Optimal Referral Auction Design

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ABSTRACT

The auction of a single indivisible item is one of the most celebrated problems in mechanism design with transfers. Despite its simplicity, it provides arguably the cleanest and most insightful results in the literature. When the information that *the auction is running* is available to every participant, Myerson [19] provided a seminal result to characterize the incentive-compatible auctions along with revenue optimality. However, such a result does not hold in an *auction on a network*, where the information of the auction is spread via the agents, and they need incentives to forward the information. In recent times, a few auctions (e.g., [12, 16]) were designed that appropriately incentivized the intermediate nodes on the network to *promulgate* the information to potentially more valuable bidders. In this paper, we provide a Myerson-like characterization of *incentive-compatible auctions on a network* and show that the currently known auctions fall within this class of randomized auctions. We then consider a special class called the *referral auctions* that are inspired by the multi-level marketing mechanisms [1, 5, 6] and obtain the structure of a revenue optimal referral auction for i.i.d. bidders.

KEYWORDS

Diffusion Auction; Optimal Revenue; Myerson Auction

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1 INTRODUCTION

Single indivisible item auction is a special mechanism design setting with monetary transfers where multiple bidders contest to collect a single item. The *true value* of the item could be different for different agents and it is their *private information*, i.e., not known to the *mechanism*[1] designer. Despite its simplicity, the single-item auction provides remarkable insights into the questions: (a) what is the structure of the mechanisms that reveal the agents' true private information, and (b) how to design mechanisms that maximize the expected revenue. In the world where the information that 'an item is being auctioned' is available to every possible bidder interested in this item, these two questions have been answered gracefully by Myerson in his seminal paper [19].

However, in various recent contexts of auctions, the network of connections plays an important role in the information flow over the network. An agent diffuses the information into the network only if they find it is *beneficial* to share. This setup is called *network auctions*, where agents *diffuse* the information of the auction only if it (strictly or weakly) improves their utilities. This problem has given birth to the domain of *diffusion auction design* on networks and has received significant attention in the recent times [8, 12, 14, 16]. Because the information about the auction *does not automatically* reach every agent in this setup, the mechanism needs to incentivize individuals to diffuse (or forward) the information. The Myerson [19] characterization does not follow here, and a fresh investigation is necessary to characterize the *truthful* and *revenue maximizing* diffusion auctions.

1.1 Our contributions

The contributions of this paper are divided into two parts. In the first part, we characterize the *truthful* network auction via certain constraints on allocation and payment. In the second part, we consider a Bayesian setup and find the class of revenue-optimal network auctions. More concretely:

(1) We provide a direct definition of truthfulness called *diffusion dominant strategy incentive compatibility* (**DDSIC**, Definition 2) and show that it is equivalent to the existing IC definition in the literature on network auction [14, e.g.].
(2) We characterize **DDSIC** (and therefore, IC) mechanisms (Lemmas 1 to 3 and Theorem 1). Note that this result is of independent interest irrespective of the revenue optimality question addressed later in this paper (similar to [19]).
(3) Our characterization includes less-explored auctions, e.g., randomized and **DDSIC** (Section 4.2).
(4) We find the revenue-optimal *referral auction* (which is a class of auctions motivated by the multi-level marketing methods) for i.i.d. bidders (Theorem 5) in Section 5.

In Section 6, we provide an example of a new mechanism called **LbLEV** that is **DDSIC** and individually rational on a tree. The purpose of introducing this mechanism is to show that when bidders are

[1]Since auctions are special cases of mechanisms, we will use these two terms interchangeably in this paper.

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non-i.i.d., the parameters of **LbLEV** can be tuned, based on the prior information of the valuations and network structure, to yield a better revenue than the currently known truthful diffusion auctions, and we show this empirically. The complete details are in [3].

1.2 Related work

The area of diffusion mechanism design is relatively new, leading to rather sparse literature. Guo and Hao [8] provides a comprehensive survey of the domain. The first works on diffusion auction are due to Li et al. [16] and Lee [11]. In particular, [16] proposed a new mechanism, called IDM, that mitigates this problem. In the following years, a few more diffusion auctions were proposed: CSM [17] for economic networks, MLDM for intermediary networks [13], TNM, CDM, WDM for the unweighted and weighted networks [12, 15]. FDM [26] and NRM [27] considered the money-burning issue in network auction and proposed schemes to redistribute the money maintaining incentive compatibility. On the characterization results, Li et al. [14] provide a characterization for deterministic diffusion auctions and find optimal payments. Our approach considers a broader approach to characterize all *randomized* diffusion auctions and shows that in this larger class, better mechanisms for revenue generation exist.

On the other hand, auctions are fairly well understood in the setting without networks, both in theory [4, 19, 21] and in practice [10]. The primary focus of the paper is to extend the theory to network auctions. We begin by understanding revenue optimality in a simpler class of mechanisms, namely the *referral auctions*, which has a close similarity with a business method called *multi-level marketing* (MLM) [22]. Direct sales firms often use this method to encourage individual distributors to recruit new distributors. A prominent example of an MLM scheme is the DARPA red balloon challenge [20].[2] MLM mechanisms are also well investigated in the mechanism design literature [1, 5, 6]. Hence, in this paper, we consider a natural candidate class of *referral auctions* for revenue optimality (§5.2).

2 BASIC PROBLEM SETUP

Consider a directed graph $G = (N \cup \{s\}, E)$, where $N = \{1, \dots, n\}$ is the set of players involved in the auction of a single indivisible item and s is a distinguished node called the *seller*. The set E is the set of edges. Each edge (i, j) denotes that node i can share information with node j. Typical examples of such graphs are online social networks where an individual can share information (selectively) with a subset of her neighbors. The direction signifies that almost all networks have asymmetric information flow (e.g., only followers receive the information from the followee). In this network, node s is a single seller that wants to sell the indivisible item. Every other node $i \in N$ is a potential buyer and the information about the auction flows only via the direction of an edge. The information cannot reach a node unless there is a directed path from s to that node and each intermediate node decides to forward the information. An intermediate node may decide not to forward the information if it reduces its *utility*.

