Fairness in Repeated House Allocation

Extended Abstract

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ABSTRACT

This article considers a house allocation setting–where exactly one object has to be assigned to each agent–in a repeated context, where the same allocation problem is decided multiple times while taking previous decisions into account. Since fairness can be rarely achieved in a one-shot decision, we study whether fairness over time can be reached. In particular, we introduce several fairness criteria and investigate whether they can be satisfied in our repeated house allocation setting. While we show that most related decision problems are hard in general, we identify restricted positive cases.

KEYWORDS

House allocation; Fairness; Repeated decision

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1 INTRODUCTION

Fair division [4, 9] is a key issue which raises many technical challenges while it tackles concrete societal problems. House allocation [1, 7], where each agent receives exactly one object, is one of the simplest settings while it still captures many real-world problems. Ensuring fairness is usually difficult in house allocation. For instance, the classical notion of *envy-freeness* [6, 10], is highly constraining in house allocation, with as many objects as agents.

However, there are many contexts where the decision on the allocation has to be repeated frequently. Typical examples are the assignment of courses to teachers, or time slots to workers. In such contexts, the set of agents and resources does not necessarily change over time but a series of allocations must be implemented at different time steps. This repetition of the decision process can be the occasion to circumvent fairness impossibilities that arise in a deterministic one-shot decision setting.

In this article, we study how fairness can be measured in repeated house allocation and how hard it is to make fair decisions. Concretely, we consider a house allocation setting, where the agents, the resources, and the agents' preferences over the resources, do not evolve over time. We explore whether we can plan the next allocations, where we assign exactly one object per agent, for a *given finite* number of next steps, and possibly a history of previous allocations, in such a way that the global allocation sequence achieves fairness over time. Ideally, one might want to use solutions from random assignments [1, 3], however these solutions may not be implementable within our finite predefined horizon and with efficient decisions at each step. While several recent works have investigated fair division over time [5, 8], they focus on additive utilities for agents' preferences. In contrast, we focus on the standard ordinal house allocation setting, which makes a significant difference in the kind of relevant fairness notions.

We study three fairness criteria that focus on different aspects of fairness over time. First, we introduce a new fairness notion called *mirrored envy*, which imposes symmetry in envy between agents. Our second notion, *equal treatment of equals*, has been mostly used in random assignments [3], and requires that similar agents should be treated the same way. Finally, we try to limit the number of times an agent is envious towards the same agent, by minimizing the maximum *cumulative envy* between any pair of agents.

2 FAIRNESS MEASURES

We are given a set N of n agents, and a set M of n objects. Each agent $i \in N$ has strict ordinal preferences over the objects, represented by a linear order $\succ_i$ over M. A preference profile $\succ$ is the set of all linear orders $\succ_i$ for all agents $i \in N$. An allocation A is a bijection $A : N \to M$. We denote the object assigned to agent $i \in N$ in allocation A by $A(i)$. An allocation A is *Pareto-optimal* if there does not exist another allocation A' such that $A'(i) \succeq_i A(i)$ for every agent $i \in N$ and there exists an agent i such that $A'(i) \succ_i A(i)$.

An allocation sequence $\mathcal{A}$ over T finite rounds is a sequence of T allocations, i.e., $\mathcal{A} = \langle A^1, \ldots, A^T \rangle$. The multiset union is represented by the symbol $\uplus$. For two allocation sequences $\mathcal{A}_1 = \langle A_1^1, \ldots, A_1^{T_1} \rangle$ and $\mathcal{A}_2 = \langle A_2^1, \ldots, A_2^{T_2} \rangle$, let $\mathcal{A}_1 + \mathcal{A}_2$ be the concatenation of the two sequences, i.e., $\mathcal{A}_1 + \mathcal{A}_2 := \langle A_1^1, \ldots, A_1^{T_1}, A_2^1, \ldots, A_2^{T_2} \rangle$.

An agent i *envies* another agent j in allocation A if $A(j) \succ_i A(i)$. For an allocation sequence $\mathcal{A} = \langle A^1, \ldots, A^T \rangle$, let the accumulated envy of agent i towards agent j be denoted by $e^{\mathcal{A}}(i, j)$, i.e., $e^{\mathcal{A}}(i, j) := |\{t \in [T] : A^t(j) \succ_i A^t(i)\}|$. We introduce a notion which aims to satisfy symmetry in envy for every pair of agents.

Definition 2.1 (Mirrored envy). An allocation sequence $\mathcal{A} = \langle A^1, \ldots, A^T \rangle$ is said to satisfy mirrored envy if $e^{\mathcal{A}}(i, j) = e^{\mathcal{A}}(j, i)$, for every pair of agents i and j.

Another way to look at fairness is to ensure that agents with the same preference are treated equally.

Definition 2.2 (Equal treatment of equals). An allocation sequence $\mathcal{A} = \langle A_1, \ldots, A^T \rangle$ satisfies equal treatment of equals if $\biguplus_{t \in [T]} A^t(i) = \biguplus_{t \in [T]} A^t(j)$ for every agents i and j with the same preferences.

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We also explore the possibility to guarantee the agents a maximum number of times they are envious towards the same agent.

Definition 2.3 (Cumulative envy). The cumulative envy of an allocation sequence $\mathcal{A} = \langle A^1, \ldots, A^T \rangle$ is given by $\max_{(i,j)\in N^2} e^{\mathcal{A}}(i, j)$.

We are searching for allocation sequences that minimize the cumulative envy in order to keep the total amount of envy low.

