# Optimal Majority Rules and Quantitative Condorcet Properties of Setwise Kemeny Voting Schemes

Extended Abstract

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## ABSTRACT

The Kemeny problem consists of computing consensus rankings of an election with respect to the Kemeny voting rule, admits important applications in biology and computational social choice [1, 2, 4, 6]. The problem was generalized recently via an interesting setwise approach by Gilbert et al. [9, 10] where not only pairwise comparisons but also the discordance between the winners of subsets of three candidates are also taken into account. We elaborate an exhaustive list of quantified axiomatic properties such as the Condorcet and Smith criteria, the 5/6-majority rule, and the Unanimity property of the 3-wise Kemeny rule. Since the 3-wise Kemeny problem is NP-hard, our results also provide some of the first useful search space reduction techniques by determining the relative orders of pairs of alternatives. Our works suggest similar interesting properties of higher setwise Kemeny voting schemes which justify the more expensive computational cost than the classical Kemeny scheme. We also establish optimal quantitative extensions of the Unanimity property and the well-known 3/4-majority rule of Betzler et al. [4] for the classical Kemeny problem.

## KEYWORDS

Kemeny problem; generalized Kendall-tau distance; consensus ranking; majority rule; computational social choice

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## 1 INTRODUCTION

We consider through out an election with a finite collection $C$ of $n = |C|$ candidates together with a voting profile $V$ consisting of a finite number of votes which are not necessarily distinct. A ranking or a vote is a complete and strict total ordering which we identify with a permutation of elements of $C$. Let $x, y$ be two candidates. If $x$ is ranked before $y$ in a vote $\pi$ then we write $x > y$ in $\pi$. We denote $x \geq_s y$, resp. $x >_s y$, if $x > y$ in at least, resp. in more than, $s|V|$ votes. Among natural distances in the space of all votes, the *Kendall-tau distance*, which is also the bubble-sort distance between two permutations, is one of the most prominent distances that counts the number of order disagreements between pairs of elements in two permutations. More generally, the $k$-*wise Kendall-tau distance* associated with the $k$-*wise Kemeny rule* recently introduced in [10] takes into account the disagreements between the winners of subsets of at most $k$ candidates where $k \geq 2$ is an integer. Let $S(C)$ be the set of all votes. Let $\Delta^k(C) \subseteq 2^C$ be the collection of all subsets containing at most $k$ candidates. Following [10], the $k$-wise Kendall-tau distance between two rankings $\pi, \sigma$ is $d^k(\pi, \sigma) = \sum_{E \in \Delta^k(C)} \left(1 - \delta_{\mathrm{top}_E(\pi), \mathrm{top}_E(\sigma)}\right)$ where $\mathrm{top}_E(\pi)$ denotes the highest ranked candidate in the induced ranking $\pi|_E$ on $E$ and $\delta_{x,y} = 1$ if $x = y$ and $\delta_{x,y} = 0$ otherwise. We say that a vote $\pi$ is a $k$-*wise median* of the election if $d^k(\pi, V) = \min_{\sigma \in S(C)} \sum_{\eta \in V} d^k(\sigma, \eta)$. For $k = 2$, we recover the classical Kendall-tau distance. In the Kemeny problem [11], [12], [19], the goal is to determine the set of medians according to the $k$-wise Kemeny voting scheme. Hence, a 2-wise median is simply a ranking that maximizes the number of pairwise agreements with the voting profile. The Kemeny rule is a maximum likelihood estimator of the correct ranking [11] and admits many important applications in biology and computational social choice [1, 2, 4, 6]. However, the 3-wise Kemeny rule may be more suitable than the 2-wise Kemeny rule in several situations since it puts more weight on candidates who are frequently ranked in top positions in the votes. Indeed, typical voters in real-world settings only pay attention to a shortlist of their preferred candidates and normally put a somewhat arbitrary order for the rest of the candidates. Thus, we should reduce the weight of the duel wins among non-preferred candidates of each vote. A possible solution for this problem is provided by the 3-wise Kemeny rule as the weight of the duel $x$ vs $y$ in a vote $LxKyR$ is at least $|K \cup R|$ by considering subsets of the form $\{x, y, z\}$ where $z \in K \cup R$. In a similar spirit, the weighted Kendall-tau distances were introduced and studied in [13].

