Pure Nash Equilibria in Weighted Congestion Games with Complementarities and Beyond

Extended Abstract

Kenjiro Takazawa
Hosei University
Tokyo, Japan
takazawa@hosei.ac.jp

ABSTRACT

Congestion games offer a primary model of non-cooperative games, and a number of generalizations have been proposed, such as weighted congestion games, congestion games with mixed costs, and congestion games with complementarities. In this paper, we prove the existence of pure Nash equilibria in weighted matroid congestion games with complementarities and their further generalization, under a simplified assumption. Some extensions of previous results on congestion games with mixed costs are also presented.

KEYWORDS

Non-Cooperative Game Theory; Pure Nash Equilibrium; Congestion Game; Matroid

ACM Reference Format:

Kenjiro Takazawa. 2024. Pure Nash Equilibria in Weighted Congestion Games with Complementarities and Beyond: Extended Abstract. In *Proc. of the 23rd International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2024), Auckland, New Zealand, May 6 – 10, 2024*, IFAAMAS, 3 pages.

1 INTRODUCTION

Congestion games [3] offer a primary model in the study of non-cooperative games in algorithmic game theory [16, 18]. The most important feature of the model is that every congestion game is a *potential game*, which implies that every congestion game has a pure Nash equilibrium [17], and the converse is also true [15].

Since those primary results, a number of generalized models have been proposed. In a *weighted congestion game*, each player has a weight and the cost of a resource is a function of the total weight of the players using it. In a *player-specific congestion game*, the resources have a cost function specific to each player.

A congestion game is referred to as a *singleton game* if every strategy of every player consists of a single resource, and as a *matroid game* if the strategy space of each player is a base family of a matroid. Every matroid player-specific congestion game has a pure Nash equilibrium [1, 14], and every weighted matroid congestion game has a pure Nash equilibrium [1]. Study on further generalized models includes [10, 19].

Proc. of the 23rd International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2024), N. Alechina, V. Dignum, M. Dastani, J.S. Sichman (eds.), May 6 – 10, 2024, Auckland, New Zealand.

In a *bottleneck congestion game*, the cost on a player is the maximum cost of the resources in her strategy. Banner and Orda [2] proved that network bottleneck congestion games of certain kinds have pure Nash equilibria. Related study appears in [8, 9].

Feldotto, Leder, and Skopalik [7] proposed a common generalization of classical congestion games and bottleneck congestion games, called *congestion games with mixed costs*, and proved the existence of pure Nash equilibria in certain classes (Theorem 4 below). In another generalized model of *congestion games with complementarities* [6], the cost imposed on a player is determined by an *aggregation function*, which a general function of the costs of the resources in her strategy. Feldotto, Leder, Skopalik [6] proved the existence of a pure Nash equilibrium in matroid games under some assumption on the monotonicity of the aggregation function, which is defined in a sophisticated manner (Theorem 5). Related work dealing with aggregation cost functions appears in [12, 13].

A central issue in this paper is a common generalization of weighted congestion games and congestion games with complementarities. On the basis of the previous results [1, 6, 19], we prove that every matroid game in this model has a pure Nash equilibrium if the aggregation function has a certain monotonicity (Theorem 8). In particular, it implies the existence of pure Nash equilibria in weighted matroid bottleneck congestion games (Corollaries 9). We remark here that the monotonicity of the aggregation function is defined in a way much simpler than that in [6]. Some extensions of the results on congestion games with mixed costs [7] are also presented.

2 PRELIMINARIES

A congestion game is represented by a tuple $(N, E, (\mathcal{S}_i)_{i\in N}, (c_e)_{e\in E})$. Here, $N = \{1, \dots, n\}$ and E denote the sets of the players and the resources, respectively. For each $i \in N$, let $\mathcal{S}_i \subseteq 2^E$ denote the strategy space of i. For each $e \in E$, a cost function $c_e \colon \mathbb{Z}_+ \to \mathbb{R}_+$ is associated. For a strategy profile $\mathrm{S} = (S_1, \dots, S_n)$ and a resource $e \in E$, let $N_e(\mathrm{S})$ denote the set of the players choosing e in her strategy, and $n_e(\mathrm{S})$ its cardinality. Now $c_e(n_e(\mathrm{S})) \in \mathbb{R}_+$ represents the cost of using a resource e. The cost $\gamma_i(\mathrm{S})$ imposed on a player $i \in N$ is defined as the total cost of the resources in her strategy S_i.

