizing the minimum utility when locations are public.*

Proof. Assume for contradiction that there exists a unanimous, strategyproof mechanism with an approximation ratio smaller than 2. Consider a profile set $\mathbf{r}$ where $k-1$ agents have the profile $(0, 1)$, $k-1$ agents have the profile $(1, 2)$, one agent has the profile $(0, 2)$, one agent has the profile $(1, 1)$, $\alpha_1 = \alpha_2 = 1$ and $k \geq 3$. Due to symmetry, let $f(\mathbf{r}) \geq \frac{1}{2}$. Then we consider another profile set $\mathbf{r}'$ where one agent with the profile $(0, 1)$ in $\mathbf{r}$ misreports their group membership to G_3 with $\alpha_3 = 0$. Then the optimal facility location of $\mathbf{r}'$ is 0, which achieves the minimum utility of 1. If the mechanism is to achieve an approximation ratio better than 2, the facility must be located in $[0, \frac{1}{2})$, implying $f(\mathbf{r}') < f(\mathbf{r})$. In such a case, the agent with $(0, 1)$ can benefit by misreporting group membership to G_3, which contradicts strategyproofness. □

5 CONCLUSION AND FUTURE WORK

We study approximate mechanism design in facility location games with multiple groups where an agent's utility is derived from using the facility and interacting with their group members. We design desirable mechanisms with (almost) tight bounds or prove the incompatibility of strategyproofness and unanimity for each setting.

There are many open questions suggested by our analysis. An immediate direction is to further strengthen the results of misreporting the location only. Moreover, we study the setting where intra-group externalities are positive. It is also reasonable to study the negative condition, i.e., a competitive relationship among the agents within the same group. Our work could be extended to social networks where an agent cares about the q distant neighborhood. We also believe that group externalities can bring more possibilities to some other fields, e.g., in resource allocation, agents can obtain additional utility by sharing goods among themselves.

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