Game Transformations That Preserve Nash Equilibria or Best-Response Sets

Extended Abstract

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ABSTRACT

In the full version of this paper, we investigate under which conditions normal-form games are (guaranteed) to be strategically equivalent. First, we show for N-player games ($N \geq 3$) that (a) it is NP-hard to decide whether a given strategy is a best response to some strategy profile of the opponents, and that (b) it is co-NP-hard to decide whether two games have the same best-response sets.

We then turn our attention to equivalence-preserving game transformations. It is a widely used fact that a positive affine (linear) transformation of the utility payoffs neither changes the best-response sets nor the Nash equilibrium set. We investigate which other game transformations also possess either of the following two properties when being applied to an arbitrary N-player game ($N \geq 2$): (i) The Nash equilibrium set stays the same; (ii) The best-response sets stay the same.

For game transformations that operate player-wise and strategy-wise, we prove that (i) implies (ii) and that transformations with property (ii) must be positive affine. The resulting equivalence chain highlights the special status of positive affine transformations among all the transformation procedures that preserve key game-theoretic characteristics.

KEYWORDS

Strategic Equivalence; Game Transformation; Nash Equilibrium; Best Responses; Positive Affine Linear Transformation

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INTRODUCTION

When faced with a strategic interaction with other agents, it is helpful for AI systems to detect when the current situation can be treated in the same way as another strategic game that has already been dealt with in the past. Du [13] has shown that this is generally a computationally hard task for the case of Nash equilibria. As we will show, this task is also computationally hard in the case of best responses.

Therefore, one may instead take an alternative approach for the currently encountered strategic interaction and generate a space of many other situations that share key game-theoretic characteristics, with the goal to find an instance in that space that can be analyzed and solved efficiently. More concretely, a classic tool that emerged in the beginnings of game theory has been to transform a given game into other strategically equivalent games that are easier to analyze [38]. Positive affine (linear) transformations (PATs) have been particularly useful in that regard [3, 5, 24]. To illustrate PATs, consider any 2-player normal-form game in which the players' utilities are measured in dollars. Then, the best-response strategies of player 1 do not change if her utility payoffs are multiplied by a factor of 5. Moreover, they also do not change if 10 dollars are added to all outcomes that involve player 2 playing his, say, third strategy. More generally, PATs have the power to rescale the utility payoffs of each player and to add constant terms to the utility payoffs of a player i for each strategy choice $\mathbf{k}_{-i}$ of her opponents.

Through leveraging PATs, previous work significantly extended the applicability of efficient Nash equilibrium solvers [2, 4, 11, 37] to classes beyond those of zero-sum and rank-1 games[1] [20, 22, 28]. The key to the success of these extensions was the well-known property of PATs that they do not change the Nash equilibrium set and best-response sets when being applied to an arbitrary game. In this paper, we investigate whether there are other (efficiently computable) game transformations with that same property.

PRELIMINARIES

Normal-Form Games. Write $[n] := \{1, \dots, n\}$ for any $n \in \mathbb{N}$. A normal-form multiplayer game G specifies (a) an integer number of players $N \geq 2$, (b) a set of pure strategies $S^i = [m_i]$ for each player i where $m_i \geq 2$ is integer, and (c) the utility payoffs for each player i given as a function $u_i : S^1 \times \ldots \times S^N \to \mathbb{R}$.

We refer to the set of (strategy) profiles in G as $S := S^1 \times \ldots \times S^N$. Throughout this paper (and unless explicitly specified otherwise) all considered multiplayer games shall have the same number of players N and the same set of profiles S. Hence, any game G will be determined by its utility functions $\{u_i\}_{i \in [N]}$. The players choose their strategies simultaneously, they cannot communicate with each other, and their goal is to maximize their personal utility. As usual, we allow the players to randomize over their pure strategies,

[1] A 2-player game, represented by its payoff matrices $A, B \in \mathbb{R}^{m \times n}$, is said to have rank 1 if $\mathrm{rank}(A + B) = 1$.