Unfortunately, the decision variant of the $k$-wise Kemeny problem is NP-complete for every constant $k \geq 2$ as shown in [3, 8, 10] which motivates the active research area of computing the medians in the last decades. Our main goal is to formulate new and asymptotically optimal quantitative majority rules in the $k$-wise Kemeny rule when $k \in \{2, 3\}$. These results provide some more refined search space reduction techniques to the Kemeny problem than several existing techniques including notably the Unanimity property, the Condorcet criterion, the Smith criterion, and the $s$-majority rule.



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## 1.1 Results overview

Table 1 summarizes the main results and compares various search space reduction criteria for the $k$-wise Kemeny rule with $k \in \{2, 3\}$. We refer to the full paper [15] for experimental results and the criteria that are not defined in the present extended abstract.

**Table 1: Quantitative properties of $k$-wise Kemeny scheme**

| Criterion | 2-wise rule | 3-wise rule |
|---|---|---|
| Monotonicity | Yes | Yes [10] |
| Condorcet loser | Yes | No [15] |
| Reversal symmetry | Yes | No [15] |
| Majority criterion | Yes | Yes [15] |
| Extended Unanimity | Yes [15] | Yes [15] |
| Condorcet | Yes [7] | No [10] |
| 2/3-Condorcet | Yes | No [15] |
| 3/4-Condorcet | Yes | Yes [15] |
| Smith | Yes [17] | No [15] |
| 2/3-Smith | Yes | No [15] |
| 3/4-Smith | Yes | Yes [15] |
| 3/4-Smith-IIA | Yes | No [15] |
| Smith-IIA | Yes | No [15] |
| Extended Condorcet | Yes [18] | No [15] |
| 3/4-Extended Condorcet | Yes | Yes [15] |
| 3/4-majority | Yes [4] | For $n \leq 5$ [16] |
| Extended $s$-majority | Yes [15] | For $s \geq 5/6$ [15] |
| Major Order | Yes [14] | No [16] |
| 3-wise Major Order | n/a | Yes [16] |

For $k \geq 2$, the Unanimity property [5], [10, Proposition 5] guarantees that if a candidate $x$ is ranked before another candidate $y$ in *every* vote then $x > y$ in every $k$-wise median. However, we can relax the above extreme condition and show that if $x >_\alpha y$ where $\alpha = 1 - \frac{1}{n}$ then $x > y$ in every 2-wise median. Our result is asymptotically optimal since for every even $n \geq 2$ and every $0 \leq \alpha \leq 1 - \frac{2}{n}$, there exists an election over $n$ candidates and a pair of candidates $(x, y)$ such that $x \geq_\alpha y$ but $x < y$ in every 2-wise median [14, Proposition 1]. Similarly, we show that if $x >_\alpha y$ where $\alpha = 1 - \frac{1}{n^2-3n+4}$ then $x > y$ in every 3-wise median.

In contrast to the 2-wise Kemeny rule, the 3-wise Kemeny rule does not satisfy the *Condorcet criterion* [7] since a *Condorcet winner* might not be the winner in some 3-wise median [10, Proposition 3]. A candidate $x$ is a Condorcet winner if $x >_{1/2} y$ for every candidate $y \neq x$. Surprisingly, we show that a candidate may still lose in some 3-wise median despite wining the 2/3 majority in every duel. Hence, it is harder for a candidate to win the election with the 3-wise Kemeny rule than with the 2-wise Kemeny rule. Nevertheless, we show that a candidate obtaining a 3/4 majority in every duel must be the unique winner in the 3-wise Kemeny voting scheme. More precisely, if $x$ satisfies $x >_\alpha y$ for every $y \neq x$ where $\alpha = \frac{3n-5}{4n-6}$, then $x$ is ranked first in every 3-wise median.

