Incentives for Early Arrival in Cooperative Games

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ABSTRACT

We study cooperative games where players join sequentially, and the value generated by those who have joined at any point must be irrevocably divided among these players. We introduce two desiderata for the value division mechanism: that the players should have incentives to join as early as possible, and that the division should be considered fair. For the latter, we require that each player's expected share in the mechanism should equal her Shapley value if the players' arrival order is uniformly at random.

When the value generation function is submodular, allocating the marginal value to the player satisfies these properties. This is no longer true for more general functions. Our main technical contribution is a complete characterization of 0-1 value games for which desired mechanisms exist. We show that a natural mechanism, Rewarding First Critical Player (RFC), is complete, in that a 0-1 value function admits a mechanism with the properties above if and only if RFC satisfies them; we analytically characterize all such value functions. Moreover, we give an algorithm that decomposes, in an online fashion, any value function into 0-1 value functions, on each of which RFC can be run. In this way, we design an extension of RFC for general monotone games, and the properties are proved to be maintained.

KEYWORDS

Cooperative Games; Early Arrival; Online Mechanisms

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1 INTRODUCTION

Consider a frequent scenario, where a group of people form a partnership for a startup [20]. They have different abilities or funds to contribute and can cooperate to create values. Sharing the value created is a classic problem studied in the literature on cooperative games [3, 8, 19, 23]. Traditional cooperative games distribute the value after the whole coalition is formed. However, in reality, people typically do not all arrive at one point of time; rather, they join sequentially. This creates two issues: first, it is often not realistic to wait until everyone arrives before distributing the value — sometimes, it is not even clear if "everyone" has joined. This requires that values be distributed in an online manner. Second, the time to join can be strategic for a player; for example, a fund may choose the best time to invest in a startup.

In this work, we propose a theory for online cooperative games that explicitly addresses these issues. Firstly, we require that, after each player joins, an irrevocable distribution of the value created so far should be immediately determined. We formalize a property called *online individually rational* to guarantee that players' shares be non-negative and non-decreasing as new players join, so that all are willing to participate till the end. Secondly, to gather resources quickly, and to prevent players from waiting indefinitely for each other to join first, we require a share-dividing mechanism to *incentivize players to join the game as early as possible.* Namely, we require the mechanism to distribute a higher reward to a player when she joins earlier (when the order of the others' arrivals remains fixed). We believe this is a critical property of an online value-sharing mechanism, which has not been discussed in the literature so far.

Incentivizing early arrival is the key property we proposed here, which also has promising applications. For example, considering a group of students working on a hard project which requires different combinations of skills to finish it, the supervisor may want to incentivize the students to join the project as early as possible so that the project can be finished earlier. Again for a startup to quickly get enough funds, they should design a proper reward sharing mechanism to incentivize investors to invest the startup as early as possible.

One may notice that there exist trivial online methods to incentivize early arrivals of players. For example, one may simply always give all the value to the first player in the game. However, such a solution is not fair (e.g., the first player may make no contribution to the value at all). Hence, we use the Shapley value [19], a well-known and widely accepted classic solution to traditional cooperative game, as a benchmark for fairness [6]. More precisely, we require every player's expected reward over all possible joining orders to be exactly her Shapley value in the game, which is referred to as *Shapley-fair* in our setting.

Taking everything together, our contributions are summarized as follows.

- We formalize the requirements mentioned above. We check two trivial ideas, including allocating the Shapley Value to the players and allocating the marginal contribution to the players, and show the limitations of them.
- For 0-1 monotone games, we propose a mechanism called Reward First Critical Player (RFC), and show it to be *complete*. Namely, we analytically characterize the set of games where RFC satisfies all the requirements, and show that any other game does not admit a mechanism with all the properties.
- We extend the method for 0-1 monotone cooperative games to deal with general games. The key idea is to decompose such a game into 0-1 monotone games in an online fashion. Properties of RFC are then extended to general games.

The remainder of the paper is organized as follows. Section 2 introduces the related work and Section 3 gives the concrete model of the problem we study. We then characterize the solution in 0-1 monotone games and a corollary of impossibility results in Section 4. Furthermore, we extend the solution to general games in Section 5. Finally, we discuss future investigations.

2 RELATED WORK

Classic Cooperative Games

Investigation on classic cooperative games can be traced back to the last century [17, 23]. One of the main goals of these studies is to discuss which value distribution should be taken with a consideration of a set of axioms. This was initiated by Shapley value [19], which is the foundation of almost all subsequent studies. For example, in [25], the author characterized the monotone solutions in cooperative games along with Shapley value, and in [12], the author studied the associated consistency among series of games with identical Shapley value. The more abundant investigations can be found in many surveys or books [1, 7].

Different from the traditional research line, our work focuses on the setting where the players can strategically control the time of arrival to the cooperation. Therefore, our approach aligns more closely with a mechanism design standpoint, as we develop a method for distributing value that encourages arriving earlier.

Cooperative Games with Hierarchies

Many studies have already considered hierarchies or dynamics among players in cooperative games. For example, cooperative games were considered where only those coalitions of players are feasible that respect a given precedence structure (denoting the precedence of players' joining order) or permission structures (where players need others' permission to work) on the set of players in [9, 11, 21, 22]. Moreover, generalized cooperative games, where different order of joining players bring different value, were considered in [18], and [28] considered this form and corresponding solutions to cooperative games with precedence structures.

The main difference of these studies from ours is that they treated the players' joining order or structural relationships as a constraint on the value function of the cooperative game, while in our work, a player's joining time is under her manipulation and we expect them to choose specific strategies (i.e., come as early as possible) by designing a proper online value distribution mechanism.

Mechanism Design in Dynamic Settings

Our approach has a similar perspective with the online mechanism design problem. For example, the auction mechanism design in dynamic environments, where players with private valuations of items will arrive or change over time was studied in [2, 15, 16]. Another interesting trend of designing mechanisms with dynamic applications is diffusion incentives [13, 14, 26, 27]. They considered to incentivize the players to invite their neighbors in a social network to join an auction or a collaboration. Furthermore, the authors in [4, 10] investigate online coalition formation games. The main objective of the study is to allocate the asynchronous joining players to groups in a way that maximizes the overall social welfare.

