Weighted Proportional Allocations of Indivisible Goods and Chores: Insights via Matchings

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ABSTRACT

We study fair allocation of indivisible goods and chores for agents with ordinal preferences and arbitrary entitlements. In the case of both goods and chores, we show that there always exist allocations that are *weighted necessarily proportional up to one item* (WSD-PROP1), that is, allocations that are WPROP1 under all additive valuations consistent with agents' ordinal preferences. We give a polynomial-time algorithm to find such allocations by reducing it to a problem of finding perfect matchings in a bipartite graph. We give a complete characterization of these allocations as extreme points of a perfect matching polytope. Using this polytope, we can optimize any linear objective function over all WSD-PROP1 allocations, for example, to find a min-cost WSD-PROP1 allocation of goods or most efficient WSD-PROP1 allocation of chores. Additionally, we show the existence and computation of sequencible (SEQ) WSD-PROP1 allocation using rank-maximal perfect matchings and show the incompatibility of Pareto optimality under all valuations with the WSD-PROP1 notion.

We also consider the notion of Best-of-Both-Worlds (BoBW) fairness. Using our characterization, we give a polynomial-time algorithm to compute Ex-ante envy-free (WSD-EF) and Ex-post WSD-PROP1 allocations for both goods and chores.

KEYWORDS

Fair Division; Matchings; Proportionality

ACM Reference Format:
Vishwa Prakash H.V. and Prajakta Nimbhorkar. 2024. Weighted Proportional Allocations of Indivisible Goods and Chores: Insights via Matchings. In *Proc. of the 23rd International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2024), Auckland, New Zealand, May 6 – 10, 2024*, IFAAMAS, 9 pages.

1 INTRODUCTION

Discrete fair allocation is a fundamental problem at the intersection of economics and computer science with applications in various multi-agent settings. Here, we are required to allocate a set of indivisible items to agents based on their preferences, such that each item is allocated to exactly one agent. This setting is commonly referred to as the *assignment problem* [5, 20, 22, 35, 44]. In this setting, there is a set A of n agents and a set B of m indivisible items with each agent $a_i \in A$ expressing an ordinal preference ordering over the items in B, given by a permutation $\pi_i(B)$ of the items in B. In addition to their ordinal preferences, agents may also have private cardinal valuations that reflect the utility or disutility of each item, ensuring compatibility with their ordinal preferences. The goal is to allocate the items to agents in a *fair* manner. In this paper, we focus on *additive valuations*, where the utility (or disutility) of a set of items is the sum of the utilitites (or disutilities) of individual items.

The set B can represent *goods*, covering scenarios such as inheritance division, house allocation, allocation of public goods, among others. Alternatively, B could be a set of *chores*, modeling situations like task allocation among employees or household chore distribution between couples and so on.

Among various notions of fairness studied in the literature, two prominent ones are *Envy-freeness* (EF) and *Proportionality* (PROP). An allocation is said to be *envy-free* if no agent would prefer to have the bundle held by any of the others. On the other hand, proportionality requires that each agent receives a set of items whose value is at least (at most, for chores) her proportional share of the total value of all the items. Unfortunately, PROP or EF allocations do not always exist and are NP-hard to compute [5, 14, 32]. Hence relaxations of these notions have been proposed in literature. PROP is relaxed as Proportionality *up to one item* (PROP1) [6, 8, 16, 19] and EF is relaxed as Envy-free up to one item (EF1)[17, 32].

In practical scenarios, agents can have varying *entitlements* in situations such as inheritance division, division of shares among investors and so on. To capture such cases, a more generalized version of these notions, namely the weighted envy-freeness WEF [6, 18] and weighted proportionality WPROP[4, 25] are considered.

Given only the ordinal preferences, these notions of fairness are further strengthened by considering the *stochastic dominance* (SD) relation. An agent prefers one allocation over another with respect to the SD relation if she gets at least as much utility from the former allocation as the latter for all cardinal utilities consistent with the ordinal preferences. An allocation is said to be weighted *necessarily* proportional (also known as weighted strong SD proportionality) (WSD-PROP) if it remains WPROP under all cardinal utilities consistent with the ordinal preferences. Similarly, the notion of envy-freeness (EF) can be extended to weighted necessarily envy-freeness (WSD-EF). Clearly, just like PROP and EF, WSD-PROP and WSD-EF allocations may not exist. In fact, as shown in [5, 39], in an SD-PROP allocation, each agent must receive their most favorite item - which is not realizable when two or more agents have the same most favorite item. For agents with varying entitlements and under ordinal preferences, these notions can be relaxed to WSD-EF1 and WSD-PROP1 (see e.g. [18, 25, 45]).

Proc. of the 23rd International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2024), N. Alechina, V. Dignum, M. Dastani, J.S. Sichman (eds.), May 6 – 10, 2024, Auckland, New Zealand. © 2024 International Foundation for Autonomous Agents and Multiagent Systems (www.ifaamas.org).

Given the non-existence of WPROP allocations, a well-studied notion of fairness is that of WPROPx. Although WPROPx allocations exists under cardinal valuations [31], an analogous notion for the ordinal instances - namely, WSD-PROPx allocations - need not exist. We give an example in the full version [42]. This further motivates the study of WSD-PROP1 allocations.

Another approach to tackle the non-existence of EF and PROP allocations is via randomization. A promising notion of fairness which has gained popularity over the recent years, is the notion of *"Best of Both Worlds Guarantees"* (BoBW) [2, 4, 7, 21, 25]. The aim is to compute a randomized fair allocation which also guarantees an approximate fairness notion in the deterministic setting. Suppose $\mathcal{P}$ and $\mathcal{Q}$ are two notions of fairness. Given a set of k allocations $A = \langle A_1, A_2, \cdots, A_k \rangle$ and a probability distribution $p = \langle p_1, p_2, \cdots, p_k \rangle$ over A, the pair $\langle A, p \rangle$ is said to be *ex-post* $\mathcal{Q}$ fair if each allocation $A_1, A_2, \cdots, A_k$ are $\mathcal{Q}$ fair and is called *ex-ante* $\mathcal{P}$ fair if $\mathcal{P}$ fairness is guaranteed in expectation.

In [2, 21], Aziz and Freeman et al. proposed polynomial time algorithms - the PS-Lottery algorithm (also called as the Eating algorithm) to compute ex-ante SD-EF and ex-post SD-EF1 allocations of goods. In their approach, the agents are asked to hypothetically *eat* the goods to produce a fractional allocation which is later decomposed into integral allocations. This approach was later modified in [25] where the agents eat the goods at varying speeds to compute allocations that are ex-ante WSD-EF and ex-post WSD-PROP1 along with *weighted transfer envy-free up to one good* WEF(1,1).

In practice, multiple allocations satisfying WSD-PROP1 might exist, and some could be *better* than others. For instance, different allocations of goods might incur different shipping/transportation costs or agents might have varying efficiency or expertise for each chore, independent of their own disutility or preference. In such cases, it becomes essential to optimize over the set of all WSD-PROP1 allocations. In this work, we particularly address this problem, providing a unified way to deal with goods and chores.

