(Submodular) Hedonic Games with Common Ranking Property

Extended Abstract

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ABSTRACT

We study hedonic games with common ranking property (HGCRP), where all members of a coalition receive the same utility. We prove the existence of partitions that are both strong individually stable (SIS) and Pareto optimal (PO), as well as partitions that are contractually Nash stable (CNS) and PO. Moreover, we show that an SIS partition can be found in polynomial time. We introduce a subclass of HGCRP with submodular joint utility functions and establish that its stability and efficiency properties align with those of general HGCRP. Finally, we show that the core price of anarchy and stability in submodular HGCRP are both n, where n is the number of agents.

KEYWORDS

Algorithmic Game Theory; Hedonic Games; Existence of Equilibrium; Computational Complexity

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1 INTRODUCTION

Hedonic games provide a framework for coalition formation, where each agent's utility depends only on their coalition [19]. They naturally model various multi-agent system problems [5], including coordination, group activity selection [16], and task allocation [25].

A key subclass of hedonic games, HGCRP, ensures that all agents in a coalition receive the same utility. Farrell and Scotchmer [20] established the existence of core stable (CS) partitions in HGCRP. This result is later extended by Banerjee et al. [7] to hedonic games with the top coalition property. Dimitrov [18] introduced semistrict core stability, proving its equivalence to core stability in HGCRP. Caskurlu and Kizilkaya [14] showed the existence of a CS, individually stable (IS), and PO partition in HGCRP.

We improve upon these results by showing that every HGCRP instance has a partition that is both SIS and PO, and that an SIS partition can be computed in polynomial time. Additionally, we prove that a CNS and PO partition always exists, though finding one remains **NP-hard** due to the intractability of computing a PO partition [14].

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A notable HGCRP subclass is Hedonic Expertise Games (HEGs) [15], which model coalition formation among agents with complementary qualities. The joint utility function in HEGs is monotone and submodular, requiring a coalition size limit κ; otherwise, the grand coalition would be a perfect solution. Caskurlu et al. [15] introduced monotone HGCRP and showed that its stability and efficiency guarantees align with those of HEGs across various stability and efficiency notions.

We introduce submodular HGCRP, where coalition utilities follow the law of diminishing returns, a fundamental economic principle [27]. This framework naturally models coalition formation in networks, such as cut games on weighted graphs [2]. Submodular HGCRP generalizes HEGs while eliminating the need for a coalition size limit. We show that it satisfies the same stability and efficiency guarantees as general HGCRP across various notions, though the complexity of finding such partitions may differ. Additionally, we establish that the core price of anarchy and stability in submodular HGCRP are n, where n is the number of agents.

Hedonic games typically require exponential space to represent utility values for all coalitions, motivating concise representations. The individually rational coalition lists (IRCL) representation stores preferences only for individually rational coalitions [6], which suffices for many solution concepts. We assume (submodular) HGCRP instances are represented using IRCL. Although most instances still need exponential space, IRCL representation helps prove hardness results by enabling reductions to polynomial-space instances in the number of agents.

HGCRP is the cooperative counterpart of the resource selection games (RSGs), a restricted form of singleton congestion games [24]. In RSGs, agents are partitioned into groups such that each group of agents utilizes from the same resource and receives identical utility [21]. The existence and computation of equilibrium in RSGs under various stipulations on coordinating agent strategies are studied heavily in the literature (see [1], [12], [13], [11]).

Related models include group activity selection [16, 17], and additively separable hedonic games [9] with fractional [3], online [10, 22], and fixed-sized coalition variants [8, 26]. See references for details.

2 GAME DEFINITION

An **HGCRP instance** is defined as a pair $\mathcal{G} = (N, U)$, where N is a finite set of n agents, and $U : 2^N \setminus \emptyset \to \mathbb{R}_+$ is a non-negative real-valued function that assigns a **joint utility** to each nonempty subset of N. For convenience, we define $U(\emptyset) = 0$.

An instance $\mathcal{G}$ is a **submodular HGCRP** if U satisfies **submodularity**, meaning that for every $X, Y \subseteq N$ with $X \subseteq Y$ and every $x \in N \setminus Y$, we have $U(X \cup \{x\}) - U(X) \geq U(Y \cup \{x\}) - U(Y)$.

A **solution** to an HGCRP instance is a **partition** (or **coalition structure**) π of N. The **coalition** containing an agent $i \in N$ in partition π is denoted by $\pi(i)$. All agents in the same coalition $S \in \pi$ receive the same utility, equal to $U(S)$. The utility of an individual agent i in partition π is denoted as $u_i(\pi)$, where $u_i(\pi) = U(\pi(i))$.

3 POSITIVE RESULTS

Caskurlu and Kizilkaya [14] proved that every HGCRP instance admits a partition that is CS, IS, and PO by constructing the following potential function.

For a fixed HGCRP instance $\mathcal{G} = (N, U)$, let $\psi(\pi)$ be the sequence of the utilities of the agents in partition π in non-increasing order. Let $\trianglerighteq$ denote the binary relation "lexicographically greater than or equal to" over the set of sequences of utilities of agents. If $\psi(\pi^*) \trianglerighteq \psi(\pi)$ for every partition π, then π^* is CS, IS, and PO [14].

Theorem 3.1 strengthen this result by proving the existence of a partition that is both SIS and PO.

