On the Computational Complexity of Quasi-Variational Inequalities and Multi-Leader-Follower Games

Extended Abstract

Bruce M. Kapron
University of Victoria
Victoria, Canada
bmkapron@uvic.ca

Koosha Samieefar
University of Victoria
Victoria, Canada
koosha021@gmail.com

ABSTRACT

We introduce a computational version of the generalized quasi-variational inequality problem and study its computational complexity, in particular proving that it is PPAD-complete. We also consider applications to multi-leader-follower games, a domain traditionally marked by the absence of general solutions. However, through the use of relaxation techniques, we obtain versions of these problems which may be formulated in terms of quasi-variational inequalities, allowing us to obtain PPAD-completeness for such games.

KEYWORDS

Quasi-variational inequality, Game theory, Computational complexity, Multi-Leader-Follower Games, Hardness results, Constrained equilibrium, PPAD

ACM Reference Format:
Bruce M. Kapron and Koosha Samieefar. 2024. On the Computational Complexity of Quasi-Variational Inequalities and Multi-Leader-Follower Games: Extended Abstract. In *Proc. of the 23rd International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2024), Auckland, New Zealand, May 6 – 10, 2024*, IFAAMAS, 3 pages.

1 INTRODUCTION

Quasi-variational Inequalities (QVI) are a class of mathematical problems that are a generalization of variational inequalities. Variational inequalities and quasi-variational inequalities are used in optimization theory, economics, engineering, and various other fields to model and solve a wide range of real-world problems, particularly those involving equilibrium or optimization under constraints [20, 22, 32, 35]. In a standard variational inequality, the goal is to find a vector that belongs to a given fixed set. QVI generalizes this concept by introducing a set-valued mapping that makes the feasible set dependent on the variables. These both arise in various fields such as optimization, equilibrium problems, and economics, where one seeks to find a solution that satisfies a certain inequality condition involving a set of functions or operators. For example, under some basic assumptions such as differentiability, Debreu-Rosen style games (see [12, 38, 41]) can be expressed as QVIs, aiding their analysis through variational techniques, and offering a unified framework for diverse multiplayer, non-cooperative games [35].

Proc. of the 23rd International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2024), N. Alechina, V. Dignum, M. Dastani, J.S. Sichman (eds.), May 6 – 10, 2024, Auckland, New Zealand.

Solving a QVI is typically more challenging than solving a VI due to the increased generality. Beginning with the formal definition proposed by Bensoussan and Lion ([3–6]), researchers have undertaken an analysis of algorithmic solutions, in particular conditions governing convergence. Various numerical and mathematical techniques, such as fixed-point methods, penalty methods, and projection methods, can be employed to find solutions to QVIs [2, 15, 20, 35, 39, 40]. In many cases, QVIs can be challenging to solve directly. Researchers often use regularization techniques or approximate the problem to make it more amenable to numerical methods. These techniques may find an approximate solution that is within a specified tolerance of the true solution. In this case, complexity depends on the chosen approximation accuracy and the convergence rate of the algorithm. The study of the computational aspects of quasi-variational inequalities is in its early stages of development. Specifically, the majority of research papers focus on examining whether a solution to a problem exists [7, 8, 28, 31, 36, 47]. The *generalized quasi-variational inequality* problem (GQVI) is an extension of QVI and the *generalized variational inequality* (GVI) studied in [16].

Definition 1.1. Given a correspondence (also called a *set-valued map* or a *a point-to-set map*) $\mathcal{F}$ from $\mathbb{R}^m$ into subsets of $\mathbb{R}^m$ and a correspondence $\mathcal{R}$ from $\mathbb{R}^m$ into subsets of $\mathbb{R}^m$, an ϵ-approximate solution the GQVI $(\mathcal{R}, \mathcal{F})$ tries to find two vectors $x^* \in \mathcal{R}(x^*)$ and $w^* \in \mathcal{F}(x^*)$ such that:

$$\left(y - x^*\right)^T w^* + \epsilon \geq 0, \quad \forall y \in \mathcal{R}\left(x^*\right)$$

Remark. In the QVI case, we assume that $\mathcal{F}$ is a function. In the VI case, we also assume that $\mathcal{R}(x) = \mathcal{R}$ for all x. In [8], an existence result for the GQVI was proved using the Eilenberg-Montgomery fixed point theorem (see [14]).