In a *weighted congestion game* $(N, E, (\mathcal{S}_i)_{i\in N}, (c_e)_{e\in E}, (w_i)_{i\in N})$, each player $i \in N$ has a nonnegative weight $w_i \in \mathbb{R}_+$. For $N' \subseteq N$, let $w(N') = \sum_{i\in N'} w_i$. In a strategy profile $\mathrm{S} = (S_1, \dots, S_n)$, the cost of using a resource e is equal to $c_e(w(N_e(\mathrm{S})))$. Hereafter, we assume that c_e is monotonically nondecreasing for each $e \in E$.

Theorem 1 ([1]). *Every weighted matroid congestion game possesses a pure Nash equilibrium.*

A further generalized model of *congestion games with set-functional costs* is proposed [19]. A distinctive feature is that c_e is a set function on 2^N. We remark that the same kind of generalization is proposed by Kiyosue and Takazawa [11] for *budget games* [4, 5]. Takazawa [19] extended Theorem 1 as follows.

Theorem 2 ([19]). *Every matroid congestion game with set-functional costs has a pure Nash equilibrium.*

In a *player-specific congestion game* $(N, E, (\mathcal{S}_i)_{i\in N}, (c_{i,e})_{i\in N,e\in E})$, a cost function $c_{i,e}\colon \mathbb{Z}_+ \to \mathbb{R}_+$ is defined for each $i \in N$ and $e \in E$.

Theorem 3 ([1]). *Every player-specific matroid congestion game possesses a pure Nash equilibrium.*

A *congestion games with mixed costs* [7] is represented by

$$(N, E, (\mathcal{S}_i)_{i\in N}, (\ell_e)_{e\in E}, (b_e)_{e\in E}, (\alpha_i)_{i\in N}).$$

For each $e \in E$, let $\ell_e\colon \mathbb{Z}_+ \to \mathbb{R}_+$ and $b_e\colon \mathbb{Z}_+ \to \mathbb{R}_+$ denote its *latency cost function* and *bottleneck cost function*, respectively. We assume that both ℓ_e and b_e are monotonically nondecreasing. For each $i \in N$, let $\alpha_i \in [0, 1]$ denote her *preference value*. In a strategy profile S, the cost $\gamma_i(\mathrm{S})$ imposed on a player $i \in N$ is defined as

$$\gamma_i(\mathrm{S}) = \alpha_i \cdot \sum_{e\in S_i} \ell_e(n_e(\mathrm{S})) + (1 - \alpha_i) \cdot \max_{e\in S_i}\{b_e(n_e(\mathrm{S}))\}.$$

The cost functions ℓ_e and b_e $(e \in E)$ have *monotone dependence* if there exists a monotonically nondecreasing function $d\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying that $b_e(x) = d(\ell_e(x))$ for each $x \in \mathbb{Z}_+$ and each $e \in E$.

Theorem 4 ([7]). *A congestion game with mixed costs has a pure Nash equilibrium in the following three cases.*

(i) *It is a singleton game.*
(ii) *It is a matroid game and $\alpha_i \in \{0, 1\}$ for each player $i \in N$.*
(iii) *It is a matroid game and the cost functions have monotone dependence.*

A *congestion game with complementarities* [6] is represented as

$$(N, E, (\mathcal{S}_i)_{i\in N}, (c_e)_{e\in E}, g).$$

In this model, g denotes the aggregation function, which is defined on $\mathbb{R}_+^r$ for a positive integer r, and we assume that the sizes of the strategies of all players are r. For $i \in N$ and a strategy profile $\mathrm{S} = (S_1, \dots, S_n)$, where $S_i = \{e_1, \dots, e_r\}$, the cost $\gamma_i(\mathrm{S})$ on i in S is

$$\gamma_i(\mathrm{S}) = g(c_{e_1}(n_{e_1}(\mathrm{S})), \dots, c_{e_r}(n_{e_r}(\mathrm{S}))).$$

Feldotto, Leder, Skopalik [6] proved the existence of a pure Nash equilibrium in a matroid game with g being a *weakly monotone function*, which is defined in a sophisticated manner.

Theorem 5 ([6]). *Every matroid congestion game with complementarities has a pure Nash equilibrium if g is weakly monotone.*

3 WEIGHTED CONGESTION GAMES WITH COMPLEMENTARITIES AND BEYOND

A *congestion game with set-functional costs and complementarities* is described as

$$(N, E, (\mathcal{S}_i)_{i\in N}, (c_e)_{e\in E}, g).$$

The difference from congestion games with complementarities is that $c_e\colon 2^N \to \mathbb{R}_+$ is a set function for each $e \in E$. This model includes that of weighted congestion games with complementarities.

