Neighborhood Stability in Assignments on Graphs

Extended Abstract

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ABSTRACT

We study the problem of assigning agents to the vertices of a graph such that no pair of neighbors can benefit from swapping assignments – a property we term *neighborhood stability.* We assume that agents' utilities are based only on their preferences over the assignees of adjacent vertices and that those preferences are binary. Having shown that even this very restricted setting does not guarantee neighborhood stable assignments, we focus on special cases providing such guarantees. We show that when the graph is a cycle or a path, a neighborhood stable assignment always exists for any preference profile. Also, we give a general condition under which neighborhood stable assignments always exist. For each of these results, we give a polynomial-time algorithm to compute a neighborhood stable assignment.

KEYWORDS

Exchange Stability, Hedonic Games, Seat Allocation

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1 INTRODUCTION

An organization has drawn up a number of roles and decided which projects each role will work on. Due to their passion for their organization's cause, the members of the organization are indifferent towards which projects they work on. Nevertheless, each member is fond of some of their colleagues, but not all. Furthermore, each member prefers roles in which they will work with more members they like. How can the organization assign its members to the roles it has designed while members swapping roles?

The problem described above was first formalized by Bodlaender et al. [5]. They investigate the problem of assigning agents to a *seat graph* in a manner that is *stable* in the sense that no two agents would prefer to exchange assignments, i.e., no two agents form a *blocking pair.* The model has attracted follow-up work, resulting in a number of recent papers (see, e.g., Berriaud et al. [2], Ceylan et al. [8], Wilczynski [13]). Besides the examples of assigning roles or seats, the problem can be summarized more broadly by its connection to hedonic games (see, e.g., Aziz and Savani [1], Cechlárová [7]), in which agents are to be partitioned into disjoint coalitions according to their preferences over agents in their coalition. Further, Berriaud et al. [3] recently investigated whether paths and/or cycles always admit a stable assignment in the seating arrangement model, and encountered numerous non-existence results.

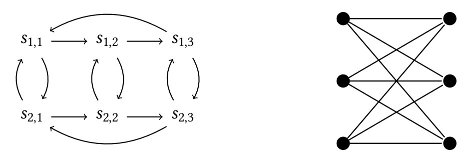

Figure 1: A preference (left) and seat graph (right) with n = 6 without neighborhood stable assignments. Theorem 3.1 generalizes this to a counterexample of arbitrary size.

Investigating the related topic of equilibria in Schelling games [11, 12], Bilò et al. [4] found that restricting agents to local swaps may significantly enlarge the set of equilibria, implying equilibrium existence for some graph classes that do not admit equilibria when allowing for arbitrary swaps. In a similar vein, we focus in this work on a property we term *neighborhood stability*, under which no two agents assigned to adjacent vertices form a blocking pair. Neighborhood stability is natural in the seating arrangement and role assignment motivating settings, as it is natural to assume that agents only have the baseline level of interaction required to agree on a swap with their neighbors. We also restrict our focus to *binary* preferences. They can be interpreted as characterizing each agent's preferences by a set of agents they 'approve', and have been studied in various domains including committee voting [10], matching (see, e.g., Bogomolnaia and Moulin [6]), and item allocation (see, e.g., Halpern et al. [9]). Binary preferences allow us to represent agents' preferences as a directed graph, and as a result, our problem takes on the character of a fundamental graph theoretic problem.

Contributions. When the seat graph is a cycle and agents have binary preferences, Berriaud et al. [2] gave an instance with five

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agents with no stable assignments, even if only blocking pairs within a distance at most two from each other are considered. However, we show that if we marginally weaken this requirement to neighborhood stability, stable assignments always exist on cycles of arbitrary length. Furthermore, even though we show that a neighborgood stable assignment cannot always be achieved through a sequence of beneficial swaps of adjacent agents, we give a polynomial-time algorithm to compute such assignments. Also, we show that neighborhood stable assignments do not always exist, even for instances with as few as six agents. However, we give a polynomial-time algorithm to compute a neighborhood stable assignment if the size of a directed feedback vertex set of the preference graph is upper bounded by the number of leaf nodes in the seat graph.