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which extends the strategy sets to all probability distributions $\Delta(S^i)$ over S^i. A tuple $\mathbf{s} = (s^1, \ldots, s^N) \in \Delta(S^1) \times \ldots \times \Delta(S^N) =: \Delta(S)$ is called a (mixed) profile[2] in G. A player then optimizes for her expected utility $u_i(\mathbf{s}) := \sum_{\mathbf{k} \in S} s^1_{k_1} \cdot \ldots \cdot s^N_{k_N} \cdot u_i(\mathbf{k})$.

We define $\mathbf{k}_{-i}, \mathbf{s}^{-i}$, and $\Delta(S^{-i})$ analogously to $\mathbf{k}, \mathbf{s}$, and $\Delta(S)$, except that player i's part is removed from it. The best-response set of player i to an opponents' profile $\mathbf{s}^{-i} \in \Delta(S^{-i})$ is then defined as $\mathrm{BR}_{u_i}(\mathbf{s}^{-i}) := \mathrm{argmax}_{t^i \in \Delta(S^i)} \left\{ u_i(t^i, \mathbf{s}^{-i}) \right\}$, where the notation $u_i(s^i, \mathbf{s}^{-i})$ stresses how player i can only influence her own strategy when it comes to her payoff. A profile $\mathbf{s} \in \Delta(S)$ is called a Nash equilibrium in G if for every player $i \in [N]$ we have $s^i \in \mathrm{BR}_{u_i}(\mathbf{s}^{-i})$. By Nash's result [29], any such multiplayer game G admits at least one Nash equilibrium.

Game Transformations. Let us define the two game transformation concepts that we study in this paper.

Definition 1. A positive affine transformation (PAT) specifies for each player i a real-valued scaling parameter $\alpha^i > 0$ and real-valued translation constants $C^i := (c^i_{\mathbf{k}_{-i}})_{\mathbf{k}_{-i} \in S^{-i}}$ for each pure profile of the opponents. The PAT H_{PAT} can then take any game $G = \{u_i\}_{i \in [N]}$ and transform it into game $H_{\mathrm{PAT}}(G) = \{u'_i\}_{i \in [N]}$ with utility functions $u'_i : S \to \mathbb{R},\ \mathbf{k} \mapsto \alpha_i \cdot u_i(\mathbf{k}) + c^i_{\mathbf{k}_{-i}}$.

Definition 2. A separable game transformation H specifies for each player i a map $H^i := \left\{ h^i_{\mathbf{k}} : \mathbb{R} \to \mathbb{R} \right\}_{\mathbf{k} \in S}$ of functions for each pure profile $\mathbf{k}$. The transformation H can then take any game $G = \{u_i\}_{i \in [N]}$ and transform it into game $H(G) = \{H^i(u_i)\}_{i \in [N]}$ with utility functions $H^i(u_i) : S \to \mathbb{R},\ \mathbf{k} \mapsto h^i_{\mathbf{k}}\big(u_i(\mathbf{k})\big)$.

Separability intuitively means that the transformed game $H(G)$ has the same number of players N and the same profile set S as the original game G, and that the utility payoff of player i in $H(G)$ from pure profile outcome $\mathbf{k}$ is only a function of the utility payoff from that same player in that same pure profile outcome in G. We discuss in the full version of this paper why it is sensible from a computational perspective to restrict our attention to this class of game transformations. We illustrate the richness of this in the following example.

Example 3. Consider the 2×2 bimatrix games, that is, the case of $N = 2$ players each with $|S^i| = 2$ strategies. A PAT may then, for example, transform such a game (A, B) into the game

$$A', B' = \begin{pmatrix} 2a_{11} + 10 & 2a_{12} - 5 \\ 2a_{21} + 10 & 2a_{22} - 5 \end{pmatrix}, \begin{pmatrix} \frac{1}{2}b_{11} & \frac{1}{2}b_{12} \\ \frac{1}{2}b_{21} - \sqrt{3} & \frac{1}{2}b_{22} - \sqrt{3} \end{pmatrix}.$$

A separable game transformation can, for example, instead transform (A, B) into the game

$$A'', B'' = \begin{pmatrix} -2a_{11} + 10 & a_{12}^5 \\ e^{a_{21}} & 0 \end{pmatrix}, \begin{pmatrix} |b_{11}| & \mathrm{sign}(b_{12}) \\ \sqrt{|b_{21}|} & \arctan(b_{22}) \end{pmatrix}.$$

It is a well-known fact that PATs do not change the best responses (and hence, not the Nash equilibrium set, either), no matter what game G they are applied to. We prove with Theorem 7 that the separable game transformation example above does not always preserve these game characteristics. In fact, each of the functions within A'' and B'' already single-handedly violates a PAT structure.

