# Cournot Queueing Games with Applications to Mobility Systems

Extended Abstract

Matthew Sheldon
ms4520@ic.ac.uk
Imperial College London
London, United Kingdom

Dario Paccagnan
d.paccagnan@ic.ac.uk
Imperial College London
London, United Kingdom

Giuliano Casale
gcasale@ic.ac.uk
Imperial College London
London, United Kingdom

## ABSTRACT

In this paper, we introduce a novel class of non-cooperative games, in which a closed network of queues is shared between multiple players. Each player receives a reward based on the throughput of their jobs, while they incur a cost that varies with the number of jobs they submit. Closed queueing networks are a commonly used stochastic formalism that model a variety of real-world situations, and this paper presents an application to competitive vehicle-sharing systems. In our vehicle-sharing system model, each provider receives revenue from each trip, but pays a cost based on the total number of vehicles. The core technical results of this paper include conditions that guarantee the existence of a pure Nash equilibrium, and an efficient equilibrium-finding algorithm. We apply this model to a case study of a hypothetical vehicle-sharing system in Oslo. The results indicate that adding a single competitor can increase the number of trips taken by up to 14.1%, adding two competitors can increase the amount by up to 18.9%, and a highly competitive market can increase this by up to 30%.

## KEYWORDS

Vehicle Sharing, Rational Queueing, Queueing Games, Mobility

**ACM Reference Format:**

Matthew Sheldon, Dario Paccagnan, and Giuliano Casale. 2024. Cournot Queueing Games with Applications to Mobility Systems: Extended Abstract. In *Proc. of the 23rd International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2024), Auckland, New Zealand, May 6 – 10, 2024*, IFAAMAS, 3 pages.

## 1 INTRODUCTION

The rise of vehicle-sharing systems has led to situations in which multiple providers must compete for ridership in a given area. Many cities feature multiple rideshare providers such as Uber, Bolt, or Lyft. The city of Milan has three distinct bikesharing providers, the city of Bangkok has four, and the city of Berlin has seven [1]. In order to accurately understand the effect of competition among providers, policy makers and operators may wish to analyze these transportation systems from a game-theoretic perspective.

The use of steady-state queueing models to analyze the performance of a vehicle-sharing system is well-established [3, 8, 9]. In [2] and [8], points at which customers can access vehicles are modeled as single-server queues within a queueing network. Transportation between stations is represented by infinite-server (IS) queues [2]. However a notable drawback of these models is that they are non-competitive, and there is a single provider that controls the entire network. To analyze competitive systems, we define and analyze a class of games in which providers decide the number of jobs that they submit, at cost, into a closed queueing network. We establish conditions for equilibrium, and introduce an algorithm for equilibrium finding. Then, we present an application of this model to an empirical case study, involving a vehicle-sharing system in Oslo with multiple competing providers.

## 2 GAME FORMULATION AND EQUILIBRIA

Definition 1. *We define a Cournot queueing game as follows,*

- *A set of $R$ players*
- *A set of strategies* $S = [x_1^{\min}, x_1^{\max}] \times [x_2^{\min}, x_2^{\max}] \times \ldots \times [x_R^{\min}, x_R^{\max}]$, *where* $[x_r^{\min}, x_r^{\max}] \subset \mathbb{N}$
- *A single-class, closed queueing network containing* $-/M/c$ *and* $-/G/\infty$[1] *queues.*
- *A fixed background quantity* $q \in \mathbb{N}$, *which represents a constant quantity of jobs in the network that is independent of the strategy of any player*
- *For each player, a function* $\psi_r(x_r) : \mathbb{N} \to \mathbb{R}$, *which represents the cost of submitting* $x_r$ *jobs into the system.*

In our competitive vehicle-sharing system model, each distinct provider is modeled by a player. The physical network is modeled by a stochastic queueing network, where $-/M/1$ queues represent stations at which customers can access vehicles, and $-/G/\infty$ delays represent the travel time between stations. Each job is a vehicle, and the service rate at an $-/M/1$ queue $i$, $\mu_i$, represents the exponentially-distributed rate at which customers arrive to use a vehicle [2, 6, 8]. In non-trivial networks, each queue will be empty with some non-zero probability, which causes lost trips and reduces throughput. Therefore, the goal of each provider $r$ is to maximize their throughput, less their cost $\psi_r$, by deciding their fleet size $x_r$.

The strategy of each player is represented by a scalar value $x_r \in \mathbb{N} \cap [x_r^{\min}, x_r^{\max}]$, which represents the number of jobs they submit into the system. Where $T$ represents the system throughput and $x_{-r} = \sum_{s \neq r} x_s$, the utility for player $r$ is,

$$U_r(x_r, x_{-r}) = \begin{cases} \frac{x_r}{x_r + x_{-r} + q} T(x_r + x_{-r} + q) - \psi_r(x_r) & x_r \neq 0 \\ 0 & x_r = 0 \end{cases} \quad (1)$$

Assumption 1. $\psi_r$ *is convex over* $x_r$.