2 THE MODEL

Let $\mathcal{A} = \{1, ..., n\}$ be a set of n agents which need to be assigned to vertices in an undirected *seat graph* $G = (V, E)$, with $|V| = n$, where an edge (v_i, v_j) indicates if seats v_i and v_j are adjacent. We consider agents with binary preferences, wherein each agent i *approves* of some subset of the other agents in $\mathcal{A}$. This preference structure is expressed by a directed preference graph $\mathcal{P}$ on $\mathcal{A}$, where (i, j) is an arc in $\mathcal{P}$ if and only if agent i approves agent j. Now, an *assignment* is a bijection $\pi : \mathcal{A} \to V$ assigning each agent to a vertex on the seat graph. For a given assignment π, the utility of an agent is as follows: $u_i(\pi) = \sum_{j \in N_\pi(i)} \mathbb{I}[i \to j]$, where $\mathbb{I}$ is an indicator function. So, agent i's utility in π is the number of i's approved neighbors.

Denote $\pi^{i \leftrightarrow j}$ as the assignment π with $\pi(i)$ and $\pi(j)$ switched. $\pi^{i \leftrightarrow j}$ represents the result of what we will call a *swap* between i and j. We say an agent i *envies* agent j if $u_i(\pi) < u_i(\pi^{i \leftrightarrow j})$ and that agents i and j form a *blocking pair* if both i envies j and j envies i. In this paper we are interested in the restricted variant of stability, in which only adjacent agents are allowed to swap.

Definition 2.1. An assignment π is *neighborhood stable* if there is no blocking pair assigned to adjacent vertices under π.

3 NEIGHBORHOOD STABILE ASSIGNMENTS

We notice first there are instances with no neighborhood stable assignments and give a general counterexample using balanced complete bipartite seat graphs to construct a family of instances without neighborhood stable assignments (see Figure 1).

Theorem 3.1. *For every $n = 2t$, where $t \geq 3$ is an odd integer, there is an instance with n agents with no neighborhood stable assignments.*

3.1 Neighborhood Stability on Cycles and Paths

Next, we turn to the natural restricted cases studied by Berriaud et al. [2] in the seat arrangement setting: cycle and path seat graphs. A natural approach to proving existence of a neighborhood stable assignment is to use a potential function argument. A *swap dynamic* is a procedure which begins with an arbitrary assignment and allows adjacent blocking pairs to swap until no such pairs remain. As show next, it cannot work when the seat graph is a cycle.

Proposition 3.2. *There is an instance $\mathcal{I}$ and an assignment on $\mathcal{I}$ from which no swap dynamics converge, even when the seat graph is a cycle of length 4.*

We now present our central result, showing that a neighborhood stable assignment always exists when the seat graph is a cycle.

Theorem 3.3. *A neighborhood stable assignment always exists and can be computed in $O(n^3)$ time when the seat graph is a cycle.*

We point out that Theorem 3.3 is a surprising result given the findings of Berriaud et al. [2]. In particular, if we additionally consider blocking pairs distance two from each other, even in an instance with five agents, a stable assignment may not exist on a cycle. Furthermore, if we consider cardinal preferences, even with only four agents, a neighborhood stable assignment may not exist.

For a path seat graph, we provide an algorithm that outputs an assignment with no blocking pairs within a distance at most two from each other. This answers a question left by Berriaud et al. [2], who gave a potential function argument for the same result on paths, but left open whether such an assignment could be computed with an explicit, polynomial-time algorithm.

Theorem 3.4. *An assignment in which no blocking pair are assigned at most distance two from each other always exists and can be computed in polynomial time $O(n^3)$ when the seat graph is a path.*

3.2 A Sufficient Condition for General Graphs

Here, we give a general, sufficient condition under which neighborhood stable assignments always exist. It constrains the relation between a measure of acyclicity (directed feedback vertex set numbe) on the preference graph and the number of *leaves* (i.e., degree-one vertices) in the seat graph. Here, the set $X \subseteq \mathcal{A}$ is a *directed feedback vertex set* (DFVS) if the subgraph of $\mathcal{P}$ induced by $\mathcal{A} \setminus X$ is acyclic. The DFVS number is the size of the smallest such set.