1.1 Our Contributions

In this paper, we investigate fair allocation problems for agents with ordinal preferences and unequal entitlements. We provide the following key contributions. The theorems in this paper are for chores, analogous result for goods are given in the full version [42].

- We show that the problem of existence and computation of WSD-PROP1 allocations reduces to that of the existence and computation of a perfect matching in a bipartite graph. We give such a reduction for both goods and chores. This gives a straightforward, matching based, polynomial-time algorithm that seamlessly adapts to computation of a WSD-PROP1 allocation of both goods and chores. In Theorem 3.6, we show the reduction for chores.
- We show that every perfect matching in the graph constructed above corresponds to a WSD-PROP1 allocation, and vice versa. Thus, we give a complete characterization of WSD-PROP1 allocations for both goods and chores as extreme points of a perfect matching polytope. This enables optimization of any linear objective function over the set of all WSD-PROP1 allocations. (See Section 4)
- We study the economic efficiencies that can be guaranteed along with WSD-PROP1. We provide a counter-example to show that Pareto optimality[1] (PO) is not compatible with WSD-PROP1.
- On the positive side, in Theorem 6.3, we show that every allocation that corresponds to a *rank-maximal* perfect matching in our perfect matchings instance is sequencible. En-route to this result, we show that rank-maximal matchings and rank-maximal perfect matchings in any bipartite graph are sequencible. (See Lemma 6.2). This may be of independent interest.
- We also consider the best of both world fairness notion, in the context of our characterization. We show that our characterization leads to a simple polynomial-time algorithm for computing ex-ante WSD-EF ex-post WSD-PROP1 allocations for both goods and chores. (See Theorem 5.3.)To the best of our knowledge, this is the first instance of such an algorithm for the case of chores. In the context of goods, our approach offers an alternative solution to the methods proposed in [4, 25].

Extensions: Our characterization of WSD-PROP1 allocations in terms of matchings paves a way to use tools from fairness in matchings to further generalization of the allocation problem. For instance, items (and also agents) can belong to various categories depending on their attributes, and there can be upper and lower quotas on each category of items that can be allotted to each agent, and also on each category of agents as to the number of items they get. Existing results from the literature on fairness in matchings (e.g. [23, 33, 38, 40, 41]) can then be used to determine the existence of WSD-PROP1 allocations satisfying the category quotas, and outputting one if it exists.

1.2 Related Work

Over the past two decades, there has been a growing interest in the study of computation of discrete fair division [1, 3, 5, 9, 13, 14, 32, 37]. In [17, 32], Budish et al. and Lipton et al. show the existence of EF1 allocations for goods. In [8, 19], Conitzer et al. and Barman et al. extensively studied PROP1 allocations for goods. While significant advancements have been made in the allocation of goods, progress in the case of chores has been notably slower, and our understanding of chore allocation remains relatively limited in comparison to goods allocation. In 2019, Brânzei and Sandomirskiy in [16] extended the notion of PROP1 to the case of chores and gave an algorithm to compute them. Their algorithm runs in polynomial time when either the number of agents or the number of chores is fixed. Subsequently, Bhaskar et al. showed the existence and polynomial time computation of EF1 allocation of chores [10].

For agents with varying entitlements, Chakraborty et al. [18] showed the existence and computation of WEF1+PO allocation in pseudo-polynomial time. They showed that, in the context of weighted allocations, WEF1 does not imply WPROP1 for goods, in contrast to the unweighted case. Li et al., in [31] showed the existence and computation of WPROPx (which implies WPROP1) allocation of chores. Recently, Ex-post WPROP1 allocations along

[1]Here, given only ordinal rankings, we call an allocation cardinally PO if it is PO under all cardinal valuations compatible with the ordinal rankings.

with Ex-ante WEF allocations for goods were studied in [4, 25]. In [4], Aziz et al. proposed the Weighted Max Nash lottery Algorithm which computes an Ex-ante PO and Ex-post WPROP1 allocation, for agents with additive cardinal valuations.

In the weighted setting with agents expressing ordinal preferences for goods, Pruhs et al. in [39] reduced the problem of WSD-PROP allocation of goods to that of finding perfect matchings in a bipartite graph. This was later generalized to preference lists with ties by Aziz et al. in [5]. While WSD-PROP allocations may not always exist, their method provides a polynomial-time algorithm to compute one when it exists. Our reductions are inspired from their work. In 2023, Wu et al. proposed the *Reversed Weighted Picking Sequence* Algorithm [45] which always computes a WSD-EF1+SEQ (thus WSD-PROP1+SEQ) allocation of chores in polynomial time.

2 PRELIMINARIES

Let $A = \{a_1, a_2, \ldots, a_n\}$ denote the set of n agents and let $B = \{b_1, b_2, \ldots, b_m\}$ be set of m indivisible items. Each agent a_i expresses ordinal preferences over the items, given by a permutation π_i of the items in B. The item set B can either be a set of goods or a set of chores. Each agent $a_i \in A$ is endowed with an *entitlement* $\alpha_i \in [0, 1]$ such that $\sum_{a_i \in A} \alpha_i = 1$.

Given an agent $a_i \in A$, we denote the ordinal preference of a_i as a rank function $\pi_i : [m] \to B$. The j^{th} rank item is given by $\pi_i(j)$ and the rank of an item b is given by $\pi_i^{-1}(b)$. In the case of goods, $\pi_i(j)$ represents the j^{th}-most favorite good and in the case of chores, $\pi_i(j)$ is the j^{th}-least favorite chore.

An instance of the allocation problem under ordinal valuations is represented by a tuple $\mathcal{I} = \langle A, B, \Pi, \mathcal{F} \rangle$, where A and B are the sets of agents and goods, respectively. $\Pi = \{\pi_1, \pi_2, \ldots, \pi_n\}$ denotes the set of rank functions, and $\mathcal{F} = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ represents the entitlements of agents.

Fractional and Randomized Allocations: We adopt the definitions of fractional and randomized allocations as outlined in [2, 4, 7, 21]. A *fractional* allocation of the items in B to the agents in A is given by a non-negative $n \times m$ matrix $X = [x_{i,j}] \in [0, 1]^{n \times m}$ such that an entry $x_{i,j}$ denotes the fraction of the item b_j allocated to the agent a_i; for each item $b_j \in B$, $\sum_{a_i \in A} x_{i,j} = 1$. We denote $X_i = \langle x_{i,1}, x_{i,2}, \ldots, x_{i,m} \rangle$ as the *fractional bundle* of items that is assigned to agent a_i. A fractional allocation is integral, if $x_{i,j} \in \{0, 1\}$ for all $a_i \in A$ and $b_j \in B$. For an integral allocation X, *bundle* X_i refers to the set of items that is assigned to agent a_i and the allocation X can be characterized by the bundles of the agents, i.e. $X = \langle X_1, X_2, \ldots, X_n \rangle$.