This setup naturally brings up an auction-like information-sharing game among the players. Each player $i \in N$ has a type $\theta_i = (v_i, r_i)$, where v_i is the valuation of agent i for the item, and r_i is the set of her directed neighbors. The sets of valuations and neighbors of player i are $\mathcal{V}_i$ and $\mathcal{R}_i$ respectively. The type set of i, Θ_i, is therefore, $\mathcal{V}_i \times \mathcal{R}_i$. The information about the auction needs to reach via directed edges to player i for her to participate in the auction. Therefore, the auction asks every agent to report their valuation for the item and to forward the information to its directed neighbors. This can be captured via their reported type $\hat{\theta}_i = (\hat{v}_i, \hat{r}_i)$ for every agent $i \in N$. We assume that the seller s is not a strategic player in this auction, rather he wants to sell the object and always forwards the information to its directed neighbors. The vector of the reported types of all the agents except i is denoted by $\hat{\theta}_{-i} = (\hat{\theta}_1, \dots, \hat{\theta}_{i-1}, \hat{\theta}_{i+1}, \dots, \hat{\theta}_n)$. We denote the set of all type profiles by $\Theta := \prod_{i \in N} \Theta_i$.

Depending on the reported types of the agents, particularly, the reported $\hat{r}_i$'s, the auction may reach only a subset of the agents in N. To denote the reported valuation and directed neighbors on the subnetwork generated by $(\hat{\theta}_i, \hat{\theta}_{-i})$, we use a *filter function* f^G for the graph G, where $f^G(\hat{\theta}_i, \hat{\theta}_{-i})$ denotes the reported valuation and directed neighbor vector of the subgraph reachable from s after the agents reported the type profile $(\hat{\theta}_i, \hat{\theta}_{-i})$. In this setup, the mechanism design goal is to incentivize each node to truthfully reveal its private valuation and forward the auction information regardless of others' actions. We consider auctions on this graph with randomized allocations. Formally, we define a *diffusion auction* as follows.

Definition 1 (Diffusion Auction). A *diffusion auction (***DA***)* is given by the tuple (g, p) where g and p are the *allocation* and *payment* functions respectively. The allocation function $g : \Theta \to \Delta_n$ is such that its i-th component $g_i(f^G(\theta))$ denotes the probability of agent i winning the object, where $\Delta_n := \{x \in \mathbb{R}^n_{\geqslant 0} : \sum_{i=1}^n x_i = 1\}$. Similarly, the payment function $p = (p_i)_{i \in N}$ is such that its i-th component $p_i : \Theta \to \mathbb{R}$ denotes the payment assigned to agent i.

Note that g_i should operate on the subnetwork that remains connected to s after the agents choose their actions $\hat{\theta}$. Hence the notation $g_i(f^G(\cdot))$ is used in the definition above. It is worth noting that the notation generalizes the one used by Li et al. [16]. The action chosen by player i may change the actions available to the other players and it is succinctly captured by the filter function which also subsumes the definition in that paper. The utility of agent i under **DA** is given by the standard quasi-linear model [24]: $u_i^{(g,p)}((\hat{\theta}_i, \hat{\theta}_{-i}); \theta_i) = v_i g_i(f^G(\hat{\theta}_i, \hat{\theta}_{-i})) - p_i(f^G(\hat{\theta}_i, \hat{\theta}_{-i}))$.

3 DESIGN DESIDERATA

The first desirable property of an auction is truthfulness. However, in the context of auctions on the network, we need to ensure that the mechanism also incentivizes the agents to forward the information in addition to being truthful about their valuations. The following definition captures both these aspects.

Definition 2 (Diffusion Dominant Strategy Incentive Compatibility). A **DA** (g, p) on a graph G is *diffusion dominant strategy incentive compatible (***DDSIC***)* if

(1) every agent's utility is maximized by reporting her true valuation irrespective of the diffusing status of herself and the

[2] https://www.darpa.mil/about-us/timeline/network-challenge

other agents, i.e., for every $i \in N, \forall r_i, \hat{\theta}_{-i}$, the following holds, $\forall v_i, v_i', \hat{\theta}_{-i}, r_i' \subseteq r_i$

$$v_i g_i(f^G((v_i, r_i'), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i'), \hat{\theta}_{-i})) \geqslant v_i g_i(f^G((v_i', r_i'), \hat{\theta}_{-i})) - p_i(f^G((v_i', r_i'), \hat{\theta}_{-i})), \text{ and,}$$

(2) for every true valuation, every agent's utility is maximized by diffusing to all its neighbors irrespective of the diffusion status of the other agents, i.e., for every $i \in N, \forall r_i, \hat{\theta}_{-i}$, the following holds, $\forall v_i, \hat{\theta}_{-i}, r_i' \subseteq r_i$

$$v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \geqslant v_i g_i(f^G((v_i, r_i'), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i'), \hat{\theta}_{-i})).$$

In the auction setup without a network, where each agent i's type is denoted only by v_i (and not r_i), this definition of **DDSIC** reduces to the standard definition of *dominant strategy incentive compatibility* (DSIC) [25] given by condition (1) with r_i and r_i' omitted. We will show in Theorem 1 that the above definition is equivalent to the following definition of incentive compatibility (restated below with the notation of this paper) given by Li et al. [14]. Hence, all the prominent mechanisms presented so far in the literature (e.g., IDM, TNM, etc.) follow this alternate definition as well.

Definition 3 (Incentive Compatibility [14]). A **DA** (g, p) on a graph G is incentive-compatible (IC) if for every $i \in N$, $\forall r_i, \hat{\theta}_{-i}$, $v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \geqslant v_i g_i(f^G((v_i', r_i'), \hat{\theta}_{-i})) - p_i(f^G((v_i', r_i'), \hat{\theta}_{-i})), \forall v_i, v_i', \forall r_i' \subseteq r_i, \forall i \in N$.

Why **DDSIC**?. A natural question can arise: why do we introduce a new definition of truthfulness when there is an existing one, given that both are equivalent? This is because the new definition provides a more direct and intuitive way to understand the truthful reporting of valuation and diffusion. **DDSIC** does this by splitting the IC condition into two sets of inequalities as given in Definition 2. In our proofs, this definition makes the analysis of truthful mechanisms simpler. We will show that IC and **DDSIC** are equivalent and both are equivalent to the conditions stated in Definition 5, and will subject all our further analyses only to **DDSIC**.

The other desirable property deals with the participation guarantee of the agents.

Definition 4 (Individual Rationality). A **DA** (g, p) on a graph G is *individually rational (IR)* if $v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \geqslant 0, \ \forall v_i, r_i, \hat{\theta}_{-i}, \forall i \in N$.

4 CHARACTERIZATION RESULTS

Our first result is to characterize the IC diffusion auctions and show equivalence between IC and **DDSIC**. For a cleaner presentation, we define the following class of auctions.

