Extended Abstract: Price of Anarchy of Traffic Assignment with Exponential Cost Functions

JAAMAS Track

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ABSTRACT

This paper is an extended abstract version of "Price of Anarchy of Traffic Assignment with Exponential Cost Functions [5]". We study a routing game where vehicles, selfish agents, independently choose routes to minimize travel delays from road congestion. We focus on exponential latency functions, unlike prior research using polynomial functions like BPR. We calculate a tight upper bound for the price of anarchy and compare it with the BPR function. Results indicate that the exponential function has a lower upper bound for traffic volumes below road capacity than the BPR function. Numerical analysis using real-world data shows that the exponential function closely approximates road latency with even tighter parameters, resulting in a relatively lower upper bound.

KEYWORDS

Price of Anarchy; Congestion Game; Traffic Assignment; Multi-agent System

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1 INTRODUCTION

The traffic assignment problem is modelled as a congestion game [2], where each player's cost depends on their chosen route and the number of others taking the same path. Road travel time is determined by the number of vehicles on each road, according to independent cost functions for each road in the network[7]. This paper explores vehicle behaviour in a road network using game theory, focusing on selfish decision-making and global optimization [8], and autonomous driving in various traffic conditions, discussing decentralized control (selfish behaviour) and centralized control (global optimization).

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2 PROBLEM FORMULATION

We define a **road network** as a directed multigraph $G = (V, P)$ with a set of nodes (positions) V and a set of edges (roads) P. An **origin-destination** $(o, d) \in V \times V$ is a pair of locations and $OD = \{(o_i, d_i) : \forall i \in [1, k], o_i \in V, d_i \in V \setminus \{o_i\}\}$ denotes the set of all such origin-destinations in G. A **route** γ is a simple path linked between an (o, d). Let $\Gamma_i \neq \emptyset$ denote all possible routes for (o_i, d_i) and $\Gamma = \bigcup_{i\in[1,k]} \Gamma_i$ define all possible routes of G. A **traffic flow** $f : \Gamma \to \mathbb{R}^+$ is a function that maps each route γ to a positive number that represents the traffic volume (number of vehicles per hour) of that route, and we use f_γ as a shorthand for $f(\gamma)$ to simplify notation. We define the **traffic demand** $r_i \in \mathbb{R}$ as the total number of vehicles per hour travelling between o_i and d_i. We say that a flow f is **feasible** if and only if it satisfies $\sum_{\gamma\in\Gamma_i} f(\gamma) = r_i$ for all $i \in [1, k]$, and we let F denote the set of all feasible flows. Furthermore, we define $f_p = \sum_{\gamma\in\Gamma:p\in\gamma} f_\gamma$ as the traffic flow of the road p for a feasible flow f. Each road $p \in P$ has a non-negative, differentiable and non-decreasing **cost function** $l_p : \mathbb{R} \to \mathbb{R}$ that takes the **traffic flow** f_p of that road as its input and that outputs the travel time (in seconds) for a vehicle to drive along that road. We use $\mathcal{L}$ to denote the set of all possible cost functions, and, for some given road network G, we use $L : P \to \mathcal{L}$ to denote the function that maps each road p to its corresponding cost function l_p. For a feasible traffic flow, the travel time of a route is $l_\gamma(f_\gamma) = \sum_{p\in\gamma} l_p(f_p)$ and the cost of a vehicle is the travel time of the route it selected. Furthermore, we define $C(f) = \sum_{p\in P} l_p(f_p) f_p$ as the **social cost** incurred by the feasible flow f. An instance of the traffic assignment problem is now defined as a tuple $(G, \vec{r}, L)$, where G and L are as above, and $\vec{r} = (r_1, \ldots, r_k)$ is a tuple containing the traffic demand r_i of each origin-destination (o_i, d_i). Given an instance $(G, \vec{r}, L)$, a feasible flow $f^* \in F$ is a **user equilibrium** (UE) **flow** if and only if for any origin-destination $i \in [1, k]$ and any $\gamma \in \Gamma_i$ with $f^*_\gamma > 0$, we have $l_\gamma(f^*_\gamma) \leq l_{\gamma'}(f^*_{\gamma'})$ for any $\gamma' \in \Gamma_i$ and a feasible flow $f' \in F$ is a **system optimum** (SO) **flow** if and only if $C(f') = \min_{f\in F} C(f)$. The **price of anarchy** (POA) of an instance is $PoA(G, \vec{r}, L) := \frac{C(UE)}{C(SO)}$ in which defined as the ratio between the social cost of the UE flow and the social cost of the SO flow.