3 COMPUTATIONAL RESULTS

In this article, we focus on computational issues for the fairness concepts we address. We combine our fairness criteria with the basic efficiency requirement of Pareto-optimality, and consider in general that some steps may have previously occurred.

Mirrored Envy Completion	
Instance:	$(N, M, \succ)$, allocation sequence $\mathcal{A}_1$, integer T
Question:	Does there exist a sequence $\mathcal{A}_2 = \langle A_2^1, \ldots, A_2^T \rangle$ such that $\mathcal{A}_1 + \mathcal{A}_2$ satisfies mirrored envy and every allocation A_2^t is Pareto-optimal for $t \in [T]$?

Firstly, Mirrored Envy Completion is solvable in polynomial time when we have only one available step.

Theorem 3.1. *Mirrored Envy Completion can be solved in polynomial time when $T \leq 1$.*

However, starting from a horizon of two steps, the problem becomes computationally hard.

Theorem 3.2. *Mirrored Envy Completion is NP-complete even when $T = 2$ and $\mathcal{A}_1$ is empty.*

Despite this hardness result, the problem is polynomial-time solvable for two steps when all agents have the same preferences.

Theorem 3.3. *Mirrored Envy Completion can be solved in polynomial time when $T = 2$ and all agents have the same preferences.*

The case of three steps and equal preferences remains open.

Equal Treatment of Equals Completion	
Instance:	$(N, M, \succ)$, sequence $\mathcal{A}_1 = \langle A_1^1, \ldots, A_1^{T_1} \rangle$, integer T
Question:	Does there exist a sequence $\mathcal{A}_2 = \langle A_2^1, \ldots, A_2^T \rangle$ such that $\mathcal{A}_1 + \mathcal{A}_2$ satisfies equal treatment of equals and every allocation A_2^t is Pareto-optimal for $t \in [T]$?

Let $EQ(N)$ be the set of maximal subsets of agents with the same preferences, i.e., for all $N' \in EQ(N)$, $\succ_i = \succ_j$ and $\succ_i \neq \succ_k$ for every $i, j \in N'$ and $k \in N \setminus N'$. To ensure equal treatment of equals, for every $N' \in EQ(N)$, every agent in N' has to get every object the same amount of times as the other agents in N'. We first observe that, when allocation sequence $\mathcal{A}_1$ is empty, if there exists a group $N' \in EQ(N)$ such that $|N'| > T$, then there exists no allocation sequence that can ensure equal treatment of equals. We use this observation to derive a positive result for two rounds.

Proposition 3.4. *Equal Treatment of Equals Completion can be solved in polynomial time when $T \leq 2$ and $\mathcal{A}_1$ is empty.*

While the exact complexity of Equal Treatment of Equals Completion has eluded us so far, we suspect the problem to be NP-complete even for three rounds and no previous rounds.

When forgetting about the Pareto-optimality requirement, observe that the Equal Treatment of Equals Completion problem boils down to solving a system of integer linear equations, which is in general an NP-hard problem. Indeed, the goal is to fill an $n \times n$ scaled bistochastic matrix, giving the number of times each agent i should get each object j, and where each row and each column needs to sum to the number of rounds T. Then, it suffices to apply the Birkhoff algorithm, which runs in polynomial time.

Thanks to this formulation, we can derive a polynomial-time algorithm with respect to some condition on the sizes of the groups.

Proposition 3.5. *If $T < \min\{|C| : C \in EQ(N), |C| > 1\}$ and we ignore Pareto-optimality, then Equal Treatment of Equals Completion can be solved in polynomial time.*

It remains unclear whether the complexity remains the same when requiring Pareto-optimality. Even if the Birkhoff algorithm efficiently computes a sequence of allocations from any scaled bistochastic matrix, these allocations may not be Pareto-optimal. It is, in fact, NP-hard to determine whether a scaled bistochastic matrix can be decomposed into Pareto-optimal allocations [2].

Cumulative Envy Bound	
Instance:	$(N, M, \succ)$, integer T, integer B
Question:	Does there exist a sequence $\mathcal{A} = \langle A^1, \ldots, A^T \rangle$ such that $e^{\mathcal{A}}(i, j) \leq B$, for every agents i and j, and every allocation A^t is Pareto-optimal for $t \in [T]$?

Firstly, we can show that an allocation sequence can be easily constructed where every agent is envious towards any agent a maximum number of times equal to half the number of rounds.

Proposition 3.6. *There always exists an allocation sequence $\mathcal{A}$ such that the cumulative envy in $\mathcal{A}$ does not exceed $\lceil \frac{T}{2} \rceil$ for T rounds.*

However, there does not always exist a sequence with a lower cumulative envy, e.g., when the agents have the same preferences. In general, deciding whether there can be a lower cumulative envy k is computationally hard, even for $k = \frac{T}{3}$.

Theorem 3.7. *Cumulative Envy Bound is NP-complete, even when $T = 3$ and $B = 1$.*

4 PERSPECTIVES

While our new fairness concepts seem promising, for most of them, an allocation satisfying them is hard to find. We have nevertheless identified some restricted easy cases. Future research might be able to find approximation schemes or relaxations of the proposed properties. An example could be to relax mirrored envy so as to allow some margin in the difference of envy in agent pairs. Moreover, one could think about redefining equal treatment of equals to allow more flexibility in the similar treatment of equals while enlarging the scope of agents who should be treated the same way. Finally, a possibility would be to integrate more "online" components in the model, by taking into account some variability in the set of agents or resources, or assuming that preferences can be altered over time.

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