A *randomized allocation* is a lottery over integral allocations. In particular, a randomized allocation R is determined by k pairs $\{(p_1, Y^1), (p_2, Y^2), \ldots, (p_k, Y^k)\}$, where each of the integral allocations Y^j for $j \in [k]$, is implemented with probability $p_j > 0$ and $\sum_{j \in [k]} p_j = 1$. We say that such an integral allocation is in the support of the randomized allocation. Moreover, we say that a fractional allocation X implements a randomized allocation Y, if the marginal probability of agent a_i receiving item b_j is $x_{i,j}$.

Cardinal Valuations: Each agent $a_i \in A$ can have a private cardinal valuation function $v_i : 2^B \to \mathbb{R}_{\geq 0}$. When B represents a set of goods, v_i is called as a utility function. When B represents a set of chores, v_i is called as a disutility function. The heaviest (least favorite) chore is assigned the highest disutility value. We consider agent's valuation function to be additive, that is $\forall S \subseteq B, v_i(S) = \sum_{b \in S} v_i(b)$.

A valuation function v_i is said to be π_i*-respecting* if v_i is consistent with the ordinal ranking π_i. That is, $\forall b, b' \in B$, $\pi_i^{-1}(b) < \pi_i^{-1}(b') \implies v_i(b) \geq v_i(b')$. We denote the set of all π_i-respecting valuations as $\mathcal{U}(\pi_i)$.

The Interval Representation of Items: Consider an agent $a_i \in A$ with an entitlement α_i. We arrange the items along a number line from 0 to m, such that the j^{th} rank item $\pi_i(j)$ occupies the interval $[j-1, j]$ for $1 \leq j \leq m$. We refer to the interval $[j-1, j]$ as the item $\pi_i(j)$ itself. Furthermore, given an interval $I = [p, q] \subseteq [0, m]$, we refer to I as a fractional bundle itself. If a δ fraction of an interval $[j-1, j]$ overlaps with the interval $[p, q]$, that is $|[j-1, j] \cap [p, q]| = \delta$, then δ fraction of the item $b = \pi_i(j)$ belongs to the fractional bundle I. For a cardinal valuation function v_i of agent a_i, the value of the bundle $I = [p, q]$ is calculated as $v_i(I) = \sum_{j \in [m]} |[j-1, j] \cap [p, q]| \cdot v_i(\pi_i(j))$.

Now, let the $[0, m]$ interval be sub-divided into $k_i = \lceil m\alpha_i \rceil$ many intervals of lengths $\frac{1}{\alpha_i}$, except possibly the last interval, which can be shorter. The the ℓ^{th} interval is given by $I_\ell^i = \left[\frac{\ell-1}{\alpha_i}, \frac{\ell}{\alpha_i}\right]$ for $1 \leq \ell \leq \lfloor m\alpha_i \rfloor$, and, if $m\alpha_i$ is not integral, then the last interval is $\left[\frac{\lfloor m\alpha_i \rfloor}{\alpha_i}, m\right]$.

Definition 2.1. For an agent $a_i \in A$, we define the set of intervals $I^i = \{I_1^i, I_2^i, \ldots, I_{k_i}^i\}$ as the *interval set* of a_i.

Note that if $\alpha_i < 1$, length of each interval $\frac{1}{\alpha_i} > 1$. Thus each interval I_ℓ^i, except possibly the last one, contains a non-zero portion of at least two consecutive items.

Stochastic Dominance(SD):. A standard way of comparing fractional/randomized allocations is through first-order stochastic dominance. This notion has been extensively studied previously in [5, 12]. An agent a_i prefers one allocation over another with respect to the SD relation if she gets at least as much value (or at most - in the case of chores) from the former allocation as the latter under all π_i-respecting cardinal valuations.

Suppose X_i and Y_i denote the fractional bundles of goods that an agent a_i receives in the allocations $X = [x_{ij}]$ and $Y = [y_{ij}]$ respectively. We say that an agent a_i SD prefers X_i to Y_i, denoted by $X \succsim_i^{\text{SD}} Y$ if the following holds:

$$\forall j^* \in [m], \sum_{j : \pi_i(j) \geq \pi_i(j^*)} x_{i,j} \geq \sum_{j : \pi_i(j) \geq \pi_i(j^*)} y_{i,j}$$

When X_i and Y_i denote bundles of chores, we say $X \succsim_i^{\text{SD}} Y$ if the following holds:

$$\forall j^* \in [m], \sum_{j : \pi_i(j) \geq \pi_i(j^*)} x_{i,j} \leq \sum_{j : \pi_i(j) \geq \pi_i(j^*)} y_{i,j}$$

2.1 Fairness and Efficiency Notions

We begin with *ex-ante - ex-post* notions as defined in [2, 4, 7, 12, 21, 25]. For any property $\langle P \rangle$ defined for an allocation, we say that a randomized allocation R satisfies $\langle P \rangle$ *ex-ante* if the allocation X that implements R satisfies $\langle P \rangle$. For any property $\langle Q \rangle$ defined for an integral allocation, we say that a randomized allocation R satisfies $\langle Q \rangle$ *ex-post* if every integral allocation in its support satisfies $\langle Q \rangle$.

We now define various notions of weighted fairness under ordinal valuations. We start with the classic notion of *envy-freeness.* Consider an instance of the allocation problem under ordinal valuations $\mathcal{I} = \langle A, B, \Pi, \mathcal{F} \rangle$.

Definition 2.2 (WSD-EF). ([12]) Let B be a set of chores. An allocation $X = \langle X_1, X_2, \ldots, X_n \rangle$ of B is said to be *weighted* SD *envy-free (WSD-EF)*, if for every pair of agents $a_i, a_k \in A$, we have

$$\frac{v_i(X_i)}{\alpha_i} \leq \frac{v_i(X_k)}{\alpha_k} \quad \forall v_i \in \mathcal{U}(\pi_i), \forall v_k \in \mathcal{U}(\pi_k)$$

If B is a set of goods, then X is *WSD-EF* if, for every pair of agents $a_i, a_k \in A$, we have

$$\frac{v_i(X_i)}{\alpha_i} \geq \frac{v_i(X_k)}{\alpha_k} \quad \forall v_i \in \mathcal{U}(\pi_i), \forall v_k \in \mathcal{U}(\pi_k)$$

We consider the following notions of relaxed proportionality defined for integral allocations under cardinal valuations.

Definition 2.3 (WPROP1 [6]). Let B be a set of chores. In an integral allocation $X = \langle X_1, X_2, \ldots, X_n \rangle$, a bundle X_i is said to be *weighted proportional up to one item (WPROP1)* for an agent a_i with a valuation function v_i, if:

$$\exists b \in X_i, \quad v_i(X_i \setminus \{b\}) \leq \alpha_i \cdot v_i(B)$$

If B is a set of goods, then a bundle X_i is said to be *WPROP1* for an agent a_i if:

$$\exists b \in B, \quad v_i(X_i \cup \{b\}) \geq \alpha_i \cdot v_i(B)$$

The allocation X is said to be *WPROP1* if, for all $i \in [n]$, bundle X_i is *WPROP1* for agent a_i.