[1] By Kendall's notation, $-/M/c$ means a queue with exponentially-distributed service and c servers, and $-/G/\infty$ means a pure delay (otherwise known as an infinite-server queue) with no queueing effects and a general service distribution.



*Proc. of the 23rd International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2024), N. Alechina, V. Dignum, M. Dastani, J.S. Sichman (eds.), May 6 – 10, 2024, Auckland, New Zealand.* 

The next assumption contains two conditions that, along with Assumption 1, guarantee the existence of an equilibrium in the price-inhomogeneous case. The first, which we title non-monopoly, ensures that no player can submit more than the majority of jobs, plus one. The second, which we term light-load, is consistent with a lightly-loaded queueing network.

Assumption 2. *The strategy set* $\mathrm{S}$ *is restricted such that at least one of these two conditions hold for all possible pure strategy profiles* $\mathbf{x}$, *where* $Z(N)$ *is the response rate* $T(N)/N$,

- *(non-monopoly)* $x_r \leq x_{-r} + q + 1$, *for all players* $r$, *or*
- *(light load)* $(\sum x_s + 1)\Delta^2 Z(q + \sum x_s) + \Delta Z(q + \sum x_s) \leq 0$

Theorem 1. *[7] Let* $G$ *be a Cournot queueing game, in which all players have identical service rates and routing probabilities, that either meets Assumptions 1 and 2, or has a shared price* $p$ *where all pricing functions are equal to* $\psi_r(x_r) = px_r$. *Then there is at least one Nash equilibrium among pure strategies.*

# 3 EQUILIBRIUM COMPUTATION

Next, we present results that establish the existence of a polynomial-time algorithm for equilibrium computation, if the pricing functions are convex. Equilibria are found via backward reply correspondences [4, 5] from the total quantity $Q$. The first result, Theorem 2, introduces necessary and sufficient conditions for a strategy profile to be an equilibrium, in terms of the total quantity $Q$.

Theorem 2. *[7] Let Assumption 1 be true. Then, the following, together, are necessary and sufficient conditions for* $x_r$ *to be the least best response of* $x_{-r} + q$, $\underline{b}_r(x_{-r} + q)$, *where* $Q = x_r + x_{-r} + q$,

1. $\Delta_{x_r} U_r \leq 0$, *if* $x_r < x_r^{\max}$
2. $x_r \in [x_r^{\min}, x_r^{\max}]$
3. $x_r + \sum_{s \neq r} x_s^{\max} + q \geq Q$
4. $x_r + \sum_{s \neq r} x_s^{\min} + q \leq Q$
5. *There is not another pair* $(Q', x_r')$ *such that* $x_r' < x_r, Q' < Q, x_r - x_r' = Q - Q'$, *and all previous conditions apply*

Next, we derive an interval, $I_3^r$, in which all strategies must fall in order to be a best-reply correspondence to a given total quantity. Of particular interest is $x_r^{pivot}$, which is the minimum strategy that meets conditions 1-4.

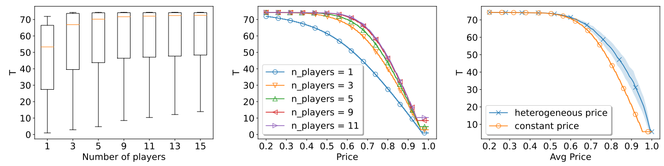
Theorem 3. *[7] Let Assumption 1 be true. Let* $Q$ *be the total job quantity under a given strategy profile,* $Q = \sum_r x_r$. *Then, the set of* $x_r$ *such that* $x_r = \underline{b}_r(Q - x_r)$ *is either* $\emptyset$, *or an interval of the form*

$$I_3^r = \left[x_r^{pivot}(Q), \bar{x}_r(Q)\right] \tag{2}$$

$$\bar{x}_r(Q) = \min(\min_{Q' < Q}\{x_r^{pivot}(Q') - Q'\} + Q - 1, \; x_r^{\max}, Q - q - \textstyle\sum_{s \neq r} x_s^{\min}) \tag{3}$$

*where* $x_r^{pivot}$ *is the least possible value of* $x_r$ *that fulfills conditions 1-4.*

Using this, the total quantity at equilibrium can be found by iterating through each possible quantity, and then checking if $Q \in \sum_r I_3^r$. This algorithm is quite efficient. In a random experiment, we analyze equilibria for $10,000$ Cournot queueing games. With 20 players, linear prices, and strategies within $[10, 190]$, we find equilibria for each game with an average runtime of $0.0578$ seconds.