[2] Not to be confused with a correlated strategy: In our notation, $\Delta(S)$ itself is not a simplex of high dimension but only the product of N lower-dimensional simplices.

Definition 4. A separable game transformation H (resp. map H^i) is said to universally preserve Nash equilibrium sets (resp. best responses) if for all games $G = \{u_i\}_{i \in [N]}$ the transformed game $H(G)$ has the same Nash equilibrium set as G (resp. the same best-response sets as G, i.e., $\mathrm{BR}_{H^i(u_i)}(\mathbf{s}^{-i}) = \mathrm{BR}_{u_i}(\mathbf{s}^{-i})$ for all profiles $\mathbf{s}^{-i} \in \Delta(S^{-i})$ of the opponents).

RESULTS

Best Responses in Many-Player Games. We investigate the computational complexity of problems involving best-response strategies. First, we consider the problem of deciding whether a (mixed) strategy of a player is ever a best response to some mixed profile of the opponents. This is related to *rationalizable* strategies [7, 30], a concept based on the idea that a rational player should eliminate any strategy that is not a best response to some belief over what her opponents may play.

Proposition 5. *It is* **NP***-hard to decide, given a* 3*-player normal-form game, whether there exist mixed strategies* $\mathbf{r}$ *and* $\mathbf{s}$ *of* P2 *and* P3 *such that the first pure strategy of* P1 *is a best response to* $(\mathbf{r}, \mathbf{s})$.

Next, we turn to best-response equivalence.

Theorem 6. *It is* **co-NP***-hard to decide whether two* 3*-player normal-form games have the same best-response sets.*

Preserving Transformations. We give two equivalent characterizations of PATs that highlight their special status among game transformations: PATs are the only separable game transformations that always preserve the Nash equilibrium set or, respectively, the best-response sets.

Theorem 7. *Let* H *be a separable game transformation. Then:*

H universally preserves Nash equilibrium sets
$\leftrightarrow$ for each player i, map H^i universally preserves best responses
$\leftrightarrow$ H is a positive affine transformation.

The novel part about Theorem 7 is the downwards implication chain. We may circumvent this result by considering non-separable game transformations, as discussed in the full version of this paper. The space of preserving transformations may also increase if we are only interested in transforming a specific subclass of N-player games (provided by, e.g., domain knowledge). For a general treatment as done in this paper, it would be preferred if such a subclass still contained "most" games.

CONCLUSION

When faced with a strategic interaction it can be highly beneficial to consider equivalent variations of it that are easier to analyze. In the full version of this paper, we shed light on why PATs have become the go-to transformation method for that purpose, reinforcing their standing as the standard off-the-shelf approach. The current literature on game theory and on decision making in AI are lacking methods to detect or generate strategically equivalent games, and we hope that our results can serve as guidance to the development of any such detection or generation toolkit.

REFERENCES

[1] Timothy G. Abbott, Daniel M. Kane, and Paul Valiant. 2005. On the Complexity of Two-PlayerWin-Lose Games. In *46th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2005), 23-25 October 2005, Pittsburgh, PA, USA, Proceedings*. IEEE Computer Society, 113–122. https://doi.org/10.1109/SFCS.2005.59

[2] Ilan Adler. 2013. The equivalence of linear programs and zero-sum games. *Int. J. Game Theory* 42, 1 (2013), 165–177. https://doi.org/10.1007/s00182-012-0328-8