Although the notion of WPROP1 is conventionally defined for integral allocations, for the sake of analysis, we extend this notion to fractional allocations as follows:

Definition 2.4 (fractional WPROP1). Let B be a set of chores. A fractional bundle $X_i = \langle x_{i,1}, x_{i,2}, \ldots, x_{i,m} \rangle$ is *WPROP1* for an agent a_i with a valuation function v_i, if $\exists b = \langle \beta_1, \beta_2, \ldots, \beta_m \rangle$, where $\|b\|_1 = 1$ and $\forall j \in [m]$ $0 \leq \beta_j \leq x_{i,j}$, we have $v_i(X_i - b) \leq \alpha_i \cdot v_i(B)$.

In the case of goods, a fractional bundle X_i is *WPROP1* for an agent a_i, if $\exists b = \langle \beta_1, \beta_2, \ldots, \beta_m \rangle$, where $\|b\|_1 = 1$, $\beta_j \geq 0$, and $\forall j \in [m]$ $x_{i,j} + \beta_j \leq 1$, we have $v_i(X_i + b) \geq \alpha_i \cdot v_i(B)$. The allocation X is *WPROP1* if bundle X_i is *WPROP1* for every agent $a_i \in A$.

We can extend these definitions to the case of ordinal valuations as follows:

Definition 2.5 (WSD-PROP1). An allocation X is said to be *WSD-PROP1*, if X is *WPROP1* for every agent $a_i \in A$ under all valuations $v_i \in \mathcal{U}(\pi_i)$.

It is straightforward to see that, for an agent a_i, if a bundle X is WSD-PROP1, then every bundle Y s.t. $Y \succsim_i^{\mathrm{SD}} X$ is also WSD-PROP1.

Along with the notions of fairness, we study the following economic efficiencies considered in literature.

Definition 2.6 (Pareto Optimailty (PO)). For agents with cardinal valuations, an allocation X is said to be *Pareto Optimal (PO)* if there is no allocation Y that Pareto dominates it. In the case of chores, an allocation Y is said to *Pareto dominate* an allocation X if $v_i(Y_i) \leq v_i(X_i)$ for all $i \in [n]$ and $\exists j \in [n]$ such that $v_j(Y_j) < v_j(X_j)$. For goods, Y is said to Pareto dominate X if $v_i(Y_i) \geq v_i(X_i)$ for all $i \in [n]$ and $\exists j \in [n]$ such that $v_j(Y_j) > v_j(X_j)$.

In the absence of Pareto Optimal allocations, a weaker notion of efficiency known as *sequencibility* (SEQ) is often considered. A *picking sequence* of n agents for m items is an m-length sequence $\sigma = \langle a'_1, a'_2, \ldots, a'_m \rangle$ where $a'_i \in A$ for $i \in [m]$. An allocation X is the result of the picking sequence σ if it is the output of the following procedure: Initially every bundle is empty; then, at time step t, agent a'_t inserts in her bundle the most preferred item among the available ones. Once an item is selected, it is removed from the set of the available items.

Definition 2.7 (Sequencibility (SEQ)). An allocation X is said to be *sequencible (SEQ)* if X is the result of some picking sequence σ.

It is known that PO implies SEQ, and when number of agents $n = 2$, then PO is same as SEQ [15].

2.2 Graphs and Matchings

In a graph $G = (V, E)$, for any $S \subset V$, $N(S)$ denotes the set of neighbors of the vertices in S. A matching is a subset of edges, no two of which share a vertex. A matching M is said to *saturate* or *match* a vertex v if M contains an edge incident on v. Given a bipartite graph $G = (A \cup B, E)$, an A-perfect matching M is a matching in G that saturates all the vertices in A. When $|A| = |B|$, an A-perfect matching is same as a perfect matching. Given a matching M and a matched vertex $a \in A$, we denote by $M(a)$ the matched partner of a.

Rank-Maximal Matchings [27]*:* Consider a bipartite graph $G = (A \cup B, E)$, s.t $|A| = n, |B| = m$, where each vertex a in A ranks its neighbours $N(a)$ from 1 to $|N(a)|$. For each edge $(a, b) \in E$, let $rank(a, b) \in [m]$ denote the rank of b in a's ranking. The graph G along with the ranking is denoted as $G = (A \cup B, E_1, E_2, \ldots, E_m)$ where $E_i = \{(a, b) \in E \mid rank(a, b) = i\}$, for all $i \in [n]$. A matching M in G can be decomposed as $M = M_1 \cup M_2 \cup \cdots \cup M_m$ where $M_i = M \cap E_i$. We define signature of a matching M in G as an m length tuple $\rho(M) = \langle |M_1|, |M_2|, \ldots, |M_m| \rangle$.

Definition 2.8 (Rank-Maximal Matching). Given a bipartite graph $G = (A \cup B, E_1, E_2, \ldots, E_m)$, a matching M in G with lexicographically highest signature $\rho(M)$ is called as a rank-maximal matching.

Note that all rank-maximal matchings have identical signature. Furthermore, a rank-maximal matching need not be a maximum size matching.

Definition 2.9 (Rank-Maximal Perfect Matching). Given a bipartite graph $G = (A \cup B, E_1, E_2, \ldots, E_m)$, a perfect matching M in G with lexicographically highest signature $\rho(M)$ among all perfect matchings in G is called as a rank-maximal perfect matching.

A matching M in G can be interpreted as an allocation of vertices in B to the vertices in A. The ranks of the edges can be interpreted as the ordinal preferences of the vertices in A. Under this interpretation, we borrow the definition of sequencibility (SEQ) (Definition 2.7) for matchings. A matching M is said to be sequencible if the corresponding allocation M is sequencible. We denote by $\rho(M)$, the picking sequence of vertices in A that constructs M.

3 EXISTENCE AND COMPUTATION OF WSD-PROP1 ALLOCATIONS FOR CHORES VIA MATCHINGS

We now show that WSD-PROP1 allocations always exist for chores. To show this, we first characterize WSD-PROP1 bundles in Lemma 3.1. Using this lemma, we construct a bipartite graph $G_c = (S \cup B, E)$ called an *allocation graph* of a chores instance. We show that a B-perfect matching in G_c corresponds to a WSD-PROP1 allocation. We then use Hall's marriage condition [24] to demonstrate that such a matching always exists, thus establishing the existence of WSD-PROP1 allocations. We extend these results to the case of goods in the full version [42].