Definition 5 (Monotone and Forwarding-Friendliness (MFF)). For a given network G, a **DA** (g, p) is *monotone and forwarding-friendly (**MFF**)* if

(a) the functions $g_i(f^G((v_i, r_i), \hat{\theta}_{-i}))$ are monotone non-decreasing in v_i, for all $r_i, \hat{\theta}_{-i}$, and $i \in N$, and for the given allocation function g, the payment p_i for each player $i \in N$ is such that, for every v_i, r_i, and $\hat{\theta}_{-i}$, the following two conditions hold.

(b) For every $r_i' \subseteq r_i$, the following payment formula is satisfied.

$$p_i(f^G((v_i, r_i'), \hat{\theta}_{-i})) = p_i(f^G((0, r_i'), \hat{\theta}_{-i})) + v_i g_i(f^G((v_i, r_i'), \hat{\theta}_{-i})) - \int_0^{v_i} g_i(f^G((y, r_i'), \hat{\theta}_{-i}))\, \mathrm{d}y.$$

(c) For every $r_i' \subseteq r_i$, the values of $p_i(f^G((0, r_i'), \hat{\theta}_{-i}))$ and $p_i(f^G((0, r_i), \hat{\theta}_{-i}))$ satisfy the following inequality.

$$p_i(f^G((0, r_i'), \hat{\theta}_{-i})) - p_i(f^G((0, r_i), \hat{\theta}_{-i})) \geqslant \int_0^{v_i} \Big(g_i(f^G((y, r_i'), \hat{\theta}_{-i})) - g_i(f^G((y, r_i), \hat{\theta}_{-i})) \Big)\, \mathrm{d}y.$$

We will refer to $p_i(f^G((0, r_i), \hat{\theta}_{-i}))$, the first term on the RHS of condition (b), as the *value independent payment component* (**VIPC**) in the rest of the paper since this component of player i is not dependent on the valuation of i.

Lemma 1. *If a* **DA** (g, p) *is IC, then it is* **MFF**.

Proof sketch: In order to prove the monotonicity of $g_i(f^G((v_i, r_i), \hat{\theta}_{-i}))$, we show that it is the sub-gradient of the utility function, which can be shown to be convex. By substituting $r_i' = r_i$ in Definition 3, we get:

$$v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i}) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i}) \geqslant v_i g_i(f^G((v_i', r_i), \hat{\theta}_{-i}) - p_i(f^G((v_i', r_i), \hat{\theta}_{-i}) \quad (1)$$

Adding and subtracting $v_i' g_i(f^G((v_i', r_i), \hat{\theta}_{-i})$ on the RHS of Equation (1) yields

$$u_i((v_i, r_i), \hat{\theta}_{-i}) \geqslant u_i((v_i', r_i), \hat{\theta}_{-i}) + (v_i - v_i') g_i(f^G((v_i', r_i), \hat{\theta}_{-i}))$$

The rest of the proof shows that the utility function is convex, then cond. (a) of Definition 5 immediately follows.

From convex analysis [23], we know that for any convex function h having subgradient ϕ, the following integral relation holds: $h(y) = h(z) + \int_z^y \phi(t) dt$ for any y, z in the domain of h. Substituting h as u_i and ϕ as g_i yields cond. b of Definition 5. To prove cond. c of Definition 5, we first put $v_i' = v_i$ in the definition of IC to get point 2 of Definition 2 and we substitute the payment expressions derived above. ■

Lemma 2. *If a* **DA** (g, p) *is* **MFF**, *then it is* **DDSIC**.

Proof sketch: This proof is straightforward since the three conditions of **MFF** yield conds. 1 and 2 of **DDSIC** via certain algebraic manipulations. The details are in the supplementary material. ■

Lemma 3. *If a* **DA** (g, p) *is* **DDSIC**, *then it is IC.*

Proof sketch: By exhaustively listing all possible cases of manipulation under Definition 3, we can see that all the inequalities can be derived from the inequalities implied by **DDSIC** (Definition 2). Due to lack of space, we present just one case here: Consider (v_i, r_i) to be the true valuation of the player and they misreport to (v_i, r_i'). Substituting this in Definition 3, we get

$$v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i}) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i}) \geqslant v_i g_i(f^G((v_i, r_i'), \hat{\theta}_{-i}) - p_i(f^G((v_i, r_i'), \hat{\theta}_{-i}) \quad (2)$$

which is implied by Item 2 of Definition 2 ■

The three lemmata lead to the following theorem.

Theorem 1. **DDSIC** $\iff$ *IC* $\iff$ **MFF**.

4.1 Difference with Myerson [19] and other DAs

The characterization result is much in the spirit of the result of Myerson [19]. However, we have the following important differences on the setup, results, and the proof techniques.

1. The setup of **DA** is that of a multidimensional mechanism design since the type of each player consists of two components (v_i, r_i) compared to Myerson's single-dimensional (only v_i) setup. The mechanism design question in this setup itself is harder [9, 18]. Our approach exploits the special structure of a single indivisible item auction on a network, and somewhat surprisingly, reduces it to a similar structure like Myerson (with the additional condition (c) on the **VIPC** terms that are absent in Myerson). The multi-dimensional setup changes the analysis at several places of the proof (and is available in [3]). In particular, the entire flow of the proof 'IC $\Rightarrow$ **MFF**, **MFF** $\Rightarrow$ **DDSIC**, **DDSIC** $\Rightarrow$ IC' differs from that of Myerson.
2. Our result is unique since we provide a characterization of all *randomized* single indivisible item diffusion auctions. The closest characterization result to our knowledge applies to only deterministic diffusion auctions [14]. In the following section, we provide an example of a **DDSIC** auction that is not covered by the characterization of [14] but is covered under the **MFF** conditions (Theorem 1).
   [3] provides a class of mechanisms that is different from the currently known **DDSIC** mechanisms.

4.2 Example to illustrate the conditions of a randomized DDSIC auction.

The distinguishing factor of the truthfulness guarantee given by **DDSIC** is in the part where an agent may not diffuse the information to its neighbors. In this example, we will focus only on that part and illustrate the meaning of the conditions of **MFF** (Definition 5), that is equivalent to **DDSIC** via Theorem 1. This example can be easily extended to a full-fledged randomized **DDSIC** auction. However, that needs the auction to be defined for *every* realized graph and for *every* type profile $((v_i, r_i), \hat{\theta}_{-i})$, which will digress a reader from the main intuition of **MFF**. Instead, we have explained how these conditions are met when the agents report their (v_i, r_i)s as shown in Figure 1.