[3] Ilan Adler, Constantinos Daskalakis, and Christos H. Papadimitriou. 2009. A Note on Strictly Competitive Games. In *Internet and Network Economics, 5th International Workshop, WINE 2009, Rome, Italy, December 14-18, 2009. Proceedings*, Stefano Leonardi (Ed.). Springer, 471–474. https://doi.org/10.1007/978-3-642-10841-9_44

[4] Bharat Adsul, Jugal Garg, Ruta Mehta, Milind A. Sohoni, and Bernhard von Stengel. 2021. Fast Algorithms for Rank-1 Bimatrix Games. *Oper. Res.* 69, 2 (2021), 613–631. https://doi.org/10.1287/opre.2020.1981

[5] Robert J. Aumann. 1961. Almost Strictly Competitive Games. *Journal of The Society for Industrial and Applied Mathematics* 9 (1961), 544–550.

[6] Tamer Başar and Geert Jan Olsder. 1998. *Dynamic Noncooperative Game Theory* (2nd. ed.). Society for Industrial and Applied Mathematics. https://doi.org/10.1137/1.9781611971132

[7] B. Douglas Bernheim. 1984. Rationalizable Strategic Behavior. *Econometrica* 52, 4 (1984), 1007–1028. http://www.jstor.org/stable/1911196

[8] Felix Brandt, Felix A. Fischer, and Yoav Shoham. 2006. On Strictly Competitive Multi-Player Games. In *Proceedings, The Twenty-First National Conference on Artificial Intelligence and the Eighteenth Innovative Applications of Artificial Intelligence Conference, July 16-20, 2006, Boston, Massachusetts, USA*. AAAI Press, 605–612. http://www.aaai.org/Library/AAAI/2006/aaai06-097.php

[9] André Casajus. 2003. Weak isomorphism of extensive games. *Math. Soc. Sci.* 46, 3 (2003), 267–290. https://doi.org/10.1016/S0165-4896(03)00064-7

[10] Chih Chang and Stef Tijs. 2006. A note on isomorphism and strategic equivalence of cooperative games. *TOP* 14 (2006), 333–342. https://doi.org/10.1007/BF02837566

[11] George B Dantzig. 1951. A proof of the equivalence of the programming problem and the game problem. In *Activity analysis of production and allocation*, Tjalling C. Koopmans (Ed.). Cowles Commission Monograph No.13, 330–335.

[12] Gaston Darboux. 1875. Sur la composition des forces en statique. *Bulletin des Sciences Mathématiques et Astronomiques* 9 (1875), 281–288.

[13] Ye Du. 2008. On the complexity of deciding bimatrix games similarity. *Theor. Comput. Sci.* 407, 1-3 (2008), 583–586. https://doi.org/10.1016/j.tcs.2008.07.021

[14] Susan Elmes and Philip J. Reny. 1994. On the Strategic Equivalence of Extensive Form Games. *Journal of Economic Theory* 62, 1 (1994), 1–23. https://doi.org/10.1006/jeth.1994.1001

[15] Joaquim Gabarró, Alina García, and Maria J. Serna. 2011. The complexity of game isomorphism. *Theor. Comput. Sci.* 412, 48 (2011), 6675–6695. https://doi.org/10.1016/j.tcs.2011.07.022

[16] Joaquim Gabarró, Alina García, and Maria J. Serna. 2013. On the hardness of game equivalence under local isomorphism. *RAIRO Theor. Informatics Appl.* 47, 2 (2013), 147–169. https://doi.org/10.1051/ita/2012024

[17] Michael R. Garey and David S. Johnson. 1990. *Computers and Intractability; A Guide to the Theory of NP-Completeness*. W. H. Freeman & Co., USA.

[18] Peter J. Hammond. 2005. Utility Invariance in Non-Cooperative Games. In *Advances in Public Economics: Utility, Choice and Welfare*, Ulrich Schmidt and Stefan Traub (Eds.). Springer, Boston, MA, 31–50. https://doi.org/10.1007/0-387-25706-3_3

[19] J.C. Harsanyi and R. Selten. 1988. *A General Theory of Equilibrium Selection in Games*. MIT Press.