Problems for which Kakutani's fixed-point theorem can establish existence results, particularly those involving the games introduced by Debreu and Rosen as well as quasi-variational inequalities, have not been thoroughly explored from an algorithmic perspective. In [37], a problem called Kakutani was introduced, and a sketch of its inclusion in the complexity class PPAD was given. The main challenge in developing a general formulation of Kakutani as a computational problem is that conventional approaches for explicitly and succinctly representing a convex set, such as the convex hull of a point set or a convex polytope defined by linear inequalities, are excessively restrictive and fail to capture key practical applications of Kakutani's theorem, such as the application to games mentioned above. Recently a more suitable computational formulation of the Kakutani problem was introduced by leveraging computational convex geometry (see [21]) and was used to settle the computational complexity of finding approximate equilibrium solutions for Debreu-Rosen style games [38]. In this approach, computational problems for Kakutani's theorem and Debreu-Rosen style games are defined using linear arithmetic circuits to represent weak/strong separation oracles for convex-valued correspondences (and also convex sets) to assure consistency between their values and their gradients in order to prevent some computational challenges. Informally, strong (weak) separation oracles can verify the membership (almost membership) of a point in a correspondence (set). Linear arithmetic circuits can efficiently approximate any polynomially computable function; in particular, they can approximate polynomials. They also have a variety of useful properties such as Lipchitzness (for more information see [17] and [38]).

Multi-leader-follower games are a class of games in which multiple agents, referred to as *leaders* and *followers*, interact strategically to achieve their respective objectives. Leaders and followers often have conflicting objectives or interests, in particular aiming to maximize their own benefits or minimize their costs. The concept of multi-leader-follower games has a variety of applications that arise from situations where there are multiple oligopoly firms operating in the market [10, 24, 25, 29, 35]. Oligopoly markets are markets dominated by a small number of suppliers. The simplest form of the multi-leader-follower game is a Stackelberg game [1, 42, 46] in which one leader and multiple followers react to the leader's strategies. These games find applications in various fields, including economics, engineering, and multi-agent systems [11, 23, 34]. Traditional game theory provides solution concepts for analyzing and solving multi-leader follower games. The multi-leader-follower (L/F)-(Nash) equilibrium is a solution concept for multi-leader-follower games and can be described as a collection of strategies employed by leaders and followers. In this equilibrium, no individual player, whether a leader or a follower, can improve their utility (or minimize their *loss* or *regret* functions) by unilaterally altering their current strategy. Stackelberg games can be seen as a specific instance of mathematical programs with equilibrium constraints (MPEC) (where there is only one leader). In this case, the followers' problems are replaced by a constraint given by their optimality conditions. In a broader context, an MPEC is an optimization problem that encompasses two sets of variables, namely decision variables and response variables[18, 30, 33]. A mathematical framework commonly used to represent the multi-leader-follower game is referred to as the equilibrium problem with equilibrium constraints (EPEC). An EPEC [13, 19, 26, 27, 43, 44] is essentially an equilibrium problem composed of multiple parametric MPECs, each of which incorporates other players' strategies as parameters. Achieving equilibria in an EPEC involves solving all the embedded MPECs simultaneously.

Although the multi-leader-follower problem offers a sound mathematical framework with a clearly defined solution concept and applications, its elevated level of complexity and technical intricacies render it computationally intractable. Specifically, they resemble an equilibrium problem in a more complex form of Debreu-Rosen style games, requiring each leader to solve a non-convex mathematical program with equilibrium constraints [35, 45]. This formulation faces two significant issues: a potential absence of an equilibrium solution due to non-convexity, and computational intractability. To address these challenges, a careful analysis and choice of *remedial models* that lead to a sensible equilibrium solution were presented in [35]. Another possible approach comes from considering a class of multi-leader-follower games [25] that satisfy some particular, but still reasonable assumptions and can be formulated in terms of variational inequalities.

2 OUR CONTRIBUTION

The fundamental results of the work primarily focus on the computational complexity of finding approximate solutions to different variants of variational inequalities, namely generalized quasi-variational inequalities, quasi-variational inequalities, and variational inequalities. We demonstrate that a general formulation of all of these result in problems that are PPAD-complete[1].