LEMMA 3.1. *Let $T \subseteq B$ be a set of m_i chores, and let $r_1 < r_2 < \cdots < r_{m_i}$ be the ranks of the chores in T in the ranking π_i of agent a_i (i.e, this set consists of the r_1-least favorite chore, r_2-least favorite chore,$\cdots$, and the r_{m_i}-least favorite chore for agent a_i). Then bundle T is* WSD-PROP1 *for a_i if and only if the following two conditions hold:*

$$m_i \leq \lfloor m\alpha_i \rfloor + 1 \tag{1}$$

$$\forall 1 \leq \ell \leq m_i, \quad r_\ell \geq \left\lceil \frac{\ell - 1}{\alpha_i} \right\rceil \tag{2}$$

PROOF. Without loss of generality, for simplifying the notation, let the chores be renumbered according to the ranking of agent a_i. Thus, $b_j = \pi_i(j)$ for $1 \leq j \leq m$. We assume $\alpha_i < 1$ as otherwise any bundle is WSD-PROP1 for agent a_i and further, $\alpha_i > 0$ as otherwise agent a_i can be removed from the instance.

First, let us prove the necessity of these conditions. If any of the two conditions are not met, we exhibit a valuation v_i according to which, the bundle T is not WPROP1 for agent a_i. Suppose T violates condition 1, i.e. $m_i \geq \lfloor m\alpha_i \rfloor + 2$. We set $v_i(b_j) = 1$ for all $b_j \in B$. Under this valuation,

$$\forall b \in T, \quad v_i(T \setminus \{b\}) \geq \lfloor m\alpha_i \rfloor + 1 > m\alpha_i = \alpha_i \cdot v_i(B)$$

Thus T is not a WPROP1 bundle. Now, suppose T violates condition 2, i.e. $r_\ell \leq \left\lceil \frac{\ell-1}{\alpha_i} \right\rceil - 1$ for some $1 \leq \ell \leq m_i$. We set $v_i(b_j) = 1$ for all $1 \leq j \leq \left\lceil \frac{\ell-1}{\alpha_i} \right\rceil - 1$ and $v_i(b_j) = 0$ for all $j \geq \left\lceil \frac{\ell-1}{\alpha_i} \right\rceil$. Under this valuation, $\forall b \in T$ we have

$$v_i(T \setminus \{b\}) \geq \ell - 1 = \left(\frac{\ell - 1}{\alpha_i}\right)\alpha_i > \left(\left\lceil \frac{\ell - 1}{\alpha_i} \right\rceil - 1\right)\alpha_i = \alpha_i \cdot v_i(B)$$

Therefore, the bundle T is not WPROP1.

We now show the sufficiency of these conditions. Suppose conditions 1 and 2 hold for the bundle T. It suffices to consider the case when both the conditions 1 and 2 are tight, except $r_1 = 1$. This is because, for any other bundle $Y_i = \{\pi(r'_1), \pi(r'_2) \cdots, \pi(r'_k)\}$ where $1 \leq r'_1 < r'_2 < \cdots < r'_k$, and at least one of the conditions 1 or 2 is not tight, we have $Y_i \succsim_i^{\mathrm{SD}} T$ since, for all $1 \leq \ell \leq k$, $r'_\ell \geq r_\ell$.

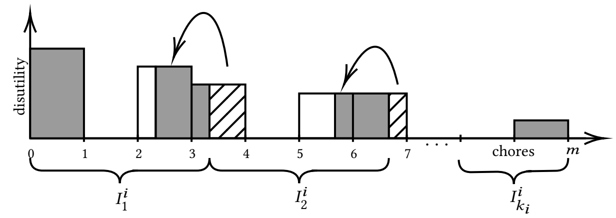

Figure 1: Construction of the fractional allocation T' (shaded in grey) that is SD dominated by T.

To show that T is WSD-PROP1 for agent a_i, we construct a fractional allocation T' using T such that $T \succsim_i^{\mathrm{SD}} T'$ and T' is a WSD-PROP1 allocation.

Consider the interval set $I^i = \{I_1^i, I_2^i, \ldots, I_{k_i}^i\}$ of agent a_i. Include the chore b_1 in T'. For any other chore $b_j \in T \setminus \{b_1\}$, we know that $\pi_i^{-1}(b_j) = \left\lceil \frac{\ell-1}{\alpha_i} \right\rceil$ for some $\ell \in [m_i]$. Therefore, a non-zero fraction of the chore b_j lies in the right end of the interval $I_{\ell-1}^i = \left[\frac{\ell-2}{\alpha_i}, \frac{\ell-1}{\alpha_i}\right]$. Suppose $\frac{\ell-1}{\alpha_i} = \left\lceil \frac{\ell-1}{\alpha_i} \right\rceil - \delta$ for some $0 \leq \delta < 1$. That is, $1 - \delta$ fraction of b_j lies in $I_{\ell-1}^i$ and the remaining δ portion lies in the interval I_ℓ^i. Then, from the interval $I_{\ell-1}^i$ include the $1 - \delta$ fraction of chore b_j and δ fraction of the preceding chore b_{j-1} in T' (as shown in Figure 1). Under any valuation $v_i \in \mathcal{U}(\pi_i)$, for every chore b_j we have $\delta v_i(b_{j-1}) + (1 - \delta)v_i(b_j) \geq v_i(b_j)$. Therefore it is clear that $T \succsim_i^{\mathrm{SD}} T'$.

From the construction of T', we know $T' \setminus \{b_1\}$ contains the least valued one unit of chore from each interval (except possibly the last interval which could have no contribution to T'). Therefore, $v_i(T - b_1) \leq \sum_{j=1}^{k_i} \alpha_i v_i(I_j^i) = \alpha_i \cdot v_i(B)$. Thus, T' is a WSD-PROP1 bundle. □

With the help of this characterization, we can now construct an *allocation graph* G_c of chores. Given a fair allocation instance $\mathcal{I} = \langle A, B, \Pi, \mathcal{F} \rangle$, we construct the allocation graph $G_c = (S \cup B, E)$ as follows:

- The set of chores B forms one bipartition of G_c with chores interpreted as vertices.
- For every agent $a_i \in A$, and every $\ell = 1, 2, \cdots, m_i$, where $m_i = \lfloor m\alpha_i \rfloor + 1$, there is a vertex $s_{i,\ell}$ in S. We call these the m_i many *slots* of agent a_i.
- From each slot $s_{i,\ell}$, draw edges to every chore b for which $\pi_i^{-1}(b) \geq \left\lceil \frac{\ell-1}{\alpha_i} \right\rceil$. That is, $(s_{i,\ell}, b) \in E \iff \pi_i^{-1}(b) \geq \left\lceil \frac{\ell-1}{\alpha_i} \right\rceil$

The allocation graph G_c exhibits several interesting properties: Firstly, we have

PROPOSITION 3.2. *Every B-perfect matching in G_c, (i.e a matching that saturates all the chores), satisfies conditions 1 and 2 and this corresponds to a WSD-PROP1 allocation of chores. Conversely, any WSD-PROP1 allocation satisfies conditions 1 and 2 and thus forms a B-perfect matching in G_c.*

Moreover, in the interval set $I^i = \{I_1^i, I_2^i, \ldots, I_{k_i}^i\}$ of an agent a_i, if non-zero fraction of a chore b lies in the interval $I_\ell^i = \left[\frac{\ell-1}{\alpha_i}, \frac{\ell}{\alpha_i}\right]$, then $\pi_i^{-1}(b) \geq \left\lceil \frac{\ell-1}{\alpha_i} \right\rceil$. Thus slot $s_{i,\ell}$ has an edge to chore b in G_c. This is formally stated in the following proposition.