**(a) Num Players vs T** **(b) Price vs T** **(c) Price vs T (6 players, with inhomogeneous comparison)**

**Figure 1: Empirical Case Study (T: Throughput)**

When prices are affine, we can find the value of $x_r^{pivot}$ by starting with an estimate,

$$\tilde{x}_r^{pivot} = \frac{\Delta\psi_r(x_r) - Z(Q+1)}{\Delta Z(Q)} \tag{4}$$

Then, we can use this quantity to find $x_r^{pivot}$ directly. If $\Delta Z(Q) = 0$, then $x_r^{pivot} = \max(x_r^{\min}, Q - q - \sum_{s \neq r} x_s^{\max})$. If $\Delta Z(Q) < 0$,

$$x_r^{pivot} = \begin{cases} \lfloor \tilde{x}_r^{pivot} \rfloor & \Delta_{x_r} U_r(\lfloor \tilde{x}_r^{pivot} \rfloor, Q) \leq 0, \lfloor \tilde{x}_r^{pivot} \rfloor \in I_1^r \\ \lceil \tilde{x}_r^{pivot} \rceil & \lceil \tilde{x}_r^{pivot} \rceil \in I_1^r, \text{and the prior case does not hold} \\ x_r^{\max} & \lfloor \tilde{x}_r^{pivot} \rfloor > I_1^r \\ \min I_1 & \lceil \tilde{x}_r^{pivot} \rceil \leq I_1^r \end{cases} \tag{5}$$

$$I_1^r = \left[\max(x_r^{\min}, Q - q - \sum_{s \neq r} x_s^{\max}), \min(x_r^{\max}, Q - q - \sum_{s \neq r} x_s^{\min})\right] \tag{6}$$

# 4 EXPERIMENTAL RESULTS

We present an empirical case study, involving the design of a new mobility system with multiple, distinct providers, in Oslo. We use a dataset[2] of 1,116,344 trips in 2021, in order to infer the service rate and routing probability at each station. We present two experiments, one with constant prices and a varying number of customers, and another with 6 providers with heterogeneous prices. In the first case, we consider numbers of players within $[1, 15]$, with prices sampled uniformly within $[0.2, 1.0]$. For each player count, $20,000$ games were analyzed. For the inhomogeneous case, we select a common price $p_{common}$ uniformly within $[0.2, 1.0]$. The price for each player, $p_r$, is distributed according to $\min(\max(p_{common} + \mathcal{N}(0, 0.1), 0.2), 1.0)$. The strategy range for each player is set to $[1, 66]$ to ensure compatibility with Assumption 2. Figure 1a shows how the number of trips changes in the homogeneous case as more players are added. Figure 1b compares the price and the throughput in the homogeneous case. Figure 1c shows the throughput of the inhomogeneous and the homogeneous cases, at each price level.

# 5 CONCLUSION

In conclusion, we have defined a novel class of games on closed queueing networks. We have proven conditions for equilibrium existence, and defined an efficient algorithm for finding equilibria. Then, we have illustrated these with a case study. In the future, this class of games may be extended to multi-class networks.

[2] https://oslobysykkel.no/en/open-data/historical

# REFERENCES


[1] List of bicycle-sharing systems. https://en.wikipedia.org/wiki/List_of_bicycle-sharing_systems.

[2] Iglesias, R., Rossi, F., Zhang, R., and Pavone, M. A BCMP network approach to modeling and controlling autonomous mobility-on-demand systems. *The International Journal of Robotics Research 38*, 2-3 (2019), 357–374.

[3] Li, Q.-L., Chen, C., Fan, R.-N., Xu, L., and Ma, J.-Y. Queueing Analysis of a Large-Scale Bike Sharing System through Mean-Field Theory. *Queueing Models and Service Management 5*, 1 (2022), 45–83.

[4] Novshek, W. Finding All n-Firm Cournot Equilibria. *International Economic Review 25*, 1 (1984), 61–70.

[5] Novshek, W. On the Existence of Cournot Equilibrium. *The Review of Economic Studies 52*, 1 (1985), 85–98.

[6] Pavone, M., Smith, S. L., Frazzoli, E., and Rus, D. Robotic Load Balancing for Mobility-on-Demand Systems. *The International Journal of Robotics Research 31*, 7 (2012), 839–854.

[7] Sheldon, M., Paccagnan, D., and Casale, G. Cournot queueing games with applications to mobility systems (in preparation).

[8] Waserhole, A., and Jost, V. Pricing in vehicle sharing systems: optimization in queuing networks with product forms. *Euro journal on transportation and logistics 5* (2014), 293–320.

[9] Zhang, R., and Pavone, M. A Queueing Network Approach to the Analysis and Control of Mobility-On-Demand Systems. In *2015 American Control Conference (ACC)* (2015), IEEE, pp. 4702–4709.