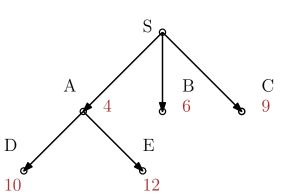

Figure 1: A randomized DDSIC auction.

For simplicity of exposition, we consider the auction where the true underlying network and the reported valuations are given by Figure 1, and r_i can take values only in $\{0, 1\}$, i.e., either forward to all its neighbors or not forwarding at all. We discuss the satisfaction of the **MFF** conditions and consider the variation of v_i and r_i of each agent i keeping the $\hat{\theta}_{-i}$ fixed at the values given in this figure. In this example, agents D and E get the information of the auction only if A forwards it at the first level of the tree. A **DDSIC DA** (g, p) needs to decide the allocations $g_i(f^G((v_i, r_i), \hat{\theta}_{-i}))$, and the **VIPC** components $p_i(f^G((0, r_i), \hat{\theta}_{-i}))$ for $r_i = 0, 1$ and for all $i \in N$. For all agents $i \neq A$, r_i's do not matter since they do not have any children in this tree. Therefore, g_i's and $\mathbf{VIPC}_i$'s of agents $i \neq A$, remain unchanged in this example auction when they set $r_i = 0$ or $r_i = 1$ given other agents' reported types are fixed. Hence, condition (c) of **MFF** is trivially satisfied for all agents except A. We discuss agent A's satisfiability of condition (c) separately. In a nutshell, this example mechanism adapts the *residual claimant* (RC) mechanism by Green and Laffont [7] to this setting at the first level of the tree. If agent A forwards, then it divides A's probability of allocation with its children and adjusts the payments according to Definition 5. If A does not forward, then it is just RC at level 1. Otherwise, the auction becomes a little nontrivial as follows.

Case 1: $r_A = 0$: When agent A does not forward the information, the auction stays limited to the agents A, B, and C. Let the auction give the object w.p. $2/3$ to the highest bidder and w.p. $1/3$ to the second highest bidder. The payment of the highest bidder is $1/3 \times$ the second highest bid. This payment is equally distributed among the non-winning agents, which, in this case, is the third highest bidder. Hence, under this case, the allocation probability of each agent is clearly monotone non-decreasing since it increases from zero to $1/3$ when it becomes the second highest bidder and from $1/3$ to $2/3$ when it becomes the highest bidder. The **VIPC** for each agent i is given by $-1/3 \times$ the second highest bid in the population except agent i. Therefore, $\mathbf{VIPC}_A(r_A = 0) = -6/3 = -2, \mathbf{VIPC}_B(r_A = 0) = -4/3, \mathbf{VIPC}_C(r_A = 0) = -4/3$. The payments follow from condition (b): $p_A(r_A = 0) = -2 + 0 + 0, p_B(r_A = 0) = -4/3 + 6 \times 1/3 - 1/3(6-4) = 0, p_C(r_A = 0) = -4/3 + 9 \times 2/3 - 1/3(6-4) - 2/3(9-6) = 2$. The allocation probabilities are zero for every valuation of agents D and E, and their **VIPC**s are zeros. Consequently, their payments are also zero in this case.

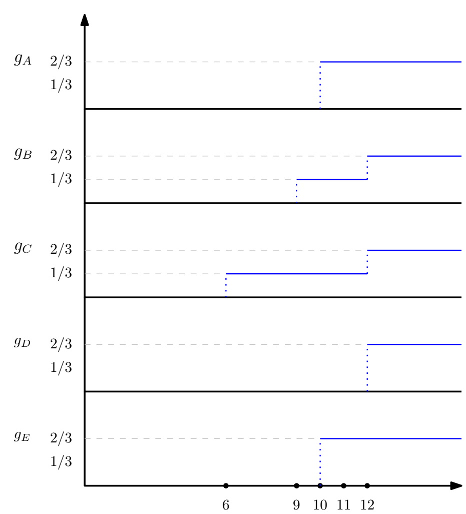

Figure 2: Allocation functions of the nodes in Figure 1.

Case 2: $r_A = 1$: i.e., A diffuses. **MFF** allows us to pick any monotone g_i's for all $i \in N$. Suppose, the g_i's be given by Figure 2, when agents except i report their valuations as shown in Figure 1. Clearly, these are monotone non-decreasing. It is simpler to just consider the allocations as given and calculate the **VIPC** and payments from it satisfying **MFF** (condition (c) in particular). However, the way the given allocation was chosen was by considering the subtree of A as a single agent which gets the item with the same probabilities as the RC mechanism, and if it gets the item, it allocates to the highest among its subtree. Now, to calculate $\texttt{VIPC}_A(r_A = 1)$ we need to find a value of that satisfying $\texttt{VIPC}_A(r_A = 0) - \texttt{VIPC}_A(r_A = 1) \geqslant \int_0^\infty (g_A(y, r_A = 0) - g_A(y, r_A = 1))dy$. Putting the values of $g_A(y, r_A = 0)$ (from case 1) and $g_A(y, r_A = 1)$ (from Figure 2) and setting the inequality as equality, we find a value of $\texttt{VIPC}_A(r_A = 1) = -2 - 1 - 2/3 = -11/3$. Let the rest of the **VIPC**'s as follows: $\texttt{VIPC}_B(r_A = 1) = -3, \texttt{VIPC}_C(r_A = 1) = -2, \texttt{VIPC}_D(r_A = 1) = 0, \texttt{VIPC}_E(r_A = 1) = 0$, which could be any number. We pick these numbers such that the sum of payments = 0. The payments are given by condition (b) as follows: $p_A(r_A = 1) = -11/3 + 0 + 0, p_B(r_A = 1) = -3 + 0 + 0, p_C(r_A = 1) = -2 + 9 \times 1/3 - 1/3(9 - 6) = 0, p_D(r_A = 1) = 0 + 0 + 0, p_E(r_A = 1) = 0 + 12 \times 2/3 - 2/3(12 - 10) = 20/3$.

The satisfiability of condition (c) for agent A warrants a separate discussion since it is the only agent which has a different **VIPC** when $r_A = 0$ and $r_A = 1$, keeping the other agents' reported types fixed. For all other agents, the satisfiability of condition (c) is trivial since both sides of the inequality reduces to zero. However, we note that the LHS of condition (c) for A is $\texttt{VIPC}_A(r_A = 0) - \texttt{VIPC}_A(r_A = 1) = -2 + 11/3 = 5/3$ which is larger than the RHS for every value of v_A. In particular, when $v_A \geqslant 10$, the RHS becomes $5/3$ and stays constant at that value for larger values of v_A. Hence, condition (c) is satisfied for agent A too.