3 POA WITH EXPONENTIAL FUNCTIONS

We are interested in instances of the traffic assignment problem where each road has an exponential cost function. Specifically, we assume that for each road $p \in P$, the cost function can be expressed as:

$$l_p(f_p) = ae^{bf_p} + c \tag{1}$$

where f_p is the traffic flow of road p, and a, b, and c are non-negative coefficients, and we will sometimes stress this by writing them as a_p, b_p, and c_p instead for each road. Note that, for any instance $(G, \vec{r}, L_{exp})$, there is a unique user equilibrium flow [3, 4], where L_{exp} is a set of exponential cost functions.

Definition 3.1. (Anarchy Value [6]) Let $(G, \vec{r}, L_{exp})$ be an instance with exponential cost functions, and let f^* denote its user equilibrium flow. Then the **anarchy value** $\phi_p(\vec{r})$ of a road p is defined as follows:

$$\phi_p(\vec{r}) := [\lambda_p \mu_p + (1 - \lambda_p)]^{-1} \tag{2}$$

where $\lambda_p \in [0, 1]$ is the solution of the equation $l_p^*(\lambda_p f_p^*) = l_p(f_p^*)$ and μ_p is defined as $\mu_p := \frac{l_p(\lambda_p f_p^*)}{l_p(f_p^*)} \in [0, 1]$.

Lemma 3.2. *For any function l of the form of Eq.(1) and for any positive value $x \in \mathbb{R}^+$, there is a unique value $\lambda \in [0, 1]$ that solves the equation $l^*(\lambda x) = l(x)$ (where $l^*(\lambda x) := \frac{d}{d\lambda x}(\lambda x \cdot l(\lambda x))$ is the marginal cost function of $l(\lambda x)$).*

Lemma 3.3. *For any instance $(G, \vec{r}, L_{exp})$, we have:*

$$PoA(G, \vec{r}, L_{exp}) \leq \phi(L_{exp})$$

3.1 Upper Bound of the POA

Lemma 3.4. *For any instance $(G, \vec{r}, L_{exp})$, let f^* denote its user equilibrium flow. The anarchy value $\phi_p(\vec{r})$ of any road $p \in P$ with cost functions of the form $l(x) = ae^{bx} + c$, where a, b, and c are non-negative coefficients, satisfies:*

$$\phi_p(\vec{r}) \leq \frac{bx^*}{bx^* + 2 - W(e^{bx^*+1}) - \frac{1}{W(e^{bx^*+1})}} \tag{3}$$

where $W(\cdot)$ is the Lambert W function [1] and $x^* = f_p^*$ is the UE flow of the road p.

Lemma 3.5. *The expression $\frac{x}{x+2-W(e^{x+1})-\frac{1}{W(e^{x+1})}}$ is monotonically increasing for $x > 0$.*

Theorem 3.6. *For any instance $(G, \vec{r}, L_{exp})$ with exponential cost functions, the price of anarchy satisfies the following.*

$$PoA(G, \vec{r}, L_{exp}) \leq \frac{\hat{b}\hat{r}}{\hat{b}\hat{r} + 2 - W(e^{\hat{b}\hat{r}+1}) - \frac{1}{W(e^{\hat{b}\hat{r}+1})}} \tag{4}$$

where $\hat{r} := \sum_{i \in [1,k]} r_i$ and $\hat{b} := \max_{p \in P} b_p$.

3.2 Tightness of the Upper Bound

Theorem 3.7. *For any positive numbers $\hat{b}$ and $\hat{r}$, there exists an instance $(G, \vec{r}, L_{exp})$ (see Fig 1) with exponential cost functions, for which $PoA(G, \vec{r}, L_{exp})$ is exactly equal to $\frac{\hat{b}\hat{r}}{\hat{b}\hat{r}+2-W(e^{\hat{b}\hat{r}+1})-\frac{1}{W(e^{\hat{b}\hat{r}+1})}}$.*