We consider a different setting for cooperative games, where players can control the time of arrival, and our goal of the mechanism design is to guarantee that they are benefited for early arrival. This has important applications to simulate swift collaborations.

Game Decomposition into 0-1 Valued Games

In this study, we use 0-1 monotone valued games as the base games and decompose any game into 0-1 monotone valued games. Previous studies have suggested various foundations for the linear space of transferable utility games, enabling each game to be represented as a linear decomposition of 0-1 games [5, 24, 25]. We proposed a novel decomposition in Section 5 which breaks down any monotone game into a sum of 0-1 monotone games. More importantly, it can be also executed online as each player joins. To the best of our knowledge, this kind of decomposition has not been studied before.

3 THE MODEL

An online cooperative game is given by a triple (N, v, π), where N is a set of players, $v : 2^N \to \mathbb{R}_+$ is a set function, and $\pi \in \Pi(N)$ is a permutation of N ($\Pi(N)$ denotes the set of all permutations of N). Players arrive sequentially, in the order given by π. A coalition is a set $S \subseteq N$ of players, who create a value $v(S)$. $v(\cdot)$ is *normalized* if $v(\emptyset) = 0$, and is *monotone* if $\forall T \subseteq S \subseteq N, v(S) \geq v(T)$. Throughout this work, we consider normalized and monotone games.

If a player i arrives earlier than j according to π, we say $i \prec_\pi j$. Let $p^\pi(i)$ denote the set of players that arrive (weakly) before i, including i: $p^\pi(i) := \{j \mid j \prec_\pi i\} \cup \{i\}$. For a subset $S \subseteq N$, v *restricted to* S, written as $v_{|S}$, is a set function $v_{|S} : 2^S \to \mathbb{R}_+$ defined as $v_{|S}(T) = v(T), \forall T \subseteq S$; π *restricted to* S, written as $\pi_{|S}$, is the permutation of S defined as $i \prec_{\pi_{|S}} j$ iff $i \prec_\pi j$, for all $i, j \in S$.

We look to divide the values in an online fashion as players join; that is, at any point of time, when the set of players that have arrived is S, we should allocate irrevocably to players in S all the value created by S, without the knowledge of v or π beyond the scope of S. We formalize this below.

Definition 3.1 (Prefix). A coalition $S \subseteq N$ is a *prefix* of π if S is the set of first $|S|$ players to arrive according to π. This is denoted as $S \sqsubseteq \pi$.

Definition 3.2 (Local Games). For a game (N, v, π) and a prefix $S \sqsubseteq \pi$, the *local game on S* is the game $(S, v_{|S}, \pi_{|S})$.

Definition 3.3. A *value-sharing policy* $\boldsymbol{\phi}$ maps a game (N, v, π) to an n-tuple of allocations, so that $\phi_i(N, v, \pi) \geq 0$ is player i's share of the value, and $\sum_i \phi_i(N, v, \pi) = v(N)$.

An *online value-sharing mechanism* is given by a value-sharing policy $\boldsymbol{\phi}$, so that after the arrival of each prefix $S \sqsubseteq \pi$, each player $i \in S$ gets a (cumulated) share of $\phi_i(S, v_{|S}, \pi_{|S})$.

When the context is clear, we often omit the first argument of a policy $\boldsymbol{\phi}$, and simply write $\phi_i(v, \pi)$.

To keep the players from quitting early, we require each player's share to weakly increase as more players arrive:

Definition 3.4. An online mechanism is *online individually rational* (OIR) for value function v if for any arrival order π and any $T, S \sqsubseteq \pi$ with $T \subseteq S$, we have $\phi_i(T, v_{|T}, \pi_{|T}) \leq \phi_i(S, v_{|S}, \pi_{|S})$ for every player $i \in T$.

To prevent players from strategically delaying their arrivals, we require each player's share of value to be no larger if she chooses to join later than her actual arrival, assuming the other players' order of arrivals is fixed. Formally,

Definition 3.5. An online mechanism is *incentivizing for early arrival* (I4EA) if for any player i, $\phi_i(N, v, \pi) \geq \phi_i(N, v, \pi')$ for all π and π' such that $\pi_{|N\setminus\{i\}} = \pi'_{|N\setminus\{i\}}$ and $p^{\pi}(i) \subsetneq p^{\pi'}(i)$.

There are trivial mechanisms satisfying OIR and I4EA; consider, e.g., allocating, at any stage, all the current value to the first player. Such a mechanism, however, is easily seen to be unfair. One of the most celebrated notions for fairness in (offline) cooperative games is *Shapley value* (SV). Intuitively, the Shapley value for a player in an offline games is defined by a mental experiment involving an online game, where players arrive in an order that is uniformly at random; each player's expected *marginal contribution* in this mental experiment is then her Shapley value. Now for the truly online games that we study, it is natural to require that, in a mechanism considered fair, a player's expected share should equal her Shapley value if the arrival order is uniformly at random. We now formalize this discussion.

Definition 3.6 (Marginal Contribution). Given a value function v, a player i's *marginal contribution* (MC) to a coalition $S \ni i$ is

$$\mathrm{MC}(i, v, S) := v(S) - v(S \setminus \{i\}).$$

Definition 3.7 (Shapley Value, [19]). Given a value function v, player i's *Shapley Value* (SV) is

$$\mathrm{SV}_i(v) := \frac{1}{|N|!} \sum_{S \subseteq N\setminus\{i\}} |S|!(|N| - |S| - 1)!\, \mathrm{MC}(i, v, S \cup \{i\});$$

equivalently,

$$\mathrm{SV}_i(v) = \frac{1}{|N|!} \sum_{\pi \in \Pi(N)} \mathrm{MC}(i, v, p^{\pi}(i)) .$$

In a monotone game, the MC of any player in any coalition is non-negative; therefore, the SV is also non-negative.

Definition 3.8 (Shapley-Fair). An online mechanism is *Shapley-fair* (SF) for a value function v if for each player $i \in N$,

$$\frac{1}{|N|!} \sum_{\pi \in \Pi(N)} \phi_i(N, v, \pi) = \mathrm{SV}_i(v).$$

In this work, we aim to design online mechanisms that are OIR, I4EA and SF in games as broad as possible.