[20] Joseph L. Heyman and Abhishek Gupta. 2023. Rank Reduction in Bimatrix Games. *IGTR* 25, 1 (2023). https://doi.org/10.1142/S0219198922500177

[21] Elon Kohlberg and Jean-Francois Mertens. 1986. On the Strategic Stability of Equilibria. *Econometrica* 54, 5 (1986), 1003–1037. http://www.jstor.org/stable/1912320

[22] Spyros C. Kontogiannis and Paul G. Spirakis. 2012. On mutual concavity and strategically-zero-sum bimatrix games. *Theor. Comput. Sci.* 432 (2012), 64–76. https://doi.org/10.1016/j.tcs.2012.01.016

[23] Luchuan A. Liu. 1996. The Invariance of Best Reply Correspondences in Two-Player Games. (1996). 93, City University of Hong Kong, Faculty of Business, Department of Economics.

[24] Luke Marris, Ian Gemp, and Georgios Piliouras. 2023. Equilibrium-Invariant Embedding, Metric Space, and Fundamental Set of 2×2 Normal-Form Games. arXiv:2304.09978 [cs.GT]

[25] Andreu Mas-Colell, Michael D. Whinston, and Jerry R. Green. 1995. *Microeconomic Theory*. Oxford University Press.

[26] Michael Maschler, Eilon Solan, and Shmuel Zamir. 2013. *Game Theory*. Cambridge University Press. https://doi.org/10.1017/CBO9780511794216

[27] Stephen Morris and Takashi Ui. 2004. Best response equivalence. *Games Econ. Behav.* 49, 2 (2004), 260–287. https://doi.org/10.1016/j.geb.2003.12.004

[28] H. Moulin and J.P. Vial. 1978. Strategically zero-sum games: The class of games whose completely mixed equilibria cannot be improved upon. *International Journal of Game Theory* 7 (1978), 201–221. https://doi.org/10.1007/BF01769190

[29] John F. Nash. 1950. Equilibrium points in n-person games. *Proceedings of the National Academy of Sciences* 36, 1 (1950), 48–49. https://doi.org/10.1073/pnas.36.1.48 arXiv:https://www.pnas.org/doi/pdf/10.1073/pnas.36.1.48

[30] David G. Pearce. 1984. Rationalizable Strategic Behavior and the Problem of Perfection. *Econometrica* 52, 4 (1984), 1029–1050. http://www.jstor.org/stable/1911197

[31] Antonin Pottier and Rabia Nessah. 2014. Berge-Vaisman and Nash Equilibria: Transformation of Games. *IGTR* 16, 4 (2014). https://doi.org/10.1142/S0219198914500091

[32] ProofWiki. 2020. Additive Function is Linear for Rational Factors. https://proofwiki.org/wiki/Additive_Function_is_Linear_for_Rational_Factors

[33] ProofWiki. 2021. Monotone Additive Function is Linear. https://proofwiki.org/wiki/Monotone_Additive_Function_is_Linear

[34] Daniel Reem. 2017. Remarks on the Cauchy functional equation and variations of it. *Aequationes mathematicae* 91 (2017), 237–264.

[35] Emanuel Tewolde and Vincent Conitzer. 2023. Game Transformations that Preserve Nash Equilibria or Best Response Sets. *CoRR* abs/2111.00076 (2023). arXiv:2111.00076 https://arxiv.org/abs/2111.00076

[36] F. B. Thompson. 1952. *Equivalence of Games in Extensive Form*. RAND Corporation, Santa Monica, CA.

[37] John von Neumann. 1928. Zur Theorie der Gesellschaftsspiele. *Math. Ann.* 100 (1928), 295–320.

[38] John von Neumann and Oskar Morgenstern. 1944. *Theory of Games and Economic Behavior*. Princeton University Press.

[39] Bernhard von Stengel. 2022. *Game Theory Basics*. Cambridge University Press.

[40] Yuhu Wu, Shuting Le, Kuize Zhang, and Xi-Ming Sun. 2022. Agent Transformation of Bayesian Games. *IEEE Trans. Automat. Control* 67, 11 (2022), 5793–5808. https://doi.org/10.1109/TAC.2021.3122372