This is a randomized **DA** that satisfies the **MFF** conditions (Definition 5) for $\hat{\theta}_{-i}$ given by Figure 1. For the $r_A = 1$ case, we could have chosen any monotone allocation rule for the agents and decided the payments according to condition (b), and set arbitrary **VIPC** terms for agents except A. But for A, we need to ensure that the differences in the **VIPC**s between $r_A = 0$ and $r_A = 1$ satisfies condition (c). This is the recipe for extending this example for every $\hat{\theta}_{-i}$.

5 BAYESIAN SETUP AND OPTIMAL AUCTION

The optimal auction is the one that maximizes the expected revenue. This is done assuming that the prior of the valuations are known to the auctioneer, which is a common assumption in classical auction literature [19, e.g.].[3] In this section, we consider the revenue-optimal auction where the prior distribution over (v_i, v_{-i}) is given by P and is a common knowledge. We define truthfulness in the prior-based setup by extending **DDSIC** in a Bayesian setting.

Definition 6 (Diffusion Bayesian Incentive Compatibility). A **DA** (g, p) on a graph G is *diffusion Bayesian incentive compatible (**DBIC**)* if

(1) every agent's expected utility is maximized by reporting her true valuation irrespective of the diffusing status of herself and the other agents, i.e., for every $i \in N$, $\forall r_i, \hat{r}_{-i}$, the following holds $\forall v_i, v_i', r_i' \subseteq r_i$

$$\mathbb{E}[v_i g_i(f^G((v_i, r_i'), (v_{-i}, \hat{r}_{-i}))) - p_i(f^G((v_i, r_i'), (v_{-i}, \hat{r}_{-i})))]$$
$$\geqslant \mathbb{E}[v_i g_i(f^G((v_i', r_i'), (v_{-i}, \hat{r}_{-i}))) - p_i(f^G((v_i', r_i'), (v_{-i}, \hat{r}_{-i})))],$$

(2) for every true valuation, every agent's expected utility is maximized by diffusing to all its neighbors irrespective of the diffusion status of the other agents, i.e., for every $i \in N$, $\forall r_i, \hat{r}_{-i}$, the following holds $\forall v_i, r_i' \subseteq r_i$

$$\mathbb{E}[v_i g_i(f^G((v_i, r_i), (v_{-i}, \hat{r}_{-i}))) - p_i(f^G((v_i, r_i), (v_{-i}, \hat{r}_{-i})))]$$
$$\geqslant \mathbb{E}[v_i g_i(f^G((v_i, r_i'), (v_{-i}, \hat{r}_{-i}))) - p_i(f^G((v_i, r_i'), (v_{-i}, \hat{r}_{-i})))].$$

The expectation in this definition and the rest of the paper is defined as $\mathbb{E}_{v_{-i}|v_i}$. Clearly, **DDSIC** implies **DBIC** since **DBIC** requires conditions 1 and 2 of Definition 2 to hold only in expectation.

5.1 Characterization of DBIC Mechanisms

Our first result is to characterize the **DBIC** auctions. For convenience, we define the *expected* allocation and payments with the shorthand notation as described below.

$$\alpha_i((v_i, r_i'), \hat{r}_{-i})) = \mathbb{E}[g_i(f^G((v_i, r_i'), (v_{-i}, \hat{r}_{-i})))] \tag{3}$$

$$\texttt{pay}_i((v_i, r_i'), \hat{r}_{-i})) = \mathbb{E}[p_i(f^G((v_i, r_i'), (v_{-i}, \hat{r}_{-i})))] \tag{4}$$

In the Bayesian setup, the notion of participation guarantee is also weakened to *interim individual rationality (IIR)* [24] where the expected utility of a player to join the mechanism is non-negative after she learns her own type. Similar to **DBIC**, it is easy to see that IR implies IIR since IIR requires the IR conditions to hold only in expectation. We define the following structure of a **DA** to succinctly characterize the **DBIC** auctions.

Definition 7 (**MFFE**). For a given network G, a **DA** (g, p) is *monotone and forwarding-friendly in expectation (**MFFE**)* if

(a) for every $i \in N$ and $r_i, \hat{r}_{-i}$, the functions $\alpha_i((v_i, r_i'), \hat{r}_{-i})$ is non-decreasing in v_i, for every $r_i' \subseteq r_i$, and for the given allocation function α, the payment $\texttt{pay}_i$ for each player $i \in N$ is such that, for every v_i, r_i, and $\hat{r}_{-i}$, the following two conditions hold.

(b) $\forall r_i' \subseteq r_i$, the following payment formula is satisfied.

$$\texttt{pay}_i((v_i, r_i'), \hat{r}_{-i}) = \texttt{pay}_i((0, r_i'), \hat{r}_{-i}) +$$
$$v_i \alpha_i((v_i, r_i'), \hat{r}_{-i}) - \int_0^{v_i} \alpha_i((y, r_i'), \hat{r}_{-i})\, \mathrm{d}y$$

(c) The values of $\texttt{pay}_i((0, r_i'), \hat{r}_{-i})$ and $\texttt{pay}_i((0, r_i), \hat{r}_{-i})$ are arbitrary real numbers that satisfies the following inequality for every $r_i' \subseteq r_i$.

$$\texttt{pay}_i((0, r_i'), \hat{r}_{-i}) - \texttt{pay}_i((0, r_i), \hat{r}_{-i})$$
$$\geqslant \int_0^{v_i} (\alpha_i((y, r_i'), \hat{r}_{-i}) - \alpha_i((y, r_i), \hat{r}_{-i}))\, \mathrm{d}y$$

Theorem 2. *A **DA** (g, p) is **DBIC** iff it is **MFFE**.*

The proof follows similar steps as Lemmas 1 and 2. For the forward implication, i.e., **DBIC** $\Rightarrow$ **MFFE**, note that the starting conditions of IC in Lemma 1 are also the same as in **DDSIC**, hence the expectation will also work in a very similar way. Full details are in [3].