Two Simple Mechanisms

As a warm-up, we discuss two simple mechanisms. The first one computes the Shapley values for the local game on each prefix $S \sqsubseteq \pi$, and allocates these to the players in S. This mechanism is I4EA, because each player's eventual share is her Shapley value, regardless of the arrival order. However, this mechanism is not OIR. Example 3.9 illustrates this. When the arrival order is (A, B, C), player A's share decreases from 3 to 2.5 when B joins, and this violates OIR. The example also illustrates the calculation of SV's by tabulating all players' MC's according to the arrival order.

Table 1: The MC of A, B, C in all joining orders in Example 3.9.

Order	1st player	MC of 1st	2nd player	MC of 2nd	3rd player	MC of 3rd
A-B-C	A	3	B	1	C	2
A-C-B	A	3	C	2	B	1
B-A-C	B	2	A	2	C	2
B-C-A	B	2	C	1	A	3
C-A-B	C	1	A	4	B	1
C-B-A	C	1	B	2	A	3

Example 3.9. For $N = \{A, B, C\}$ and $v = [v(A), v(B), v(C), v(AB), v(AC), v(BC), v(ABC)] = [3, 2, 1, 4, 5, 3, 6]$, Table 1 shows the MC of each player to the existing coalition at her arrival, according to the 6 arrival orders. The SV of player A is $\mathrm{SV}_A(v) = (3+3+2+4+3+3)/6 = 3$ and the SVs of B, C are both 1.5.

The second simple mechanism awards each player, at her arrival, her MC to the existing coalition, and gives out no more share to this player in the future.

Definition 3.10. In the *Distributing MC* (DMC) mechanism,

$$\phi_i(v_{|S}, \pi_{|S}) = \mathrm{MC}(i, v, p^{\pi}(i)).$$

DMC is Shapley-fair by definition of SV. It is also easily seen to be OIR for monotone games. However, it is not I4EA in general: in Example 3.9, A receives 3 in DMC when π is $[A, C, B]$. However, for $\pi = [C, A, B]$, A's share becomes 4, violating I4EA. The following theorem shows that DMC is I4EA iff v is submodular.

Theorem 3.11. *DMC is I4EA if and only if the value function v is submodular.* [1]

Proof. "⇒": if DMC is I4EA under v, then given any $i, j \in N$ and any $S \subseteq N \setminus \{i, j\}$, construct two joining orders π, π' such that $\pi_{|N\setminus\{i\}} = \pi'_{|N\setminus\{i\}}$, $p^{\pi}(i) \setminus \{i\} = S$ and $p^{\pi'}(i) \setminus \{i\} = S \cup \{j\}$. Then, we have

$$\begin{aligned} 0 &\leq \phi_i(v, \pi) - \phi_i(v, \pi') \\ &= \mathrm{MC}(i, v, S \cup \{i\}) - \mathrm{MC}(i, v, S \cup \{i, j\}) \\ &= [v(S \cup \{i\}) - v(S)] - [v(S \cup \{i, j\}) - v(S \cup \{j\})] \end{aligned}$$

which concludes that v is submodular.

"⇐": If v is submodular, then for any two joining orders π, π' such that $\pi_{|N\setminus\{i\}} = \pi'_{|N\setminus\{i\}}$, $p^{\pi'}(i) = p^{\pi}(i) \cup \{j\}$, we have

$$\begin{aligned} &\phi_i(v, \pi) - \phi_i(v, \pi') \\ =\ &\mathrm{MC}(i, v, p^{\pi}(i)) - \mathrm{MC}(i, v, p^{\pi'}(i)) \\ =\ &\mathrm{MC}(i, v, p^{\pi'}(i) \setminus \{j\}) - \mathrm{MC}(i, v, p^{\pi'}(i)) \\ =\ &\left[v(p^{\pi'}(i) \setminus \{j\}) - v(p^{\pi'}(i) \setminus \{i, j\})\right] \\ &\quad - \left[v(p^{\pi'}(i)) - v(p^{\pi'}(i) \setminus \{j\})\right] \geq 0 \end{aligned}$$

which concludes that DMC is I4EA.

□

4 0-1 VALUED MONOTONE GAMES

In this section, we focus on valuation functions that take value only 0 or 1. Even for such simple functions, it is not a priori clear whether every function admits a mechanism that is OIR, I4EA and SF. A corollary of this section answers this question in the negative. The main technical contribution in this section is a mechanism, *Rewarding The First Critical Player* (RFC, Definition 4.2), which we show to be complete for 0-1 valuation functions, in the sense that for any 0-1 valued v that admits an OIR, I4EA and SF mechanism, RFC also satisfies these properties (Theorem 4.4). We also analytically characterize all such valuation functions (Theorem 4.5). We give a few examples in Section 4.3. In Section 5, we discuss extensions to general valuation functions.

4.1 The RFC Mechanism

When v takes values only 0 or 1 and is monotone, for any arrival order π, there is at most one player whose arrival makes the current coalition's value jump from 0 to 1. We call this player the *marginal player* of (N, v, π). Note that the DMC mechanism allocates all the value to the marginal player. The RFC mechanism, in contrast, considers players that are indispensable in creating the positive value, and allocates the value to the first such player. Such indispensable players are called *critical*. Formally,

Definition 4.1. Given a 0-1 valued v, for any S with $v(S) = 1$, define $S^* := \{j \in S \mid \mathrm{MC}(j, v, S) = 1\}$. For a 0-1 valued v and arrival order π, let i be the marginal player; the set of *critical players* is

$$\mathrm{CR}(\pi, v) := (p^{\pi}(i))^*.$$

[1] A value function v is *submodular* if for every $S, T \subseteq N$ with $T \subseteq S$ and every $i \in N \setminus S$, we have $v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S)$. v is *supermodular* if this inequality goes the other way for all such S, T and i.

Recall that $p^{\pi}(i)$ is the coalition formed after i's arrival. In plain language, a player is critical if she is in $p^{\pi}(i)$ and if her removal makes the coalition's value drop to 0. By definition, the marginal player must be critical, but the set of critical players may include others. In the DMC mechanism, a critical player arriving earlier than the marginal player does not get allocated anything but may choose to delay her arrival to become the marginal player herself; this destroys incentive for early arrival. The RFC mechanism redresses this by awarding to the earliest among the critical players. Crucially, the set of critical players is fully determined by $v_{|p^{\pi}(i)}$ and $\pi_{|p^{\pi}(i)}$.