Next, we prove the existence of a neighborhood stable assignment for every instance in which the DFVS number is upper bounded by the number of leaves in the seat graph.

Theorem 3.5. *For an instance in which the number of leaves of the seat graph G exceeds the DFVS number γ of the preference graph $\mathcal{P}$, there is a neighborhood stable assignment on G. When $\gamma = 0$, i.e., $\mathcal{P}$ is a DAG, a stable assignment can be computed in polynomial time.*

4 CONCLUSION

We have initiated the study of neighborhood stability in problems where we assign agents to vertices of graphs. We prove that for cycle and path seat graphs, a neighborhood stable assignment can be computed in polynomial time. A natural next step is to identify other seat graph restrictions guaranteeing the existence of a neighborhood stable assignment for all preference graphs, or to establish how neighborhood stability affects *efficiency concepts*.

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REFERENCES

[1] H. Aziz and R. Savani. 2016. Hedonic Games. In *Handbook of Computational Social Choice*, F. Brandt, V. Conitzer, U. Endriss, J. Lang, and A. D. Procaccia (Eds.). Cambridge University Press, Chapter 15.

[2] Damien Berriaud, Andrei Constantinescu, and Roger Wattenhofer. 2023. Stable Dinner Party Seating Arrangements. In *Web and Internet Economics - 19th International Conference, WINE*, Jugal Garg, Max Klimm, and Yuqing Kong (Eds.), Vol. 14413. 3–20.

[3] Damien Berriaud, Andrei Constantinescu, and Roger Wattenhofer. 2023. Stable Dinner Party Seating Arrangements. *arXiv preprint arXiv:2305.09549* (2023).

[4] Davide Bilò, Vittorio Bilò, Pascal Lenzner, and Louise Molitor. 2022. Topological influence and locality in swap schelling games. *Autonomous Agents and Multi-Agent Systems* 36, 2 (2022), 47.

[5] Hans L. Bodlaender, Tesshu Hanaka, Lars Jaffke, Hirotaka Ono, Yota Otachi, and Tom C. van der Zanden. 2020. Hedonic Seat Arrangement Problems. In *Proceedings of the 19th International Conference on Autonomous Agents and MultiAgent Systems*. 1777–1779.

[6] Anna Bogomolnaia and Hervé Moulin. 2004. Random matching under dichotomous preferences. *Econometrica* 72, 1 (2004), 257–279.

[7] K. Cechlárová. 2008. Stable Partition Problem. In *Encyclopedia of Algorithms*. Springer, 885–888.

[8] Esra Ceylan, Jiehua Chen, and Sanjukta Roy. 2023. Optimal Seat Arrangement: What Are the Hard and Easy Cases?. In *Proceedings of the Thirty-Second International Joint Conference on Artificial Intelligence, IJCAI*. 2563–2571.

[9] Daniel Halpern, Ariel D Procaccia, Alexandros Psomas, and Nisarg Shah. 2020. Fair division with binary valuations: One rule to rule them all. In *Web and Internet Economics: 16th International Conference, WINE 2020*. 370–383.

[10] Martin Lackner and Piotr Skowron. 2023. *Multi-Winner Voting with Approval Preferences*. Springer International Publishing.

[11] Thomas C. Schelling. 1969. Models of Segregation. *The American Economic Review* 59, 2 (1969), 488–493.

[12] Thomas C. Schelling. 1971. Dynamic models of segregation. *The Journal of Mathematical Sociology* 1, 2 (1971), 143–186.

[13] Anaëlle Wilczynski. 2023. Ordinal hedonic seat arrangement under restricted preference domains: swap stability and popularity. In *Proceedings of the Thirty-Second International Joint Conference on Artificial Intelligence*. 2906–2914.