Proposition 3.3. *In the allocation graph G_c of a chore allocation instance $\mathcal{I}$, each slot $s_{i,\ell}$ of each agent $a_i \in A$, has edges to every chore with a non-zero portion in the intervals I_k^i, $\forall k \geq \ell$ in the interval set of a_i.*

We also observe the following property regarding edge relationships in G_c:

Proposition 3.4. *Let $S_i = \{s_{i,\ell} \mid \ell \in [m_i]\}$ represent the set of slots in S that belong to an agent $a_i \in A$. For any two chores b_j and b_k in B, such that $\pi_i^{-1}(b_k) \geq \pi_i^{-1}(b_j)$, the neighbourhood of b_k in S_i contains the neighbourhood of b_j in S_i. That is, $N(b_j) \cap S_i \subseteq N(b_k) \cap S_i$. Therefore, for any set of chores $Y = \{b_1, b_2, \cdots, b_k\}$, the following holds:*

$$S_i \cap N(\{\pi_i(1), \pi_i(2), \cdots, \pi_i(k)\}) \subseteq S_i \cap N(Y)$$

We now show the existence of WSD-PROP1 allocations by showing that a B-perfect matching always exists in G_c. To prove this, we rely on the Hall's marriage theorem [24] which characterizes the existence of perfect matchings in bipartite graphs.

Theorem 3.5 (Hall's Theorem [24]). *Given a bipartite graph $G = (A \cup B, E)$, there exists an A-perfect matching in G if and only if $\forall S \subseteq A,\ |N(S)| \geq |S|$.*

Now, we establish the following main result:

Theorem 3.6. *For any fair allocation instance of chores $\mathcal{I} = \langle A, B, \Pi, \mathcal{F} \rangle$, there always exists a WSD-PROP1 allocation.*

Proof. Consider the allocation graph G_c of $\mathcal{I}$. We show that G_c always has a B-perfect matching and thus $\mathcal{I}$ has a WSD-PROP1 allocation. Let $T = \{b_1, b_2, \cdots, b_k\} \subseteq B$ be a set of k vertices in B. The goal is to show that $|N(T)| \geq |T|$. From Proposition 3.4, we can assume *w.l.o.g* that, for each agent $a_i \in A$, the chores $b_1, b_2, \cdots, b_k$ are the first (lowest rank) k chores, since it minimizes the size of the neighbourhood of the chores. For an agent a_i, let s_{i,ℓ_i} be the highest index slot which has an edge to b_k. Thus, all the slots $s_{i,1}, s_{i,2}, \cdots, s_{i,\ell_i}$ have edges to b_k. Therefore, $|N(T)| \geq \sum_{i \in [n]} \ell_i$. Since the slot s_{i,ℓ_i+1} does not have an edge to b_k, condition 2 is violated. Thus,

$$\begin{aligned}
& k < \left\lceil \frac{(\ell_i + 1) - 1}{\alpha_i} \right\rceil \\
\Longrightarrow\ & k < \frac{\ell_i}{\alpha_i} + 1 \\
\Longrightarrow\ & (k-1)\alpha_i < \ell_i \\
\Longrightarrow\ & \sum_{i \in [n]} (k-1)\alpha_i < \sum_{i \in [n]} \ell_i && \text{(Summing over all agents)} \\
\Longrightarrow\ & k - 1 < |N(T)| && \text{(We know } \textstyle\sum_{i \in [n]} \ell_i \leq |N(T)|\text{)} \\
\Longrightarrow\ & k \leq |N(T)| && \text{(As both } |N(T)| \text{ and } k \text{ are integers)}
\end{aligned}$$

Therefore, for any set of k chores the size of the neighbourhood is more than or equal to k. From Theorem 3.5, the allocation graph G_c always has a B-perfect matching - which corresponds to a WSD-PROP1 allocation of chores. □

The graph G_c has $O(m)$ vertices and $O(m^2)$ many edges, assuming $m \geq n$. Therefore, using the famous Hopcroft-Karp algorithm [26] to find perfect matchings, we can compute a WSD-PROP1 allocation in time $O(m^{2.5})$.

4 OPTIMIZING OVER ALLOCATIONS

Recall that any B-perfect matching in the allocation graph G_c corresponds to a WSD-PROP1 allocation and vice versa. In this section, we extend the allocation graph G_c to G_c^+ by balancing the two parts of the bipartite graph while maintaining the correspondence between WSD-PROP1 allocations and perfect matchings in G_c^+. We can then optimize any linear objective function over all WSD-PROP1 allocations using the perfect matching polytope.

Extending the Allocation Graph: Consider the allocation graph $G_c = (S \cup B, E)$ of an instance $\mathcal{I}$ of chores allocation. For each agent $a_i \in A$, there are $m_i = \lfloor m\alpha_i \rfloor + 1$ many slots in S. Therefore the total number of slots $|S| = n + \sum_{i \in [n]} \lfloor m\alpha_i \rfloor$. To construct $G_c^+ = (S \cup B', E')$, we create $|S| - m = q$ many additional *dummy* chores $b'_1, b'_2, \cdots, b'_q$ in B' to balance the bipartite graph. Draw additional edges from all the slots in S to every dummy chore.

A WSD-PROP1 allocation of chores gives a B-perfect matching in G_c. We can extend this matching to a perfect matching in G_c^+ by matching the dummy chores in any manner as all the dummy chores have edges to every slot. Conversely, given a perfect matching in G_c^+, we can ignore the edges from dummy chores to get a B-perfect matching in G_c and thus a WSD-PROP1 allocation.

Given a bipartite graph $G = (X \cup Y, E)$, the following constraints define the matching polytope:

$$\begin{aligned}
\sum_{x \in N(y)} e_{xy} &= 1 \quad \forall y \in Y \\
\sum_{y \in N(x)} e_{xy} &= 1 \quad \forall x \in X \\
e_{xy} &\geq 0
\end{aligned} \tag{3}$$

We know that above matching polytope is integral [34] and hence a matching that maximizes a given objective function is computable in polynomial-time [29, 30]. We now use this fact to compute WSD-PROP1 allocations while considering agents' *efficiency* in doing the chores.

4.1 Considering Agent Competence

Regardless of how each agent personally values any given chore, it is important to acknowledge that their skills and proficiency in performing them can vary significantly across different tasks. For any specific agent-chore pairing a_i, b, we can quantify the agent's competence in performing chore b as $u_i(b) \in [0, 1]$, where 0 indicates low competency and 1 indicates high competency. This efficiency metric helps us assess how well-suited each agent is to tackle a particular chore, guiding us in achieving a fair and efficient chore allocation.