Definition 4.2 (RFC). The *Rewarding The First Critical Player* (RFC) mechanism is defined by the following value-sharing policy: for any prefix $S \sqsubseteq \pi$ with $v(S) = 1$, and player $i \in S$,

$$\phi_i(v_{|S}, \pi_{|S}) = \begin{cases} 1, \text{ if } i \in \mathrm{CR}(\pi_{|S}, v_{|S}) \text{ and} \\ \quad \forall j \in \mathrm{CR}(\pi_{|S}, v_{|S}) \setminus \{i\}, i \preceq j, \\ 0, \text{ otherwise.} \end{cases}$$

For prefix S with $v(S) = 0$, no player gets allocated anything.

Theorem 4.3. *For all 0-1 valued, monotone v, RFC is OIR and SF.*

Proof. **OIR**: For any prefix $S, T \sqsubseteq \pi$ with $T \subseteq S$, if $v(T) = 1$, then $v(S) = 1$; the marginal player is the same in the local games $(T, v_{|T}, \pi_{|T})$ and $(S, v_{|S}, \pi_{|S})$, and so is the set of critical players. The earliest critical player is therefore also the same in both local games, and her share is 1 in both local games, and the other players get 0. If $v(T) = 0$, all players get 0 in the local game $(T, v_{|T}, \pi_{|T})$, and their shares cannot get worse later.

SF: We show that the expected share of any player in RFC is the same as her expected share in DMC, and DMC is SF by definition of Shapley value (see Section 3). For joining order $\pi_1 = [..., j, ..., i, ...]$, let i be the marginal player and j the player that wins the value in RFC. Consider $\pi_2 = [..., i, ..., j, ...]$ which is identical to π_1 except that the positions of j and i are exchanged. As we have $p^{\pi_1}(i) = p^{\pi_2}(j)$, observe that $\mathrm{CR}(\pi_1, v) = \mathrm{CR}(\pi_2, v)$, and the earliest arrival in $\mathrm{CR}(\pi_2, v)$ is i. Therefore, $\phi^{RFC}(v, \pi_1) = \phi^{DMC}(v, \pi_2)$. Furthermore, the mapping from π_1 to π_2 is one-to-one, so we have

$$\frac{1}{|N|!} \sum_{\pi_1 \in \Pi(N)} \phi^{RFC}(v, \pi_1) = \frac{1}{|N|!} \sum_{\pi_2 \in \Pi(N)} \phi^{DMC}(v, \pi_2) = \mathrm{SV}(v)$$

□

4.2 Completeness of RFC

The RFC mechanism was motivated to redress an incentive issue in the DMC mechanism. Perhaps surprisingly, we show that RFC not only outperforms DMC in the sense that it is I4EA for broader 0-1 valued games, but it is the best among all mechanisms for such valuation functions: whenever a 0-1 valued v admits an OIR, SF and I4EA mechanism, RFC is such a mechanism as well (Theorem 4.4). We then precisely characterize all such valuation functions(Theorem 4.5). Figure 1 illustrates the corresponding categorization of 0-1 valuation functions.

Theorem 4.4. *For any 0-1 valued monotone v, if there exists a mechanism satisfying OIR, SF and I4EA, then RFC is such a mechanism.*

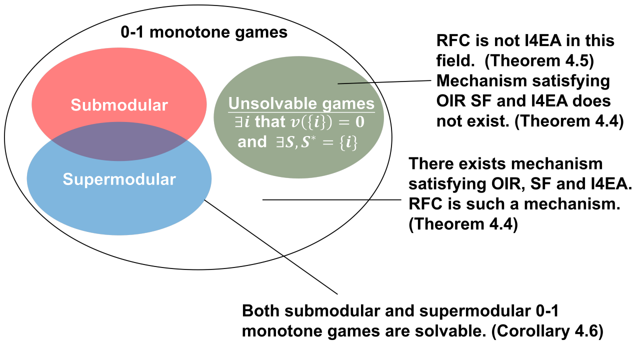

Figure 1: Summary of the theorems mentioned in Section 4.2

Theorem 4.5. *For any 0-1 valued monotone v, RFC is not I4EA if and only if there exists i such that $v(\{i\}) = 0$ and $\exists S, S^* = \{i\}$. (Recall the definition of S^* from Definition 4.1.)*

Corollary 4.6. *RFC is OIR, SF and I4EA on submodular and supermodular 0-1 valued monotone games.*

To show the correctness of above conclusions, we start with the proof of Theorem 4.5. The key is to show that a player who loses by being truthful in RFC can strategically delay her arrival to obtain a positive share if and only if she can become the unique critical player by doing so. Then we prove Theorem 4.4 by showing that no mechanisms can be OIR, SF and I4EA for a valuation function with the property in Theorem 4.5. Corollary 4.6 is straightforward.

Proof of Theorem 4.5. We prove by showing the following statements are equivalent. For any v,

(1) RFC is not I4EA.
(2) There exists S such that $\exists T \subsetneq S$ and $\exists i \in T$, $S^* = \{i\}$ and $T^* \supsetneq \{i\}$.
(3) There exists i such that $v(\{i\}) = 0$ and $\exists S, S^* = \{i\}$.

$1 \Rightarrow 2$. As violating I4EA, there exist $\pi_1 = [...i, j, ...], \pi_2 = [...j, i, ...]$ where $\phi_i(v, \pi_1) < \phi_i(v, \pi_2)$. [2] We discuss by categories as following:

- If $i \notin \mathrm{CR}(\pi_1, v)$, then we can infer $i \notin \mathrm{CR}(\pi_2, v)$ as $v(p^{\pi_2}(i) \setminus \{i\}) = v(p^{\pi_1}(i) \cup \{j\} \setminus \{i\}) \geq v(p^{\pi_1}(i) \setminus \{i\}) = 1$, which means i is not critical in both orders. Therefore, $\phi_i(v, \pi_1) = \phi_i(v, \pi_2) = 0$, which leads to a contradiction.
- If $i \in \mathrm{CR}(\pi_1, v)$ but i is not the marginal player, then we can infer that $\mathrm{CR}(\pi_1, v) = \mathrm{CR}(\pi_2, v)$. The reason is that if j is the marginal player in π_1, then i is the marginal player in π_2 and $\mathrm{CR}(\pi_1, v) = (p^{\pi_1}(j))^* = (p^{\pi_2}(i))^* = \mathrm{CR}(\pi_2, v)$; if j is not the marginal player in π_1, then $\mathrm{CR}(\pi_1, v) = \mathrm{CR}(\pi_2, v)$ as the players before then marginal player are same in two orders. Overall, $\phi_i(v, \pi_1) \geq \phi_i(v, \pi_2)$ as i moves to a not prior position comparing to other critical players. This leads to a contradiction.
- For i being the marginal player in π_1, to ensure $\phi_i(v, \pi_1) < \phi_i(v, \pi_2)$, we have $\phi_i(v, \pi_1) = 0$ and $\phi_i(v, \pi_2) = 1$. That means i is not the unique critical player in π_1 and moreover, i is the first critical player in π_2. Together with $v(p^{\pi_2}(i)) = 1$, we can infer i is the unique critical player in π_2.

In conclusion, we can find $T = p^{\pi_1}(i)$ and $S = p^{\pi_2}(i)$ that $T^* \supsetneq \{i\}$ and $S^* = \{i\}$.

$1 \Leftarrow 2$. Construct two joining orders $\pi_1 = [..., i, ...j, ...], \pi_2 = [..., j, i, ...]$ where only the position of i is moved and $p^{\pi_1}(i) = T$, $p^{\pi_1}(j) = p^{\pi_2}(i) = S$, then $v(p^{\pi_1}(i)) = 0$ and $v(p^{\pi_2}(i)) = 1$. Start with π_1 and delay i's arrival one position backwards each time until π_2, there exists an order where i is the marginal player.

$2 \Rightarrow 3$. For S that $S^* = \{i\}$, if $v(\{i\}) = 1$, then $\forall T \subseteq S \setminus \{i\}$ and for every $j \in T$, we have $v(T \setminus \{j\} \cup \{i\}) \geq v(\{i\}) = 1$ and $v(T) \leq v(S \setminus \{i\}) = 0$. Then $(T \cup \{i\})^* = \{i\}$, which is contradict to $T \supsetneq \{i\}$. Therefore, $v(\{i\}) = 0$.

$2 \Leftarrow 3$. Construct $\pi = [i, ..., j, ...]$ where $p^{\pi}(j) = S$. As we have $v(\{i\}) = 0$ and $v(S) = 1$, we can infer that there is a marginal player arrives later than i and not later than j. We denote that player as i', then for $T = p^{\pi}(i')$, we have $v(T) = 1$ and $v(T \setminus \{i\}) \leq v(S \setminus \{i\}) = 0$. Therefore, $\{i, i'\} \subseteq T^*$. □

Lemma 4.7. *For every T, S satisfying $T \subseteq S$, we have $T^* = \emptyset$ if $v(T) = 0$ and $T^* \supseteq S^*$ otherwise.*

Proof. If $v(T) = v(S) = 1$, for any $i \in S^*$, $v(T \setminus \{i\}) \leq v(S \setminus \{i\}) = 0$, so $i \in T^*$. Notice that there exists $T \subseteq S$ satisfying $v(T \setminus \{i\}) = 0 < v(S \setminus \{i\})$, so $T^* \not\subset S^*$.

If $v(T) = 0$, it's obvious that $T^* = \emptyset$. □

Proof of Theorem 4.4. We prove this theorem by showing no mechanisms satisfy OIR, SF and I4EA together in the games characterized by Theorem 4.5. For such i in Theorem 4.5, we can find S that $S^* = \{i\}$ and $\forall j \in S$ satisfying $j \neq i$, $(S \setminus \{j\})^* \supsetneq \{i\}$. This can be guaranteed because if there is j that $(S \setminus \{j\})^* = \{i\}$, we can eliminate j from S. As we have $v(\{i\}) = 0$, so the coalition after the eliminations would be a superset of $\{i\}$. What's more, we can infer $\forall T \subseteq S, T^* \supsetneq \{i\}$ or $T^* = \emptyset$ with Lemma 4.7.

We consider the existence of mechanism on $v_{|S}$. The SV of i can be derived as

$$\mathrm{SV}_i(v_{|S}) = \frac{1}{|S|!} \sum_{T \subseteq S \setminus \{i\}} |T|!(|S| - |T| - 1)! \,\mathrm{MC}(i, v_{|S}, T \cup \{i\})$$

$$= \frac{1}{|S|!} \sum_{T \subseteq S, i \in T^*} (|T| - 1)!(|S| - |T|)!$$

$$= \frac{1}{|S|!} \sum_{T \subseteq S, T^* = \{i\}} (|T| - 1)!(|S| - |T|)!$$

$$+ \frac{1}{|S|!} \sum_{T \subseteq S, T^* \supsetneq \{i\}} (|T| - 1)!(|S| - |T|)!$$

$$= \frac{1}{|S|!} \left((|S| - 1)! + \sum_{T \subsetneq S, v_{|S}(T) \neq 0} (|T| - 1)!(|S| - |T|)! \right)$$

For joining order $\pi_{|S}$ where i is not the last to arrive, an OIR and SF mechanism will allocate all the value to the players before j because $v_{|S}(S \setminus \{j\}) = 1$. In such joining orders, the expected value of i is equal to $\mathrm{SV}_i(v_{|S \setminus \{j\}})$. Then, if we only consider the joining orders where i is not the last to join, the expected value of i is to average the expected value of i in each joining order. Let

[2] When the context is clear, we denote $\pi_1 = [..., i, j, ...]$ and $\pi_2 = [..., j, i...]$ as two orders where only i is delayed one position backwards and the order is same for everyone else.