5.2 Referral auctions

Multi-level marketing (MLM) is a marketing approach that incentivizes individuals who not only adopt a product but advertise it also [1, 5, 6]. On a social network, it creates a viral effect where

[3] This assumption is primarily due to two reasons: (a) for prior-free auctions, the worst-case revenue can be arbitrarily bad, hence revenue maximization does not yield any useful result, and (b) in practice, the prior on the users' valuation can be estimated from the historical data.

the information regarding a product reaches far beyond what traditional marketing can do. Due to its similarity with the objective of diffusion auctions, i.e., to spread the information of the auction to more individuals on a network, in this section, we consider a natural adaptation of MLM into auctions and call this class of auctions as *referral auctions* (**RA**).

In a referral auction, the seller invites its immediate neighbors in the network to report their valuations and invite all of their neighbors. These agents are also suggested, in turn, to spread the same message, i.e., to ask their neighbors to report their valuations and forward the information to their neighbors. Each time a node i reports and forwards the information to its neighbors, the information of $(\hat{v}_i, \hat{r}_i)$ is recorded by the seller along with its (system-generated) timestamp τ_i. Note that one can implement this auction in various possible ways, e.g., via inviting each node to register on the seller's site and providing the information of their neighbors. In all possible such cases, the seller can record the timestamp which cannot be manipulated by the agents. This information will be used by the class of referral auctions.

In the class **RA**, all agents are sorted w.r.t. their timestamps and a referral tree is formed via a first-invite-first-served policy (breaking ties in a fixed order, e.g., w.r.t. their social IDs). This implies that the unique parent of every node is determined by the earliest timestamp of those inviting nodes. Note that, this is the principle of multi-level marketing as well – only those individuals on a network are considered for referral bonuses that invited a new customer *first* to the seller's system.

Once the referral tree is formed, the mechanism runs an auction at every level of the tree through a general *deterministic* allocation rule $\mathcal{g}$ which is monotone non-decreasing and runs only on the agents at a given level. Define the corresponding payment as

$$p_i(\rho_i, \rho_{-i}) = \rho_i g_i(\rho_i, \rho_{-i}) - \int_0^{\rho_i} g_i(y, \rho_{-i})\,\mathrm{d}y. \quad (5)$$

We note that the payment formula in Equation (5) is the same as the payment formula in the classical result of Myerson [19] with the **VIPC** term being zero. Based on different choices of g, we obtain the class **RA**, described algorithmically in Algorithm 1. Given an instance of the reported types and the execution of Algorithm 1, we partition the agents into *three* exhaustive classes: (i) *winner* – agent who gets the item (**RA** is deterministic, hence there will be a deterministic winner), (ii) *on-path non-winner* – agents that lie on the path from the seller to the winner, and (iii) *not-on-path non-winner* – agents that are not on the winning path from the seller to the winner.

We show that each member of **RA** also follows the desirable properties as mentioned in Section 3. The proofs are in [3]. Note that the mechanisms in **RA** generate a referral tree $\hat{T}$ from an arbitrary underlying network. Hence, to prove truthfulness of the auctions in this class, we need to show that no agent can profit by underreporting her set of *true* neighbors in the underlying graph.

Theorem 3. *In each auction in* **RA***, no agent* $i \in N$ *gets a higher utility by reporting* $\hat{r}_i \subset r_i$.

Proof sketch: Each auction in **RA** is designed in such a way that only the *on-path non-winner* or the *winner* gets a non-negative utility. Each *not-on-path non-winner* node gets a utility of zero. Also, note that the auctions in **RA** create the referral tree in a first-invite-first-served manner. Since the agents cannot alter their timestamps, if they under-report their neighbor set, they can potentially stop becoming a *on-path non-winner*, which does not improve their

Algorithm 1: Referral Auctions (**RA**)

Input: reported types $\hat{\theta}_i = (\hat{v}_i, \hat{r}_i), \hat{r}_i \subseteq r_i$, and recorded timestamps τ_i, for all $i \in N$

Parameter: an arbitrary monotone non-decreasing deterministic allocation g

Output: winner of the auction (which can be $\emptyset$), payments of each agent

1 **Preprocessing:** Create the referral tree $\hat{T}$ rooted at s such that the neighbors of s is **`children`**(s), and **`parent`**$(i) = \text{argmin}\{\tau_k : i \in \hat{r}_k\}$, for all $i \in N \setminus$ **`children`**(s). Ties are broken w.r.t. a fixed order over the nodes.

2 **if** $\hat{v}_i = 0, \forall\, i \in N$ **then**

3   Item is not sold and payment is set to zero for all agents, STOP

4 *Initialization:* all agents are `non-winners` and their `actual payments` are zeros, set `offset` = 0, `level` = 1, `parent` = s, $v_{\texttt{parent}} = 0$

5 In this level of $\hat{T}$:

6   **for** *each node* $i \in$ **`children`**(**`parent`**) **do**

7   Set `effective valuation` $\rho_i := \max\{\hat{v}_j : j \in \hat{T}_i\} -$ **`offset`**

8   Remove the nodes that have $\rho_i < 0$, denote the rest of the agents with $N_{\texttt{remain}}$

9   **if** $|N_{\texttt{remain}}| \geqslant 2$ **then**

10   Find i^* where $g_{i^*}(\rho_{i^*}, \rho_{N_{\texttt{remain}} \setminus \{i^*\}}) = 1$

11   Compute $z := p_{i^*}(\rho_{i^*}, \rho_{N_{\texttt{remain}} \setminus \{i^*\}})$, given by Equation (5)

12   **else**

13   Set $z = 0$

14   **if** $v_{\texttt{parent}} \geqslant$ **`offset`** $+ z$ **then**

15   STOP and go to Step 23

16   Set agent i^* as the `tentative winner` and its `effective payment` to be z

17   All nodes and their subtrees except i^* are declared `non-winners`

18   The `actual payment` of i^* to `parent` = `effective payment` + `offset`

19   `parent` = i^*, `offset` = `actual payment` of i^*

20   `level` = `level` + 1

21 Repeat Steps 5 to 20 with the updated `parent` and `offset` for the new level

22 STOP when no agent i has $\rho_i \geqslant 0$ OR the leaf nodes are reached

23 Set `tentative winner` as `final winner`; final payments are the `actual payments` that are paid to the respective parents of $\hat{T}$

utility. This observation is the key to this proof. We consider the following cases for an agent i.