We adopt the following definition of weakly monotone functions, which is simpler than that in [6].

Definition 6. Let r be a positive integer. A function $g\colon \mathbb{R}_+^r \to \mathbb{R}$ is *weakly monotone* if, for each $x, y \in \mathbb{R}_+$, it holds that

$$g(v', x) \le g(v', y) \quad \text{for each } v' \in \mathbb{R}_+^{r-1}, \text{ or}$$
$$g(v', x) \ge g(v', y) \quad \text{for each } v' \in \mathbb{R}_+^{r-1}.$$

Associated with Definition 6, the monotonicity of the cost function $c_e\colon 2^N \to \mathbb{R}$ $(e \in E)$ is adjusted in the following way.

Definition 7. Let $g\colon \mathbb{R}_+^r \to \mathbb{R}$ be a weakly monotone function. A function $c\colon 2^N \to \mathbb{R}_+$ is *monotonically nondecreasing with respect to g* if, for each $X, Y \subseteq N$ with $X \subseteq Y$, it holds that

$$g(v', c(X)) \le g(v', c(Y)) \quad \text{for each } v' \in \mathbb{R}_+^{r-1}.$$

Our main technical contribution is the following theorem, which amounts to a common extension of Theorems 1, 2, and 5.

Theorem 8. *A matroid congestion game $(N, E, (\mathcal{S}_i)_{i\in N}, (c_e)_{e\in E}, g)$ with set-functional costs and complementarities has a pure Nash equilibrium if the aggregation function $g\colon \mathbb{R}_+^r \to \mathbb{R}$ is weakly monotone and the cost function $c_e\colon 2^N \to \mathbb{R}_+$ is monotonically nondecreasing with respect to g for each resource $e \in E$.*

The following corollary is immediately derived from Theorem 8.

Corollary 9. *Every weighted matroid bottleneck congestion game in which the strategies of all players have the same cardinality has a pure Nash equilibrium.*

4 GENERALIZATIONS OF CONGESTION GAMES WITH MIXED COSTS

A *player-specific congestion game with mixed costs* is represented by

$$(N, E, (\mathcal{S}_i)_{i\in N}, (\ell_{i,e})_{i\in N,e\in E}, (b_{i,e})_{i\in N,e\in E}, (\alpha_i)_{i\in N}).$$

The following theorem extends Theorem 4(i)(ii).

Theorem 10. *A player-specific congestion game G with mixed costs has a pure Nash equilibrium in the following two cases.*

(i) *G is a singleton game.*
(ii) *G is a matroid game and $\alpha_i \in \{0, 1\}$ for each player $i \in N$.*

A *congestion game with mixed and set-functional costs*, including a *weighted congestion game with mixed costs*, is represented by

$$(N, E, (\mathcal{S}_i)_{i\in N}, (\ell_e)_{e\in E}, (b_e)_{e\in E}, (\alpha_i)_{i\in N}),$$

where $\ell_e\colon 2^N \to \mathbb{R}_+$ and $b_e\colon 2^N \to \mathbb{R}_+$ are set functions. We generalize the definition of the monotone dependence of ℓ_e and b_e $(e \in E)$ as the existence of a monotonically nondecreasing function $d\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying that

$$b_e(N') = d(\ell_e(N')) \quad (N' \subseteq N).$$

For this model, Theorem 4(iii) is extended as follows.

Theorem 11. *A congestion game G with mixed and set-functional costs has a pure Nash equilibrium if G is a matroid game and the cost functions have monotone dependence.*

ACKNOWLEDGMENTS

This work is partially supported by JSPS KAKENHI Grant Number JP20K11699, Japan.