$\pi_{|S} \sim [..., i, ...]$ and $\pi_{|S} \sim [..., i]$ denote that i is not the last to arrive and i is the last to arrive respectively. Then, we have

$$\mathbf{E}_{\pi_{|S}\sim[...,i,...,j]}\phi_i(\pi_{|S}, v_{|S}) = \frac{\sum_{j\in S\setminus\{i\}} \mathrm{SV}_i(v_{|S-\{j\}})}{|S|-1}$$

By classifying the joining orders, we can calculate the expected value of i as

$$\begin{aligned}
&\frac{1}{|S|!}\sum_{\pi_{|S}\in\Pi(S)} \phi_i(v_{|S}, \pi_{|S}) \\
=& \frac{(|S|-1)!}{|S|!}\mathbf{E}_{\pi_{|S}\sim[...,i]}\left[\phi_i(v_{|S}, \pi_{|S})\right] \\
&+ \frac{(|S|-1)(|S|-1)!}{|S|!}\mathbf{E}_{\pi_{|S}\sim[...,i,...]}\left[\phi_i(v_{|S}, \pi_{|S})\right] \\
=& \frac{(|S|-1)!}{|S|!}\mathbf{E}_{\pi_{|S}\sim[...,i]}\left[\phi_i(v_{|S}, \pi_{|S})\right] \\
&+ \frac{(|S|-1)(|S|-1)!}{|S|!}\left(\frac{\sum_{j\in S\setminus\{i\}} \mathrm{SV}_i(v_{|S-\{j\}})}{|S|-1}\right) \\
=& \frac{(|S|-1)!}{|S|!}\mathbf{E}_{\pi_{|S}\sim[...,i]}\left[\phi_i(v_{|S}, \pi_{|S})\right] \\
&+ \frac{1}{|S!|}\sum_{j\in S\setminus\{i\}}\ \sum_{T\subseteq S\setminus\{j\}, i\in T^*}(|T|-1)!(|S|-1-|T|)! \qquad (1) \\
=& \frac{(|S|-1)!}{|S|!}\mathbf{E}_{\pi_{|S}\sim[...,i]}\left[\phi_i(v_{|S}, \pi_{|S})\right] \\
&+ \frac{1}{|S!|}\sum_{T\subsetneq S, v_{|S}(T)\neq 0}(|T|-1)!(|S|-|T|)! \qquad (2)
\end{aligned}$$

Here, the term in (1) equals to the term in (2) as they are both counting the number of joining orders where i can create MC but not being the last to join with different aspects. More concretely, the term in (1) counts by classifying the last member of the order while the term in (2) counts directly. Therefore, to ensure the SF, the mechanism should guarantee $\mathbf{E}_{\pi_{|S}\sim[...,i]}[\phi_i(v_{|S}, \pi_{|S})] = 1$ which means allocating 1 to i when i is the last to join. However, when i is not the last to join, i is expected to receive less than 1, so there exists chance for i to increase the value by delay. In conclusion, when RFC is not I4EA on v, then any mechanism satisfying OIR and SF is not I4EA. □

Proof of Corollary 4.6. Assume there exists i satisfying $v(\{i\}) = 0$ and there exists S such that $S^* = \{i\}$. Then we have $v(\{i\}) - v(\emptyset) = 0 < 1 = v(S) - v(S \setminus \{i\})$, which leads to a contradiction to submodularity. For $T \subsetneq S$ that $T^* \supsetneq \{i\}$, for every $j \in T^* \setminus \{i\}$, we have $v(T) - v(T \setminus \{j\}) = 1 > 0 = v(S) - v(S \setminus \{j\})$, which leads to a contradiction to supermodularity. □

4.3 Examples

Example 4.8 shows a three-player valuation function that satisfies the condition in Theorem 4.5. We show in Proposition 4.9 that this is in fact the only three-player valuation with this property. We then give another valuation function in Example 4.10 that is neither submodular nor supermodular, for which RFC is OIR, SF and I4EA.

Example 4.8. Consider a game where $N = \{A, B, C\}$ and $v = [v(A), v(B), v(C), v(AB), v(AC), v(BC), v(ABC)] = [0, 0, 0, 0, 1, 1, 1]$, the marginal player and the critical players are listed in the 2nd column and 3rd column of Table 2. In the 4th column, we list the receivers of the values determined by RFC. In this game, *RFC* is not I4EA as we have $v(C) = 0$ and $\{A, B, C\}^* = \{C\}$. More specifically, in order $[A, C, B]$, C is the marginal player but not the unique critical player when she joins, so the value would be allocated to A. However, in order $[A, B, C]$, C is both the marginal player and the unique critical player when she joins, so she would get the value.

Table 2: The marginal player, critical players and the value receiver determined by RFC of game where $N = \{A, B, C\}$ and $v = [0, 0, 0, 0, 1, 1, 1]$ in every order.

Joining Order	Marginal Player	Critical Players	Value Receiver
[A,B,C]	C	C	C
[A,C,B]	C	A,C	A
[B,A,C]	C	C	C
[B,C,A]	C	B,C	B
[C,A,B]	A	C,A	C
[C,B,A]	B	C,B	C

Table 3: 3-player 0-1 valued monotone games and the interpretation of the value allocation on them.

v	Allocation of RFC	I4EA
[1,1,1,1,1,1,1]	To the first joining player	√
[0,1,1,1,1,1,1]	To the first of $\{B, C\}$	√
[0,0,1,1,1,1,1]	To C if she is the first or second. To first of $\{A, B\}$ in other case.	√
[0,0,0,1,1,1,1]	To the first joining player	√
[0,0,1,0,1,1,1]	To C.	√
[0,0,0,0,1,1,1]	To C if she is the first or third. To first of $\{A, B\}$ in other case.	
[0,0,0,0,0,1,1]	To the first of $\{B, C\}$	√
[0,0,0,0,0,0,1]	To the first joining player	√

Let $N = \{A, B, C\}$ and consider all possible 0-1 valued monotone games on N. Without loss of generality, we assume $v(C) \geq v(B) \geq v(A)$ and $v(BC) \geq v(AC) \geq v(AB)$. The games satisfying our assumption and the interpretations of the value allocations are listed in Table 3. $v = [0, 0, 0, 0, 1, 1, 1]$ is the only one that RFC is not I4EA.

Proposition 4.9. *For $N = \{A, B, C\}$ and v satisfying $v(C) \geq v(B) \geq v(A)$ and $v(BC) \geq v(AC) \geq v(AB)$, RFC is not I4EA if and only if $v = [0, 0, 0, 0, 1, 1, 1]$.*

Example 4.10. Consider a game where $N = \{A, B, C\}$ and $v = [0, 0, 0, 1, 1, 1, 1]$, notice that v is neither submodular nor supermodular as $v(AB) - v(B) = 1 > 0 = v(ABC) - v(BC) = v(A) - v(\emptyset)$. The RFC always allocates the value to the first joining player after the second player joins, which is I4EA.