The *Smith criterion* [17] and the *Extended Condorcet criterion* [18] are stronger than the Condorcet criterion. A *Smith set* is the smallest non-empty subset $S \subseteq C$ such that $x >_{1/2} y$ for every $x \in S$ and $y \in C \setminus S$. An election satisfies the Smith criterion if the winner in every median belongs to the Smith set. An election satisfies the Extended Condorcet criterion if for every partition $C = I \cup J$ with $x >_{1/2} y$ whenever $x \in I$ and $y \in J$, we must have $x > y$ in every median for all $x \in I$ and $y \in J$. While the Kemeny rule satisfies the Smith and Extended Condorcet criteria [17, 18], the 3-wise Kemeny rule fail both of them [15]. However, we obtain the following 3-wise 3/4-Smith and 3/4-Extended Condorcet criteria. If $C = I \cup J$ is a partition such that $0 < 2|I| \leq |J| + 4$ and $x \geq_{3/4} y$ for all $x \in I$ and $y \in J$, then in every 3-wise median, $x > y$ for all $x \in I$ and $y \in J$.

Following [4], a candidate $x$ is *non-dirty* with threshold $s$ if either $x \geq_s y$ or $y \geq_s x$ for every candidate $y \neq x$. An election satisfies the *$s$-majority rule* if for any non-dirty candidate $x$ with threshold $s$, we have $x > y$ in every median whenever $x \geq_s y$. Recent results in [16] show that for the 3-wise Kemeny rule, the 3/4-majority rule is only valid in general when $n \leq 5$. In contrast, it is well-known [4] that the Kemeny rule satisfies the 3/4-majority rule. By taking into account the number $n$ of candidates, we extend and improve the 3/4-majority rule by showing that for every $s < 3/4$, the Kemeny rule satisfies the $s$-majority rule whenever $n < \frac{6-5s}{4(3-4s)}$. For instance, when $n = 14$, we can take $s = 0.74 < 3/4$. The counterexample to the 2-wise $s$-majority rule given in [4, Proposition 1] requires at least $n \geq \frac{3s}{3-4s}$ candidates. Therefore, our extension of the $s$-majority rule is also asymptotically optimal for the Kemeny rule. Beside results for the 3-wise 5/6-majority rule obtained in [15] without restriction on the number of candidates, we prove the following when $n \leq 6$: for a 3-wise median $\pi$ and a non-dirty candidate $x$ with threshold 5/6 such that $z > x$ in $\pi$ whenever $z \geq_{5/6} x$, then we must have $x > y$ in $\pi$ for every $y \neq x$ with $x \geq_{5/6} y$.

## 2 CONCLUSION

In the full version of this paper [15], we study at length the fundamental Condorcet properties of the 3-wise Kemeny rule in comparison to the classical Kemeny rule. Quantitatively, we show that the 3-wise Kemeny rule presents stronger manipulation-proof properties than the Kemeny rule. We thereby further confirm the interests and the advantages of the setwise approach proposed in [10]. As the Unanimity property, the Smith criterion, the Extended Condorcet criterion, and the $s$-majority rule are particularly useful for real-world data where the positions of the candidates tend to be stable among the votes, our results for such criteria should find a wide range of applications. It is worth noting that recent algorithms based on the so-called 3-wise Major Order theorems [16] provide significant search space reductions for the 3-wise Kemeny problem.

Future research directions include quantifying the conditions under which the 3-wise Kemeny rule is more suitable than other $k$-wise rules, determining the best $k$-wise Kemeny rule according to a given set of criteria, and establishing new techniques exploiting different theoretical points of view and heuristics (e.g. determining the relative order of a pair of candidates using not only the head-to-head majority but also the interaction between 3 or more candidates).

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