REFERENCES

[1] Heiner Ackermann, Heiko Röglin, and Berthold Vöcking. 2009. Pure Nash equilibria in player-specific and weighted congestion games. *Theor. Comput. Sci.* 410, 17 (2009), 1552–1563. https://doi.org/10.1016/j.tcs.2008.12.035

[2] Ron Banner and Ariel Orda. 2007. Bottleneck routing games in communication networks. *IEEE J. Sel. Areas Commun.* 25, 6 (2007), 1173–1179. https://doi.org/10.1109/JSAC.2007.070811

[3] Vittorio Bilò and Cosimo Vinci. 2023. *Coping with Selfishness in Congestion Games—Analysis and Design via LP Duality*. Springer. https://doi.org/10.1007/978-3-031-30261-9

[4] Maximilian Drees, Matthias Feldotto, Sören Riechers, and Alexander Skopalik. 2015. On existence and properties of approximate pure Nash equilibria in bandwidth allocation games. In *8th International Symposium on Algorithmic Game Theory, SAGT 2015 (Lecture Notes in Computer Science, Vol. 9347)*, Martin Hoefer (Ed.). Springer, 178–189. https://doi.org/10.1007/978-3-662-48433-3_14

[5] Maximilian Drees, Matthias Feldotto, Sören Riechers, and Alexander Skopalik. 2019. Pure Nash equilibria in restricted budget games. *J. Comb. Optim.* 37, 2 (2019), 620–638. https://doi.org/10.1007/s10878-018-0269-7

[6] Matthias Feldotto, Lennart Leder, and Alexander Skopalik. 2017. Congestion games with complementarities. In *10th International Conference on Algorithms and Complexity, CIAC 2017 (Lecture Notes in Computer Science, Vol. 10236)*, Dimitris Fotakis, Aris Pagourtzis, and Vangelis Th. Paschos (Eds.). 222–233. https://doi.org/10.1007/978-3-319-57586-5_19

[7] Matthias Feldotto, Lennart Leder, and Alexander Skopalik. 2018. Congestion games with mixed objectives. *J. Comb. Optim.* 36, 4 (2018), 1145–1167. https://doi.org/10.1007/s10878-017-0189-y

[8] Tobias Harks, Martin Hoefer, Max Klimm, and Alexander Skopalik. 2013. Computing pure Nash and strong equilibria in bottleneck congestion games. *Math. Program.* 141, 1-2 (2013), 193–215. https://doi.org/10.1007/s10107-012-0521-3

[9] Tobias Harks, Max Klimm, and Rolf H. Möhring. 2013. Strong equilibria in games with the lexicographical improvement property. *Int. J. Game Theory* 42, 2 (2013), 461–482. https://doi.org/10.1007/s00182-012-0322-1

[10] Tobias Harks, Max Klimm, and Britta Peis. 2018. Sensitivity analysis for convex separable optimization over integral polymatroids. *SIAM J. Optim.* 28, 3 (2018), 2222–2245. https://doi.org/10.1137/16M1107450

[11] Fuga Kiyosue and Kenjiro Takazawa. 2022. A Common Generalization of Budget Games and Congestion Games. In *15th International Symposium on Algorithmic Game Theory, SAGT 2022 (Lecture Notes in Computer Science, Vol. 13584)*, Panagiotis Kanellopoulos, Maria Kyropoulou, and Alexandros A. Voudouris (Eds.). Springer, 258–274. https://doi.org/10.1007/978-3-031-15714-1_15

[12] Nikolai S. Kukushkin. 2007. Congestion games revisited. *Int. J. Game Theory* 36, 1 (2007), 57–83. https://doi.org/10.1007/s00182-007-0090-5

[13] Nikolai S. Kukushkin. 2015. Rosenthal's potential and a discrete version of the Debreu-Gorman Theorem. *Autom. Remote. Control.* 76, 6 (2015), 1101–1110. https://doi.org/10.1134/S0005117915060144

[14] Igal Milchtaich. 1996. Congestion games with player-specific payoff functions. *Game. Econ. Behav.* 13 (1996), 111–124.

[15] Dov Monderer and Lloyd S. Shapley. 1996. Potential games. *Games Econ. Behav.* 14 (1996), 124–143. https://doi.org/10.1006/game.1996.0044

[16] Noam Nisan, Tim Roughgarden, Éva Tardos, and Vijay V. Vazirani (Eds.). 2007. *Algorithmic Game Theory*. Cambridge University Press.

[17] Robert W. Rosenthal. 1973. A class of games possessing pure-strategy Nash equilibria. *Int. J. Game Theory* 2 (1973), 65–67. https://doi.org/doi.org/10.1007/BF01737559

[18] Tim Roughgarden. 2016. *Twenty Lectures on Algorithmic Game Theory*. Cambridge University Press.

[19] Kenjiro Takazawa. 2019. Generalizations of weighted matroid congestion games: pure Nash equilibrium, sensitivity analysis, and discrete convex function. *J. Comb. Optim.* 38, 4 (2019), 1043–1065. https://doi.org/10.1007/s10878-019-00435-9