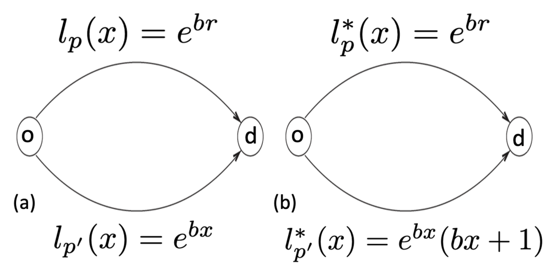

Figure 1: A Variant of Pigou's example

3.3 Alternative Upper Bound

Lemma 3.8. *For any non-negative x, we have*

$$\frac{x}{log(x+1)} \leq \frac{x}{x+2-W(e^{x+1})-\frac{1}{W(e^{x+1})}} \leq \frac{2x}{log(x+1)}$$

Theorem 3.9. *For any instance $(G, \vec{r}, L_{exp})$ with exponential cost functions, its price of anarchy satisfies:*

$$PoA(G, \vec{r}, L_{exp}) \leq \frac{2\hat{b}\hat{r}}{log(\hat{b}\hat{r}+1)}$$

where $\hat{r} := \sum_{i \in [1,k]} r_i$ and $\hat{b} := \max_{p \in P} b_p$.

4 NUMERICAL RESULTS

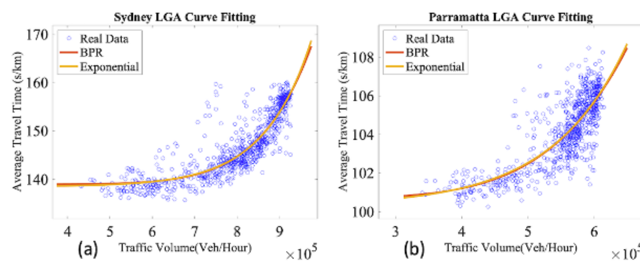

Figure 2: Real-world Data Curve Fit

We used two regions of the Australian NSW data as a basis. We consider each of these regions as a single road segment of a road network, and for each of them, we try to fit the real-world data with an exponential function and with a BPR function. Looking at the results in Fig.2 and Table 1 we see that the fit of the exponential function is slightly better than that of the BPR function, which is evidence to verify the validity of the exponential function.

	Region	m	n	t_0	R^2
BPR	Sydney	0.04671	8.12	139	0.7847
	Parramatta	0.03452	5.598	100.7	0.6434
EXP	Sydney	0.004602	7.325	138.5	0.7856
	Parramatta	0.02734	4.956	100.3	0.6443

Table 1: Curve Fit Results

5 CONCLUSION

The paper investigates the upper bound price of anarchy in road networks with exponential latency functions. It explores the impact of changes in traffic demand on this upper bound, comparing it to the BPR function. Real traffic data supports the validity of the exponential cost function, which provides higher accuracy than the BPR function, especially when traffic rates are lower than capacity.

REFERENCES

[1] Robert M Corless, Gaston H Gonnet, David EG Hare, David J Jeffrey, and Donald E Knuth. 1996. On the LambertW function. *Advances in Computational mathematics* 5, 1 (1996), 329–359.

[2] Robert Gibbons et al. 1992. A primer in game theory. (1992).

[3] Hideo Konishi. 2004. Uniqueness of user equilibrium in transportation networks with heterogeneous commuters. *Transportation science* 38, 3 (2004), 315–330.

[4] Igal Milchtaich. 2005. Topological conditions for uniqueness of equilibrium in networks. *Mathematics of Operations Research* 30, 1 (2005), 225–244.

[5] Jianglin Qiao, Dave de Jonge, Dongmo Zhang, Simeon Simoff, Carles Sierra, and Bo Du. 2023. Price of anarchy of traffic assignment with exponential cost functions. *Autonomous Agents and Multi-Agent Systems* 37, 2 (2023), 42.

[6] Tim Roughgarden. 2003. The price of anarchy is independent of the network topology. *J. Comput. System Sci.* 67, 2 (2003), 341–364.

[7] Xuehe Wang, Nan Xiao, Lihua Xie, Emilio Frazzoli, and Daniela Rus. 2015. Analysis of Price of Anarchy in Traffic Networks With Heterogeneous Price-Sensitivity Populations. *IEEE Transactions on Control Systems Technology* 23, 6 (2015), 2227–2237. https://doi.org/10.1109/TCST.2015.2410762

[8] J G Wardrop. 1952. ROAD PAPER. SOME THEORETICAL ASPECTS OF ROAD TRAFFIC RESEARCH. *Proceedings of the Institution of Civil Engineers* 1, 3 (1952), 325–362. https://doi.org/10.1680/ipeds.1952.11259