5 EXTENSION TO GENERAL VALUATION FUNCTIONS

In this section, we propose an extension of RFC for general valuation functions. We give a procedure (Algorithm 1) that decomposes any monotone valuation function into a weighted sum of 0-1 monotone valuation functions. Importantly, this decomposition is done in an online fashion as players arrive. An RFC is then run, simultaneously, on each 0-1 valued component, and each player's share is the weighted sum of her shares from the decomposed 0-1 games.

5.1 GM-Decomposition

We firstly introduce the decomposing process in the mechanism, which is called the *greedy monotone decomposition* (GM). GM gives a non-negative linear combination of a monotone game as $v = \sum_k c_k g_k$ where $\{g_k\}$ are the 0-1 game components and $\{c_k\}$ are the coefficients. We denote $D(v) = \{(g_k, c_k)\}$ as the set of component-coefficient pairs which determines a decomposition. In each iteration, we greedily split a scaled 0-1 valued monotone set function from the current set function until it becomes zero. An example for this decomposition is

$$\begin{aligned} v =&[1, 2, 3, 4, 5, 6, 7] \\ =&[1, 1, 1, 1, 1, 1, 1] + [0, 1, 1, 1, 1, 1, 1] + [0, 0, 1, 1, 1, 1, 1] \\ &+ [0, 0, 0, 1, 1, 1, 1] + [0, 0, 0, 0, 1, 1, 1] + [0, 0, 0, 0, 0, 1, 1] \\ &+ [0, 0, 0, 0, 0, 0, 1]. \end{aligned}$$

We formalize the GM in Algorithm 1, where v_k, c_k, g_k denote the function to be split, the coefficient and the component we get respectively in the kth iteration. Intuitively, in each iteration of Algorithm 1, we find the coalition with minimum non-zero value in v_k and assigned it to c_k and g_k has value 1 for all non-zero value coalitions in v_k. Then we decrease all the positive values in v_k by c_k and start next iteration. At the end of the iterations, we get the GM of the input function v.

It is easy to check that GM has the following properties, which is the reason why we choose it for extending the RFC: (1) the GM provides a positive linear combination of a set function; (2) the component functions are monotone, shown in Proposition 5.1; (3) a game is decomposed consistently in both global and local games, shown in Proposition 5.2.

Algorithm 1 Greedy Monotone Decomposition (GM)

Input: monotone v.
Output: a decomposition $D(v)$.

```
Let D be an empty list.
Let v_1 be a copy of v.
while max(v_k) > 0 do
    S ← argmin_{T⊆N, v(T)>0} v_k(T)
    c_k ← v_k(S)
    Let g_k be a set function.
    for T ⊆ N do
        if v_k(T) > 0 then
            g_k(T) ← 1
        else if v_k(T) = 0 then
            g_k(T) ← 0
    v_{k+1} ← v_k − c_k g_k
    Put (g_k, c_k) into D.
    k ← k + 1
return D.
```

Table 4: The GMs of $v = [1, 2, 3, 4, 5, 6, 7]$ and its restriction $v_{|AB}$.

Coalition	A	B	C	AB	AC	BC	ABC	Coefficient
v	1	2	3	4	5	6	7	
Components of $D(v)$	1	1	1	1	1	1	1	1
		1	1	1	1	1	1	1
			1	1	1	1	1	1
				1	1	1	1	1
					1	1	1	1
						1	1	1
							1	1
$v_{\|AB}$	1	2		4				
Components of $D(v_{\|AB})$	1	1		1				1
		1		1				1
				1				2

Proposition 5.1. *GM outputs a decomposition $v = \sum_k c_k g_k$ where $\{g_k\}$ are 0-1 valued monotone functions.*

Proof. Notice that in GM, we have the relation that $\forall S_1, S_2$, if $v(S_1) \le v(S_2)$, then $\forall k, v_k(S_1) \le v_k(S_2)$ and $g_k(S_1) \le g_k(S_2)$. Therefore, we can infer that $\forall k$, if v is monotone, then v_k, g_k are monotone games. □

Proposition 5.2. *Given v with $D(v) = \{(g_k, c_k)\}$, then for any S, then $D(v_{|S})$ is equivalent to $\{(g_{k|S}, c_k)\}$ where $g_{k|S}$ is g_k restricted to S.*

Proof. Let $D(v_{|S}) = \{f_{k'}, e_{k'}\}$ be the GM of $v_{|S}$. We claim that for any k', there exists k that $g_{k|S} = f_{k'}$. Otherwise, there exists S and T that $v(S) \ge v(T)$ while $v_{|S}(S) < v_{|S}(T)$, which leads to a contradiction. Now we can rewrite the decomposition of $v_{|S}$ as

$$v_{|S} = \sum_{k'} f_{k'} \left(\sum_{k: g_{k|S} = f_{k'}} c_k \right).$$

Notice that $\{f_{k'}\}$ are linear independent, so the corresponding decomposition is unique, which means $e_{k'} = \sum_{k: g_{k|S} = f_{k'}} c_k$. Therefore, $D(v_{|S})$ is equivalent to $\{(g_{k|S}, c_k)\}$ if we merge the coefficients of homogeneous components. □

Example 5.3. To better show the Proposition 5.2, we build Table 4 for the previous decomposition example. It contains the GM-decompositions of $v = [1, 2, 3, 4, 5, 6, 7]$ and $v_{|AB} = [1, 2, 4]$. The red cells are the restrictions to $\{A, B\}$ of the components in $D(v)$. In $D(v)$, only 4 component functions have non-zero restrictions on $\{A, B\}$. The blue cells are decomposition of $v_{|AB}$. If we merge the homogeneous components' coefficients, the red cells are equivalent to the blue cells. That means $v_{|AB}$ are decomposed consistently in $D(v)$ and $D(v_{|AB})$.

5.2 The Extended RFC

Now we propose the extended RFC mechanism based on GM. The mechanism firstly does GM-decomposition on input set function v. Then it calculates the value in each 0-1 valued monotone games by RFC and accumulates them with coefficients to be the value in v. The properties of RFC are maintained through this process.