Theorem 2.1 (informal). *Finding an approximate solution to computational variants of GQVI, QVI, VI are PPAD-complete where:*

- *The correspondences are convex-valued and given by linear arithmetic circuits which represent either a strong or weak separation oracle.*
- *The functions are represented by linear arithmetic circuits.*
- *The sets are convex and given by linear arithmetic circuits which represent either a strong or weak separation oracle.*

Proof sketch. We combine the techniques of [8, 38] to show the inclusion of the GQVI problem in PPAD, leveraging the computational version of Kakutani's fixed point theorem and the robust version of Berge's maximum theorem [38]. PPAD-hardness of this problem can be shown by converting a game in which finding an approximate equilibrium is hard to the QVI format[35, 38]. □

Building upon the remedial model that was introduced in [35] for multi-leader-follower games, we formulate a computational version of finding remedial equilibrium solutions in multi-leader-follower games. By using the machinery that was provided by the computational version of quasi-variational inequality, we prove the PPAD-completeness of the aforementioned computational problem.

Theorem 2.2 (informal). *Finding an approximate solution to remedial solution of [35] to an L/F equilibrium in a multi-leader follower game is PPAD-complete given the conditions of Theorem 5 of [35] and:*

- *The correspondences are convex-valued given by linear arithmetic circuits that represent a strong or weak separation oracle.*
- *The functions are represented by linear arithmetic circuits.*
- *The sets are convex given by linear arithmetic circuits which represent either a strong or weak separation oracle.*

Proof sketch. We show that this problem can be converted to a QVI problem in polynomial time. PPAD-hardness of this problem is implied by the hardness of finding a mixed Nash equilibrium (see [9]) in a game with 2 leaders where the utilities represent the expected payoff of mixed strategies and the followers have only one strategy and no restrictions.

□

3 ACKNOWLEDGEMENTS

We would like to acknowledge the support of NSERC discovery grant RGPIN-2021-02481.

[1] The formal theorems and proofs appear in the full version.

REFERENCES

[1] Tamer Başar and Geert Jan Olsder. 1998. *Dynamic Noncooperative Game Theory, 2nd Edition*. Society for Industrial and Applied Mathematics. https://doi.org/10.1137/1.9781611971132

[2] Heinz H. Bauschke, Regina S. Burachik, Patrick L. Combettes, Veit Elser, D. Russell Luke, and Henry Wolkowicz (Eds.). 2011. *Fixed-Point Algorithms for Inverse Problems in Science and Engineering*. Springer New York. https://doi.org/10.1007/978-1-4419-9569-8

[3] Alain Bensoussan and Jacques Louis Lions. 1973. Nouvelle formulation de problèmes de contrôle impulsionnel et applications.

[4] A. Bensoussan and J. L. Lions. 1975. Nouvelles Methodes en Controle Impulsionnel. *Applied Mathematics & Optimization* 1, 4 (Dec. 1975), 289–312. https://doi.org/10.1007/bf01447955

[5] A. Bensoussan and J. L. Lions. 1975. Problemes de Temps D'arret Optimaux et de Perturbations Singulieres Dans Les Inequations Variationnelles. In *Lecture Notes in Economics and Mathematical Systems*. Springer Berlin Heidelberg, 567–584. https://doi.org/10.1007/978-3-642-46317-4_40

[6] Alain Bensoussan and Jacques Louis Lions. 1982. Contrôle impulsionnel et inéquations quasi variationnelles. https://api.semanticscholar.org/CorpusID:118306059

[7] P. Bhattacharyya and V. Vetrivel. 1994. An Existence Theorem on Generalized Quasi-Variational Inequality Problem. *J. Math. Anal. Appl.* 188, 2 (1994), 610–615. https://doi.org/10.1006/jmaa.1994.1448

[8] D. Chan and J. S. Pang. 1982. The Generalized Quasi-Variational Inequality Problem. *Mathematics of Operations Research* 7, 2 (1982), 211–222. http://www.jstor.org/stable/3689542

[9] Xi Chen, Xiaotie Deng, and Shang-Hua Teng. 2009. Settling the Complexity of Computing Two-Player Nash Equilibria. *J. ACM* 56, 3 (May 2009), 1–57.