We can use the above given linear programming formulation to maximize efficiency over all WSD-PROP1 allocations by setting the objective function as: maximize $\sum_{(s_{i,\ell}, b) \in E'} u_i(b) \cdot e_{(i,\ell),b}$ Similarly, we can optimize for time spent on doing chores and other linear objective functions.

5 BEST OF BOTH WORLDS

In this section, using the perfect matchings in the extended allocation graph G_c^+, we give a polynomial time algorithm to compute an ex-ante WSD-EF and ex-post WSD-PROP1 allocation of chores.

We begin by constructing a WSD-EF allocation $X = \langle X_1, \cdots, X_n \rangle$. Give each agent $a_i \in A$, an α_i fraction of every chore $b \in B$. In this allocation, for each pair of agents a_i and a_k, for any two valuations $v_i \in \mathcal{U}(\pi_i), v_k \in \mathcal{U}(\pi_k)$, we know that $\frac{v_i(X_i)}{\alpha_i} = \frac{v_i(X_k)}{\alpha_k} = v_i(B)$ and hence X is a WSD-EF allocation. We now show that this fractional allocation can be realized as a fractional perfect matching in the extended allocation graph G_c^+.

Lemma 5.1. *Given an instance $\mathcal{I} = \langle A, B, \Pi, \mathcal{F} \rangle$ of chore allocation, there exists a fractional perfect matching in the extended allocation graph $G_c^+ = (S \cup B', E')$ of $\mathcal{I}$ that corresponds to a WSD-EF chore allocation where each agent a_i receives α_i fraction of every chore.*

Proof. We first construct a fractional matching M that saturates all the *real* chores (non-dummy chores). Such a matching can always be extended to a fractional perfect matching by assigning the dummy chores in any manner, as all the dummy chores have edges to all the slots.

Consider the interval set I^i of an agent $a_i \in A$. From Proposition 3.3, we know that slot $s_{i,\ell}$ has edges to every chore in the interval I^i_ℓ. With the help of this fact, we construct a fractional matching M in G_c^+ as follows:

Let $x_{i,\ell,b}$ denote the fraction of the edge $(s_{i,\ell}, b)$ in M. Let $\delta_{b,\ell}$ denote the fraction of a chore $b \in B$ that is present in the interval I^i_ℓ. For every edge $(s_{i,\ell}, b)$, we set $x_{i,\ell,b} = \alpha_i \cdot \delta_{b,\ell}$. A slot $s_{i,\ell}$ receives non-zero fractions of the chores from the interval I^i_ℓ. Each slot receives at most 1 unit of chore because total chores assigned for a slot $s_{i,\ell}$ is :

$$\sum_{b \in B} x_{i,\ell,b} = \alpha_i \sum_{b \in B} \delta_{b,\ell} \leq \alpha_i \frac{1}{\alpha_i} = 1$$

The fraction of a given real chore b received by agent i across all the slots is:

$$\sum_{\ell=1}^{m_i} x_{i,\ell,b} = \alpha_i \sum_{\ell=1}^{m_i} \delta_{b,\ell} = \alpha_i$$

Thus the matching M saturates all the real chores. Since the graph G_c^+ is a balanced bipartite graph, and as all the dummy chores have edges to all the slots, the matching M can be extended to a fractional perfect matching by dividing the dummy chores across the remaining spaces of all the slots in any arbitrary way. □

Let us denote this fractional perfect matching as M^*. Note that M^* lies inside the matching polytope of G_c^+. We now decompose this fractional perfect matching into convex combination of integral perfect matchings using the Birkhoff-von Neumann decomposition. Given a perfect matching M (fractional or otherwise) of a balanced bipartite graph $G = (P \cup Q, E)$ with $2n$ vertices, M can be represented as a $n \times n$ bi-stochastic matrix $X = (x_{ij})$ where an entry x_{ij} denotes the fraction of the edge (i, j) present in M. Given a fractional perfect matching, we can decompose it as a convex combination of integral perfect matchings with the help of Birkhoff-von-Neumann theorem [11, 28, 34, 43].

Theorem 5.2 (Birkhoff-von Neumann). *Let X be a $p \times p$ bi-stochastic matrix. There exists an algorithm that runs in $O(p^{4.5})$ time and computes a decomposition $X = \sum_{k=1}^{q} \lambda_k X_k$ where $q \leq p^2 - p + 2$; for each $k \in [q]$, $\lambda_k \in [0, 1]$, X_k is a $p \times p$ permutation matrix; and $\sum_{k=1}^{q} \lambda_k = 1$.*

Using Theorem 5.2, we design Algorithm 1. We call it *The Uniform Lottery Algorithm*, which gives an ex-ante WSD-EF and ex-post WSD-PROP1 allocation of chores using only the ordinal valuations.

Algorithm 1 Uniform Lottery Algorithm for chores

Input: A chore allocation instance $\mathcal{I} = \langle A, B, \Pi, \mathcal{F} \rangle$, where $|A| = n$ and $|B| = m$.

Output: A fractional WSD-EF allocation $X = \sum_{k=1}^{q} \lambda_k X_k$ where each X_k represents a deterministic WSD-PROP1 allocation and $q \in O(m^c)$.

1: $G_c^+ \leftarrow$ extended allocation graph of $\mathcal{I}$
2: $Y \leftarrow$ fractional perfect matching in G_c^+ where each agent $a_i \in A$ gets α_i fraction of every real chore ▷ (As in Lemma 5.1)
3: Invoke Theorem 5.2 to compute a decomposition $Y = \sum_{k=1}^{q} \lambda_k Y_k$ where $q \leq (m+n)^2 - (m+n) - 2$
4: Convert $Y = \sum_{k=1}^{q} \lambda_k Y_k$ to $X = \sum_{k=1}^{q} \lambda_k X_k$ where all the dummy chores are ignored.
5: **return** Allocation X and its decomposition $\sum_{k=1}^{q} \lambda_k X_k$

Theorem 5.3. *The randomized allocation implemented by Algorithm 1 is ex-ante WSD-EF and ex-post WSD-PROP1.*

Proof. Algorithm 1 returns an allocation X and its decomposition $\sum_{k=1}^{q} \lambda_k X_k$. From Lemma 5.1, we know that the allocation X returned by the algorithm is WSD-EF. Each of the X_ks in the decomposition is a B-perfect matching in the allocation graph G_c. Therefore, from Proposition 3.2, each X_k is WSD-PROP1. □

6 BEYOND FAIRNESS: ECONOMIC GUARANTEES

In Section 3, we discussed the reduction from WSD-PROP1 allocations to matchings. In this section, we investigate the incorporation of additional economic efficiency notions alongside fairness. In the full version of this paper [42], we give an example instance where no WSD-PROP1 allocation is Pareto optimal under all valuations. Furthermore, we give examples of instances where given the cardinal valuations, there does not exist a Pareto optimal allocation that is WSD-PROP1 for the underlying ordinal instance. Therefore, we explore a more relaxed concept known as *sequencibility* (SEQ). We prove a general graph theoretic lemma, showing that every rank-maximal A-perfect matching is sequencible. This lemma could be of independent interest, and may find other applications. Using this result, we establish that computing a rank-maximal perfect matching, rather than an arbitrary one, yields WSD-PROP1+SEQ allocations.