Definition 5.4. The *extended rewarding first critical player mechanism* (eRFC) is defined by

$$\bar{\phi}_i(v_{|S}, \pi^S) = \sum_{(g_k, c_k) \in D(v_{|S})} c_k \phi_i^{RFC}(g_k, \pi^S)$$

where ϕ_i^{RFC} is the value-sharing policy of RFC and $D(\cdot)$ is the GM-decomposition.

THEOREM 5.5. *eRFC is SF and OIR.*

PROOF. **SF**: SF is maintained as the SV satisfies *additivity*, i.e. $\mathrm{SV}_i(v + w) = \mathrm{SV}_i(v) + \mathrm{SV}_i(w)$ for any v, w. Therefore, we have

$$\begin{aligned}\bar{\phi}_i(v, \pi) &= \sum_{(g_k, c_k) \in D(v)} c_k \phi_i^{RFC}(g_k, \pi) \\ &= \sum_{(g_k, c_k) \in D(v)} c_k \, \mathrm{SV}_i(g_k) = \mathrm{SV}_i(v)\end{aligned}$$

OIR: Given π, for any $T, S \sqsubseteq \pi$ with $T \subseteq S$, we have

$$\begin{aligned}\bar{\phi}_i(v_{|S}, \pi_{|S}) &= \sum_{(g_k, c_k) \in D(v_{|S})} c_k \phi_i^{RFC}(g_k, \pi_{|S}) \\ &\geq \sum_{(g_k, c_k) \in D(v_{|S})} c_k \phi_i^{RFC}(g_{k|T}, \pi_{|T}) \\ &= \bar{\phi}_i(v_{|T}, \pi_{|T}).\end{aligned}$$

□

THEOREM 5.6. *eRFC is I4EA for monotone v if for every g_k in $D(v)$, RFC is I4EA on g_k.*

PROOF. For $\pi_1 = [..., i, j, ...]$ and $\pi_2 = [..., j, i, ...]$, we have

$$\begin{aligned}\bar{\phi}_i(v, \pi_1) &= \sum_{(c_k, g_k) \in D(v)} c_k \cdot \phi_i(g_k, \pi_1) \\ &\geq \sum_{(c_k, g_k) \in D(v)} c_k \cdot \phi_i(g_k, \pi_2) = \bar{\phi}_i(v, \pi_2)\end{aligned}$$

□

COROLLARY 5.7. *For symmetric monotone games, eRFC is I4EA.*

PROOF. Notice that for symmetric games, we have $v(S_1) = v(S_2)$ if $|S_1| = |S_2|$. Therefore, in $D(v)$, every g_k is also symmetric. In symmetric g_k, $\forall i, j \in S$ satisfying $i \neq j$, $g_k(S \setminus \{i\}) = g_k(S \setminus \{j\})$. We can infer that if $i \in S^*$, then $j \in S^*$. This leads to an impossibility of the condition $S^* = \{i\}, g_k(\{i\}) = 0$. □

In Theorem 5.6, the condition is sufficient but not necessary, which is different from the results in 0-1 valued monotone games. In the following example, eRFC satisfies the desired properties even there is a component function where RFC is not I4EA.

Example 5.8. Let $v = [0, 1, 1, 1, 3, 2, 3]$ which can be decomposed as $[0, 1, 1, 1, 1, 1, 1, 1] + [0, 0, 0, 0, 1, 1, 1] + [0, 0, 0, 0, 1, 0, 1]$. Here we have $[0, 0, 0, 0, 1, 1, 1]$ violating the condition. The allocations to players in different joining orders are shown in Table 5. Even C could delay to get a higher value in a component game, she might lose in the other component games. Therefore, the delay would not bring an increase on the total value to C in original game.

Table 5: The allocations determined by eRFC to A, B, C in all possible joining orders in the game where $N = \{A, B, C\}$ and $v = [0, 1, 1, 1, 3, 2, 3]$. The allocations of RFC to the components of v are listed in the middle part of the table.

Joining Order	Components	Allocations in Components			Total Value Allocation		
		A	B	C	A	B	C
[A,B,C]	[0,1,1,1,1,1,1]		1		1	1	1
	[0,0,0,0,1,1,1]			1			
	[0,0,0,0,1,0,1]	1					
[A,C,B]	[0,1,1,1,1,1,1]			1	2	0	1
	[0,0,0,0,1,1,1]	1					
	[0,0,0,0,1,0,1]	1					
[B,A,C]	[0,1,1,1,1,1,1]		1		1	1	1
	[0,0,0,0,1,1,1]			1			
	[0,0,0,0,1,0,1]	1					
[B,C,A]	[0,1,1,1,1,1,1]		1			2	1
	[0,0,0,0,1,1,1]		1				
	[0,0,0,0,1,0,1]			1			
[C,A,B]	[0,1,1,1,1,1,1]			1			3
	[0,0,0,0,1,1,1]			1			
	[0,0,0,0,1,0,1]			1			
[C,B,A]	[0,1,1,1,1,1,1]			1			3
	[0,0,0,0,1,1,1]			1			
	[0,0,0,0,1,0,1]			1			

It is also worth to note that GM might not be the best choice for decomposition from a global point of view. For example, if we decompose a submodular function $v = [1, 1, 1, 1, 2, 2, 2]$ as the sum of $[1, 1, 1, 1, 1, 1, 1]$ and $[0, 0, 0, 0, 1, 1, 1]$, then running eRFC on the components and accumulating the reward is not I4EA. However, with global information, we can also decompose the function as $[1, 1, 0, 1, 1, 1, 1]$ and $[0, 0, 1, 0, 1, 1, 1]$ satisfying the condition. For this work, we do not extend RFC with such a decomposition as we require a mechanism that only uses local information. It is an open question that if there exist better decomposition methods to extend RFC so the I4EA can be maintained on more games.

6 FUTURE WORK

There are several future directions worth investigation. For 0-1 valued monotone games, one may consider to characterize a whole set of mechanisms satisfying all the properties on solvable games. For general monotone games, the characterization of all solvable games is still open and there might exist other decomposition that works on more games. It is also worth studying the I4EA property in offline (make the decisions until everyone joined) or other (such as cost-sharing and hedonic game) settings.

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