[10] Yihsu Chen, Benjamin F. Hobbs, Sven Leyffer, and Todd S. Munson. 2006. Leader-Follower Equilibria for Electric Power and NO x Allowances Markets. *Computational Management Science* 3, 4 (June 2006), 307–330. https://doi.org/10.1007/s10287-006-0020-1

[11] S. Chan Choi, Wayne S. Desarbo, and Patrick T. Harker. 1990. Product Positioning Under Price Competition. *Management Science* 36, 2 (feb 1990), 175–199. https://doi.org/10.1287/mnsc.36.2.175

[12] Gerard Debreu. 1952. A Social Equilibrium Existence Theorem. *Proceedings of the National Academy of Sciences* 38, 10 (1952), 886–893. https://doi.org/10.1073/pnas.38.10.886 arXiv:https://www.pnas.org/doi/pdf/10.1073/pnas.38.10.886

[13] Andreas Ehrenmann. 2005. *Equilibrium problems with equilibrium constraints and their applications in electricity markets*. Ph.D. Dissertation. Cambridge University.

[14] Samuel Eilenberg and Deane Montgomery. 1946. Fixed Point Theorems for Multi-Valued Transformations. *American Journal of Mathematics* 68, 2 (1946), 214–222. http://www.jstor.org/stable/2371832

[15] Francisco Facchinei and Jong-Shi Pang (Eds.). 2004. *Finite-Dimensional Variational Inequalities and Complementarity Problems*. Springer New York. https://doi.org/10.1007/b97543

[16] S. C. Fang and E. L. Peterson. 1982. Generalized variational inequalities. *Journal of Optimization Theory and Applications* 38, 3 (Nov. 1982), 363–383. https://doi.org/10.1007/bf00935344

[17] John Fearnley, Paul W. Goldberg, Alexandros Hollender, and Rahul Savani. 2021. The complexity of gradient descent: CLS = PPAD ∩ PLS. In *Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing*. ACM. https://doi.org/10.1145/3406325.3451052

[18] M Ferris, S Dirkse, and A Meeraus. 2002. *Mathematical programs with equilibrium constraints: automatic reformulation and solution via constrained optimization*. Technical Report.

[19] Steven A. Gabriel, Antonio J. Conejo, J. David Fuller, Benjamin F. Hobbs, and Carlos Ruiz. 2012. Equilibrium Problems with Equilibrium Constraints. In *Complementarity Modeling in Energy Markets*. Springer New York, 263–321. https://doi.org/10.1007/978-1-4419-6123-5_7

[20] F. Giannessi and A. Maugeri (Eds.). 1995. *Variational Inequalities and Network Equilibrium Problems*. Springer US. https://doi.org/10.1007/978-1-4899-1358-6

[21] Martin Grötschel, László Lovász, and Alexander Schrijver. 1993. *Geometric Algorithms and Combinatorial Optimization*. Springer Berlin Heidelberg. https://doi.org/10.1007/978-3-642-78240-4

[22] Jaroslav Haslinger, Markku Miettinen, and Panagiotis D. Panagiotopoulos. 1999. *Finite Element Method for Hemivariational Inequalities*. Springer US. https://doi.org/10.1007/978-1-4757-5233-5

[23] B.E. Hobbs. 2001. Linear complementarity models of Nash-Cournot competition in bilateral and POOLCO power markets. *IEEE Transactions on Power Systems* 16, 2 (2001), 194–202. https://doi.org/10.1109/59.918286

[24] B.F. Hobbs, C.B. Metzler, and J.-S. Pang. 2000. Strategic gaming analysis for electric power systems: an MPEC approach. *IEEE Transactions on Power Systems* 15, 2 (2000), 638–645. https://doi.org/10.1109/59.867153

[25] Ming Hu and Masao Fukushima. 2011. Variational Inequality Formulation of a Class of Multi-Leader-Follower Games. *Journal of Optimization Theory and Applications* 151, 3 (Aug. 2011), 455–473. https://doi.org/10.1007/s10957-011-9901-8

[26] Xinmin Hu. 2003. *Athematical programs with complementarity constraints and game theory models in electricity markets*. Ph.D. Dissertation. university of Melbourne.