If i is the *winner* or a *not-on-path non-winner* after reporting r_i, her true neighbor set, then her forwarding information is irrelevant to her utility. If i is an *on-path non-winner* when it reports r_i, and continues to be a *on-path non-winner* when it reports $r'_i \subset r_i$, then according to Algorithm 1, agent i gets the same utility in both these cases. Finally, if agent i is an *on-path non-winner* when it reports r_i, but a *not-on-path non-winner* when it reports $r'_i \subset r_i$. In this case, the utility is zero when agent i is a *not-on-path non-winner*, but her utility is non-negative when she is an *on-path non-winner*. Hence, agent i cannot improve her utility in any of these cases. ■

Theorem 4. *Each auction in* **RA** *is* **DDSIC** *and IR.*

Proof sketch: Consider an arbitrary auction $f \in$ **RA**. We showed in Theorem 3 that an agent cannot manipulate the referral tree to her favor. Hence, we need to show that for the formed referral tree $\hat{T}$, f satisfies **DDSIC** and IR. We show this by proving that f satisfies each of the three conditions of **MFF** (Definition 5). ■

Since each $f \in$ **RA** is **DDSIC** and IR, they are **DBIC** and IIR. For simplicity of terminology, we will call each member of the class **RA** simply an **RA** (referral auction) henceforth.

We will now find an **RA** that maximizes the expected revenue of the seller when the valuations of the buyers are i.i.d. Briefly, we observe from Equation (5) that the revenue of the seller in any auction in **RA** is the sum of the payments made by the buyers at the first level, when their valuations are replaced with the maximum valuation in their subtree. This allows us to "replace" each buyer at the first level, with the buyer having a maximum valuation in its subtree.

5.3 Optimal referral auction for i.i.d. valuations

In the objective of finding the *revenue-optimal* mechanism on a network, we address the problem in steps. In this section, we consider the mechanisms in the class **RA**, assuming that the priors on the valuations are known to the designer and that all v_i's are i.i.d. with distribution F that follows the *monotone hazard rate* (MHR) condition, i.e., $f(x)/(1 - F(x))$ is non-decreasing in x.[4] In this section, we find the optimal mechanism for this setup. To do that, first, we need to define a *transformed auction* (**TA**) of an **RA** as follows.

Definition 8 (Transformed Auction). A *transformed auction* (**TA**) of an **RA** is the auction where each subtree $\hat{T}_i, i \in$ **children**(s) is replaced with a node with a valuation of $\max_{j\in\hat{T}_i} v_j$, and the allocation and payments are given by $(\mathcal{g}_i, \mathcal{p}_i), i \in$ **children**(s).

Note that a **TA** does not specify the allocations beyond the first level of the tree. This is because, we will only be interested in the revenue generated by a **TA**, and every **TA**, regardless of how it allocates the object and extracts payments in the subsequent levels, will earn the same revenue, as shown formally in the following result.

Lemma 4. *The revenue earned by an* **RA** *is identical to its* **TA**.

Given the above lemma, we can, WLOG, look only at the **TA**s for revenue maximization. In the **TA** of a given **RA**, the revenue maximization problem is restricted to the first level of the tree. However, the nodes of this restricted tree are the *transformed nodes* whose valuations are the maximum valuations of their respective subtrees. For notational simplicity, we use a fresh index ℓ to denote these transformed nodes at the first level, i.e., for **children**(s). The *transformed valuation* of ℓ is denoted by $v_\ell := \max_{j\in\hat{T}_\ell} v_j$. Again, to reduce notational complexity, the set of the players in this **TA** is represented by $\tilde{N} :=$ **children**(s). In the following, we state the fact that the v_ℓ's also follow the MHR property.

Fact 1. If the distribution of a finite number of i.i.d. random variables satisfies MHR condition, then the distribution of the maximum of those random variables also satisfies MHR condition.

We now focus on the revenue maximization problem. Note that, $\mathcal{g}$ and $\mathcal{p}$ are particular choices of the allocations g and p respectively. Therefore, the expected allocations and payments are given by Equations (3) and (4) with g and p replaced with $\mathcal{g}$ and $\mathcal{p}$ respectively. In particular, the **VIPC** term in Equation (4) is zero for the nodes in the **TA** since the payment $\mathcal{p}$ sets it to zero for the nodes in the first level of the class **RA**.[5] Also, in the **TA**, the **offset** is zero. Therefore, $\rho_\ell = v_\ell$, $\forall \ell \in \tilde{N}$. The neighbor component of the types r_ℓ are no longer relevant since the mechanism is restricted to the first level in the **TA**. Hence, we can reduce the arguments of $\textbf{pay}_\ell$ and α_ℓ to only v_ℓ in Equations (3) and (4). Since, the only variable parameter in the payment of the agents is the allocation function $\mathcal{g}$, the optimization problem for revenue maximization in the **RA** class is given by

$$\begin{aligned} \max \quad & \sum_{\ell\in\tilde{N}} \int_{v_\ell=0}^{b_\ell} \textbf{pay}_\ell(v_\ell) f_\ell(v_\ell)\, \mathrm{d}v_\ell \\ \text{s.t.} \quad & \mathcal{g} \text{ is monotone non-decreasing and deterministic} \end{aligned} \tag{6}$$

In the above equation, f_ℓ is the density of v_ℓ, which is assumed to have a bounded support of $[0, b_\ell]$. We will denote the corresponding distribution with F_ℓ. This optimization problem now reduces to the classic single item auction setting of Myerson [19]. Following that analysis, we find that the individual terms in the sum of the objective function of Equation (6) can be written as follows

$$\begin{aligned} &\int_{v_\ell=0}^{b_\ell} \textbf{pay}_\ell(v_\ell) f_\ell(v_\ell)\, \mathrm{d}v_\ell = \int_0^{b_\ell} w_\ell(v_\ell)\alpha_\ell(v_\ell) f_\ell(v_\ell)\, \mathrm{d}v_\ell \\ &= \int_0^{b_\ell} w_\ell(v_\ell) \left(\int_{v_{-\ell}} \mathcal{g}_\ell(v_\ell, v_{-\ell}) f_{-\ell}(v_{-\ell})\, \mathrm{d}v_{-\ell} \right) f_\ell(v_\ell)\, \mathrm{d}v_\ell \\ &= \int_v w_\ell(v_\ell) \mathcal{g}_\ell(v_\ell, v_{-\ell}) f(v)\, \mathrm{d}v. \end{aligned}$$

The expression $w_\ell(x) := x - (1 - F_\ell(x))/f_\ell(x)$ is defined as the *virtual valuation* of agent ℓ and for completeness, the derivation of the first equality is provided in [3]. The second equality holds after expanding $\alpha_\ell(v_\ell)$ from Equation (3). The last equality holds since the valuations are independent (but may not be identically distributed as the number of nodes in the subtree of ℓ can be different from that of ℓ'), and f denotes the joint probability density of $(v_\ell, v_{-\ell})$.