We begin with the following simple observation about rank-maximal matchings:

Proposition 6.1. *Given a graph $G = (A \cup B, E = E_1 \cup \ldots \cup E_r)$, if M is a rank-maximal matching in G, then M is sequencible (SEQ).*

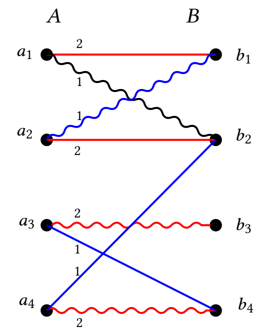

Figure 2: The edges in red form a perfect matching - but it is not sequencible. The blue edges correspond to a rank-maximal matching, but it is not perfect. The squiggly edges corresponds to a rank-maximal A-perfect matching. This is sequencible, and the picking sequence is $\langle a_1, a_2, a_4, a_3 \rangle$

Proof. Define $G_i = (A \cup B, E_1 \cup \ldots \cup E_i)$. Let the rank-maximal matching M be decomposed as $M = M_1 \cup M_2 \cup \ldots M_r$ where $M_i = M \cap E_i$ for $i \in [r]$. For a vertex a, we denote its matched partner in M as $M(a)$. A picking sequence for M is $\langle A_1, A_2, \ldots, A_r \rangle$ where, for $i \in [r]$, A_i is an arbitrary ordering of vertices in A that are matched by an edge in M_i. This sequence results in M, for the following reason. Firstly, M_1 is a maximum matching in G_1. For each $i \in [r], i > 1$, after the agents in $A_1 \cup \ldots \cup A_{i-1}$ pick the items they are matched to in M, M_i is a maximum matching on rank i edges in the remaining graph. Thus, when all the agents in $A_1 \cup \ldots \cup A_i$ pick their favorite item among the available items, there are no items left, which are ranked between 1 and i for any of the remaining agents, this results in agents in A_{i+1} picking their rank $i + 1$ items. □

Our interest is in finding an A-perfect matching that is also sequencible. In general, a rank-maximal matching need not be an A-perfect matching and all perfect matchings are not sequencible. Figure 2 shows one such example. We show that rank-maximal perfect matchings are sequencible. Unlike a rank-maximal matching, a rank-maximal perfect matching M may not satisfy the properties mentioned in Proposition 6.1 i.e., M_1 may not be a maximum matching in G_1, and in general, after agents in $A_1 \cup \ldots \cup A_{i-1}$ pick their respective choices, M_i may not be a maximum matching on rank i edges in the remaining graph. Hence the ordering of vertices in $A_1, \ldots, A_r$ needs to be carefully chosen while constructing the picking sequence.

Lemma 6.2. *Given a graph $G = (A \cup B, E_1 \cup \ldots \cup E_r)$ which is an instance of the rank-maximal matchings problem, and an A-perfect matching M in G, if M is a rank-maximal A-perfect matching then M is sequencible.*

Kindly refer to the full version [42] for the proof of this lemma.

Using Lemma 6.2, we now show that a rank-maximal perfect matching in the extended allocation graph G_c^+ gives a sequencible WSD-PROP1 allocation.

Theorem 6.3. *There always exists a WSD-PROP1+SEQ allocation of chores.*

Proof. Let $G_c^+ = (S, B', E')$ be the extended allocation graph of an instance $\mathcal{I}$. Recall that B is the set of real chores and $B' \setminus B$ is the set of dummy chores and $|B| = m, |B'| = m + q$. For each slot $s_{i,\ell}$, we first rank the real chores from 1 to m as $rank(s_{i,\ell}, b) = m + 1 - \pi_i(b)$ for all $b \in B$. The dummy chores are ranked from $m + 1$ to $m + q$ in an arbitrary way.

For any two slots $s_{i,p}$ and $s_{i,q}$ of an agent a_i, if $p > q$, then $N(p) \subseteq N(q)$. This is because G_c^+ satisfies condition 2. Therefore, given a matching M, if $rank(s_{i,p}, M(s_{i,p})) > rank(s_{i,q}, M(s_{i,q}))$, then we can interchange $M(s_{i,p})$ and $M(s_{i,q})$ without altering the signature of the matching. Thus, given a rank-maximal perfect matching M, we can assume w.l.o.g. that for any agent $a_i \in A$, and $p, q \le m_i$, if $p > q$ then $rank(s_{i,p}, M(s_{i,p})) < rank(s_{i,q}, M(s_{i,q}))$.

Given a rank-maximal perfect matching M in G_c^+, from Lemma 6.2 we obtain a sequence $\sigma(S)$ of slots. To construct a sequence of agents, replace each $s_{i,\ell}$ with the corresponding agent a_i. Since dummy chores are ranked higher than real chores, all the slots that are matched to dummy chore forms the tail of the sequence $\sigma(M)$ and hence they can be safely ignored. Therefore, a rank-maximal perfect matching in G_c^+ gives a WSD-PROP1+SEQ allocation. □

Therefore, using the algorithm to find rank-maximal perfect matchings [27, 36], we can compute a WSD-PROP1+SEQ allocation in time $O((m + n)^{3.5})$.

7 CONCLUSION

In this paper, we consider the fairness notion of weighted necessarily proportionality up to one item (WSD-PROP1). We show that finding WSD-PROP1 allocations can be reduced to finding perfect matchings in a bipartite graph - namely the allocation graph. This insight provides a practical framework for leveraging tools and techniques from the field of matching theory. We show that rank-maximal perfect matchings give picking sequences for finding WSD-PROP1+SEQ allocations. We show that the perfect matching polytope of the allocation graph captures all the WSD-PROP1 allocations, thus enabling us to optimize any linear objective function over WSD-PROP1 allocations. We then create a fractional perfect matching in the allocation graph, corresponding to a WSD-EF allocation. Decomposing this allocation, equivalent to decomposing the fractional matching into integral matchings, results in a randomized algorithm for computing an Ex-ante WSD-EF Ex-post WSD-PROP1 allocation, both in the case of goods and chores. Our works raises the open question of the existence of WSD-PROP1 allocations in the mixed setting, where the set B includes both goods and chores.

ACKNOWLEDGMENTS

We express gratitude to the anonymous reviewers of AAMAS 2024 for their valuable comments. H.V. acknowledges support from TCS-RSP for their grant. Special thanks to friends and colleagues at CMI - Archit Chauhan, Asif Khan, Soumodev Mal, and Sheikh Shakil Akhtar for their time and expertise in proofreading and discussing the contents of this paper.

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