[27] Xinmin Hu and Daniel Ralph. 2007. Using EPECs to Model Bilevel Games in Restructured Electricity Markets with Locational Prices. *Operations Research* 55, 5 (Oct. 2007), 809–827. https://doi.org/10.1287/opre.1070.0431

[28] Michal Kočvara and Jiří V. Outrata. 1995. On a class of quasi-variational inequalities. *Optimization Methods and Software* 5, 4 (1995), 275–295. https://doi.org/10.1080/10556789508805617 arXiv:https://doi.org/10.1080/10556789508805617

[29] Sven Leyffer and Todd Munson. 2010. Solving multi-leader–common-follower games. *Optimization Methods and Software* 25, 4 (aug 2010), 601–623. https://doi.org/10.1080/10556780903448052

[30] Zhi-Quan Luo, Jong-Shi Pang, and Daniel Ralph. 1996. *Mathematical Programs with Equilibrium Constraints*. Cambridge University Press. https://doi.org/10.1017/cbo9780511983658

[31] Monica Milasi. 2013. Existence theorem for a class of generalized quasi-variational inequalities. *Journal of Global Optimization* 60, 4 (Oct. 2013), 679–688. https://doi.org/10.1007/s10898-013-0114-6

[32] D. Motreanu and P. D. Panagiotopoulos. 1999. *Minimax Theorems and Qualitative Properties of the Solutions of Hemivariational Inequalities*. Springer US. https://doi.org/10.1007/978-1-4615-4064-9

[33] Jiři Outrata, Michal Kočvara, and Jochem Zowe. 1998. *Nonsmooth Approach to Optimization Problems with Equilibrium Constraints*. Springer US. https://doi.org/10.1007/978-1-4757-2825-5

[34] Jiří V. Outrata. 2004. A note on a class of equilibrium problems with equilibrium constraints. *Kybernetika* 40, 5 (2004), [585]–594. http://eudml.org/doc/33721

[35] Jong-Shi Pang and Masao Fukushima. 2005. Quasi-variational inequalities, generalized Nash equilibria, and multi-leader-follower games. *Computational Management Science* 2, 1 (Jan. 2005), 21–56. https://doi.org/10.1007/s10287-004-0010-0

[36] Jong-Shi Pang and Jen-Chih Yao. 1995. On a Generalization of a Normal Map and Equation. *SIAM Journal on Control and Optimization* 33, 1 (1995), 168–184. https://doi.org/10.1137/S0363012992241673 arXiv:https://doi.org/10.1137/S0363012992241673

[37] Christos H Papadimitriou. 1994. On the Complexity of the Parity Argument and Other Inefficient Proofs of Existence. *J. Comput. Syst. Sci.* 48, 3 (June 1994), 498–532.

[38] Christos H. Papadimitriou, Emmanouil-Vasileios Vlatakis-Gkaragkounis, and Manolis Zampetakis. 2022. The Computational Complexity of Multi-player Concave Games and Kakutani Fixed Points. https://doi.org/10.48550/ARXIV.2207.07557

[39] B. Perthame. 1985. Some remarks on quasi-variational inequalities and the associated impulsive control problem. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire* 2, 3 (June 1985), 237–260. https://doi.org/10.1016/s0294-1449(16)30404-8

[40] R. Tyrrell Rockafellar and Roger J. B. Wets. 1998. *Variational Analysis*. Springer Berlin Heidelberg. https://doi.org/10.1007/978-3-642-02431-3

[41] J. B. Rosen. 1965. Existence and Uniqueness of Equilibrium Points for Concave N-Person Games. *Econometrica* 33, 3 (1965), 520–534. http://www.jstor.org/stable/1911749

[42] Hanif D. Sherali. 1984. A Multiple Leader Stackelberg Model and Analysis. *Operations Research* 32, 2 (apr 1984), 390–404. https://doi.org/10.1287/opre.32.2.390

[43] Che-Lin Su. 2005. *EQUILIBRIUM PROBLEMS WITH EQUILIBRIUM CONSTRAINTS:STATIONARITIES, ALGORITHMS, AND APPLICATIONS*. Ph.D. Dissertation. STANFORD UNIVERSITY.

[44] Che-Lin Su. 2007. Analysis on the forward market equilibrium model. *Operations Research Letters* 35, 1 (2007), 74–82. https://doi.org/10.1016/j.orl.2006.01.006

[45] Luis N. Vicente and Paul H. Calamai. 1994. Bilevel and multilevel programming: A bibliography review. *Journal of Global Optimization* 5, 3 (Oct. 1994), 291–306. https://doi.org/10.1007/bf01096458

[46] Heinrich von Stackelberg. 1952. *The Theory of Market Economy*. Oxford University Press.

[47] Jen-Chih Yao. 1991. The generalized quasi-variational inequality problem with applications. *J. Math. Anal. Appl.* 158, 1 (1991), 139–160. https://doi.org/10.1016/0022-247X(91)90273-3