[4] The intuitive meaning of this condition is that the distribution is not *heavy-tailed*. Many distributions, e.g., uniform and exponential, follow the MHR condition [2].

[5] The **VIPC** term needs to be non-positive for the auction to be IIR, and since our objective is to maximize revenue, it must be zero. This is ensured by $\mathcal{p}_i$.

The objective function of Equation (6) can therefore be written

$$\int_v \left(\sum_{\ell \in \tilde{N}} w_\ell(v_\ell) g_\ell(v_\ell, v_{-\ell}) \right) f(v)\, dv.$$

The solution to the unconstrained version of the optimization problem given by Equation (6) is rather simple.

$$\begin{aligned} &\text{if } w_\ell(v_\ell) < 0, \forall \ell \in \tilde{N}, \text{ then } g_\ell(v_\ell, v_{-\ell}) = 0, \forall \ell \in \tilde{N} \\ &\text{else } g_\ell(v_\ell, v_{-\ell}) = \begin{cases} 1 & \text{if } w_\ell(v_\ell) \geqslant w_k(v_k), \forall k \in \tilde{N} \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{7}$$

The ties in $w_\ell(v_\ell)$ are broken arbitrarily. Since the distributions of $v_\ell, \ell \in \tilde{N}$ satisfy MHR, the virtual valuations, w_ℓ, are monotone non-decreasing. Also, since this mechanism breaks the tie arbitrarily in favor of an agent, the allocation is also deterministic. Therefore, the optimal solution of the unconstrained problem of Equation (6) also happens to be the optimal solution of the constrained problem. We find the payments of the winner from Equation (5) as follows.

$$\begin{aligned} \text{define } \kappa^*_\ell(v_{-\ell}) &= \inf\{y : g_\ell(y, v_{-\ell}) = 1\}, \\ p_\ell(v_\ell, v_{-\ell}) &= \kappa^*_\ell(v_{-\ell}) \cdot g_\ell(v_\ell, v_{-\ell}), \end{aligned} \tag{8}$$

where $\kappa^*_\ell(v_{-\ell})$ is the minimum valuation of agent ℓ to become the winner. Formally, we define the auction as follows.

Definition 9 (Maximum Virtual Valuation Auction (**maxViVa**)). The *maximum virtual valuation* auction is a subclass of **RA**, where the **TA**s of that subclass follow the allocation and payments given by Equations (7) and (8) respectively.

We consolidate the arguments above in the form of the following theorem.

THEOREM 5. *For agents having i.i.d. MHR valuations, the revenue-optimal* **RA** *is* **maxViVa**.

Since multiple **RA**s can reduce to the same **TA**, the revenue optimal **RA** is a class of auctions, all belonging to **RA**, that has the same **TA** given by Definition 9. Note that neither IDM nor TNM (and other mechanisms in the literature) is **maxViVa** because they do not use any priors. Therefore, the revenue-maximizing auctions in this setting are a new class of mechanisms that have not been explored in the literature.

5.4 Extension to non-i.i.d. agents and general graphs

When we migrate from i.i.d. valuations, it is not clear if the nodes in the **TA** satisfy MHR or a relatively weaker condition of *regularity* (which only requires the virtual valuations to be non-decreasing). Hence, the revenue maximization problem becomes far more challenging. We provide an experimental study in the supplementary material that shows that if the i.i.d. assumption does not hold, a special auction from the **LbLEV** class can yield more revenue than the currently known network auctions.

To generalize our results beyond **RA** for revenue maximization, we need to consider the revenue maximization problem (Equation (6)) with the constraints of **MFFE** (Definition 7). This optimization problem seems to have much less structure than that in **RA**. Therefore, we need more structural results about the $\mathbf{pay}_i$ terms so that this optimization problem can be simplified. Also, we cannot work with only **TA**s anymore since the revenue-optimal **DA** may not be reducible to a **TA** (*à la* Lemma 4). These are interesting open questions to address.

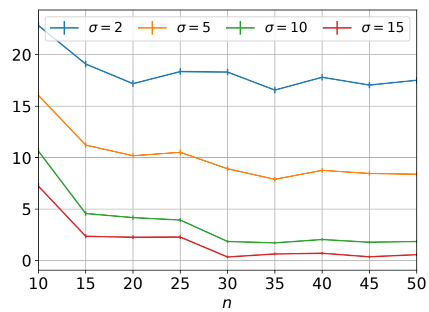

Figure 3: Percentage increase in revenue for LbLEV over IDM for different σ and *optimal* t.

6 DESIGNING DA FOR IMPROVING REVENUE

Based on our characterization results, we constructed new **DA**s that improve revenue over the currently known **DA**s. Due to space constraints, we only mention the results in a nutshell (full details in [3]). We introduce a class of **DA**s called *Level-by-Level Exponential Valuation (***LbLEV***)* that is **DDSIC** and IR. As the name suggests, the mechanism is run at every level of the referral tree $\hat{T}$ rooted at s. **LbLEV** is parametrized by a vector $t \in \mathbb{R}^n_{>0}$, where t_i denotes the *exponent* of agent i. Different choices of t for the same input instance create a class of mechanisms, and we call each of them **LbLEV**.

We generate the trees randomly. The valuations are drawn independently from $\mathcal{N}(\mu, \sigma^2)$. We assume that there are *three* classes of agents: `high`, `medium`, and `low`, having μ to be 100, 70, and 50 respectively, and the same σ. For a "suitably" chosen exponent vector t (picked when the distribution of the valuations are known, but before both the actual valuations and the tree realizes), the expected revenue of **LbLEV** performs significantly better than the currently known IDM [16] which is the only comparison candidate when the network is a referral tree (Figure 3).

7 SUMMARY AND PLANS OF EXTENSION

We provided a characterization of randomized truthful single indivisible item auctions on a network. Our results are the network counterpart of Myerson's result [19]. The question of finding the revenue optimal mechanism for a general network is still open and is an interesting future work.

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ETHICS STATEMENT

This paper adheres to the principles of research ethics, research integrity, and social responsibility. The research conducted in this paper is purely theoretical, and no human or animal subjects were involved in the research. Therefore, there were no ethical concerns related to the treatment of participants or the use of personal data. Also, designing auctions that maximize revenue is quite a standard goal both in computer science and economics. Therefore the results presented in this paper does not consider anything that adversely impact the society.

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