Theorem 3.1. *Every HGCRP instance has a coalition structure that is strong individually stable and Pareto optimal at the same time.*

The proof of Theorem 3.1 uses the same potential function. We show that any deviation violating SIS would contradict the function's maximization, ensuring that an SIS and PO partition exists. Finding such a partition is **NP-hard**, but Theorem 3.2 guarantees that an SIS partition can be found efficiently.

Theorem 3.2. *A strong individually stable partition of any given HGCRP instance $\mathcal{G} = (N, U)$ can be found in polynomial time by a simple greedy algorithm.*

Theorem 3.3. *Every HGCRP instance has a coalition structure that is both contractually Nash stable and Pareto optimal.*

The proof of Theorem 3.3 constructs a sequence of partitions, starting from any initial partition, where each partition in the sequence either contractually Nash dominates or Pareto dominates the previous one. Given the finite number of partitions, this process must eventually reach a CNS and PO partition. Otherwise, it would result in an infinite improving sequence, forming a cycle and leading to a contradiction.

4 NEGATIVE RESULTS

HEGs are a special class of hedonic games with monotone and submodular utility functions, limited to coalitions of size at most κ. Any HEG instance can be transformed into an equivalent HGCRP instance, extending the nonexistence results from HEGs [15] to both submodular and general HGCRP, as stated in Theorem 4.1.

Theorem 4.1. *The following statements are true:*

- *A submodular HGCRP (and thus, an HGCRP) instance may not have a core stable and socially optimal partition.*
- *A submodular HGCRP (and thus, an HGCRP) instance may not have an individually stable and socially optimal partition.*
- *A submodular HGCRP (and thus, an HGCRP) instance may not have a strict core stable partition.*

Theorem 4.2. *A submodular HGCRP (and thus, an HGCRP) instance may not have a contractually Nash stable and individually stable partition.*

Proof. Let $\mathcal{G} = (N, U)$ be a submodular HGCRP instance, where $N = \{1, 2, 3\}$ and U is defined as $U(\{1, 2\}) = 3$, $U(\{1\}) = U(\{2\}) = 2$, $U(\{1, 2, 3\}) = U(\{1, 3\}) = 1$, and $U(\{2, 3\}) = U(\{3\}) = 0$.

Let π be an IS partition. $\{1\} \notin \pi$, since agent 2 would deviate by moving into coalition $\{1\}$ to form $\{1, 2\}$. $\{2\} \notin \pi$, since agent 1 would deviate by moving into coalition $\{2\}$ to form $\{1, 2\}$. $\{1, 3\} \notin \pi$, since agent 1 would deviate to form a singleton coalition. $\{2, 3\} \notin \pi$, since agent 2 would deviate to form a singleton coalition. $\{1, 2, 3\} \notin \pi$, since either agent 1 or agent 2 would deviate to form a singleton coalition. Thus, $\pi = \{\{1, 2\}, \{3\}\}$ is the unique IS solution. π is not CNS, since agent 3 strictly benefits from moving into $\{1, 2\}$. □

We now present a submodular HGCRP instance that we use as a counterexample in the proofs of Theorem 4.3 and Theorem 4.4.

Example 1. Let $\mathcal{G}_1 = (N_1, U_1)$ be a submodular HGCRP instance, where $N_1 = \{1, 2, 3\}$, and U_1 is defined as follows: $U_1(\{3\}) = U_1(\{1, 3\}) = 0$, $U_1(\{1\}) = U_1(\{2\}) = U_1(\{2, 3\}) = U_1(\{1, 2, 3\}) = 1$, $U_1(\{1, 2\}) = 2$.

Theorem 4.3. *A submodular HGCRP (and thus, an HGCRP) instance may not have a contractually Nash stable and socially optimal (SO) partition.*

Proof. The unique SO solution of $\mathcal{G}_1$ is $\pi = \{\{1, 2\}, \{3\}\}$. π is not a CNS partition, since agent 3 strictly benefits from joining $\{1, 2\}$. □

Theorem 4.4. *A submodular HGCRP (and thus, an HGCRP) instance may not have a contractually Nash stable and core stable partition.*

Proof. The unique CS solution of $\mathcal{G}_1$ is $\pi = \{\{1, 2\}, \{3\}\}$. This is because $\{1, 2\}$ is the only coalition with maximum joint utility, and thus has to be contained in any CS solution. However, π is not a CNS partition since agent 3 strictly benefits from joining $\{1, 2\}$. □

Finding a PO partition of an HGCRP instance is **NP-hard** [14]. We show this hardness holds even for submodular HGCRP (Corollary 4.7), using Theorem 4.5.

Theorem 4.5 (Aziz et al. [4]). *For any hedonic game class where perfectness can be checked in polynomial time, the* ***NP-hardness*** *of finding a perfect partition (if it exists) implies the* ***NP-hardness*** *of computing a Pareto optimal partition.*

Since perfectness in submodular HGCRP can be checked in polynomial time, it suffices to prove the hardness of finding a perfect partition. The proof of Theorem 4.6 follows via a polynomial-time reduction from the *Exact Cover by 3-Sets* (X3C) problem [23].

Theorem 4.6. *Finding a perfect partition (if it exists) for a submodular HGCRP instance is* ***NP-hard****.*

Thus, by Theorems 4.5 and 4.6, we obtain:

Corollary 4.7. *Finding a Pareto optimal partition of a submodular HGCRP instance is* **NP-hard**.

Theorem 4.8 establishes the core price of anarchy and the core price of stability of submodular HGCRP.

Theorem 4.8. *Both the core price of anarchy and the core price of stability of submodular HGCRP are n.*

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