Weighted Envy Freeness With Bounded Subsidies

Extended Abstract

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ABSTRACT

We explore solutions for fairly allocating indivisible items among agents assigned weights representing their entitlements. Our fairness goal is **weighted-envy-freeness (WEF)**, where each agent deems their allocated portion relative to their entitlement at least as favorable as any other's relative to their own. In many cases, achieving WEF necessitates monetary transfers, which can be modeled as third-party subsidies. The goal is to attain WEF with bounded subsidies.

Previous work in the unweighted setting of subsidies relied on basic characterizations of EF that fail in the weighted settings. This makes our new setting challenging and theoretically intriguing. We present polynomial-time algorithms that compute WEF-able allocations with an upper bound on the subsidy per agent in three distinct additive valuation scenarios: (1) general, (2) identical, and (3) binary. When all weights are equal, our bounds reduce to the bounds derived in the literature for the unweighted setting.

The full version is available at **[20]**.

KEYWORDS

Envy-Freeness; Entitlements

ACM Reference Format:
Noga Klein Elmalem, Rica Gonen, and Erel Segal-Halevi. 2025. Weighted Envy Freeness With Bounded Subsidies: Extended Abstract. In *Proc. of the 24th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2025), Detroit, Michigan, USA, May 19 – 23, 2025*, IFAAMAS, 3 pages.

1 INTRODUCTION

The mathematical theory of fair item allocation among multiple agents has practical applications in scenarios like inheritance and partnership dissolutions. When agents have equal entitlements, as in inheritance cases, each agent naturally expects their allotment to be at least as good as others'. An allocation satisfying this requirement is called *envy-free (EF)*.

When the items available for allocation are indivisible, an EF allocation might not exist. A solution often applied in practice is to use *money* to compensate for the envy. In the recent literature, it is common to assume that a hypothetical third-party is willing to subsidize the process such that all agents receive a non-negative amount, and ask what is the *minimum amount of subsidy* required to attain envy-freeness.

Proc. of the 24th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2025), Y. Vorobeychik, S. Das, A. Nowé (eds.), May 19 – 23, 2025, Detroit, Michigan, USA. © 2025 International Foundation for Autonomous Agents and Multiagent Systems (www.ifaamas.org).

It is common to assume that the agents are *quasilinear*. This means that their utility equals their total value for the items they receive, plus the amount of money they receive (which may be positive or negative).

The subsidy minimization problem was first studied by Halpern and Shah [18]. They showed that, for any given allocation, there exists a permutation of the bundles that is *envy-freeable (EF-able)*, that is, can be made *envy-free* with subsidies. The total required subsidy is at most $(n-1)mV$, where m is the number of items, n the number of agents, and V the maximum item value for an agent, and this bound is tight in the worst case. Brustle et al. [7] considered the case in which the allocation is not given, but can be chosen. They presented an algorithm that finds an envy-freeable allocation through iterative maximum matching, requiring a total subsidy of at most $(n-1)V$, which is tight too.

This paper extends previous work by considering agents with different entitlements, which we call *weights*. This extension is useful in partnership dissolutions, where partners often hold varying numbers of shares, entitling them to different proportions of the asset. In such cases, each agent expects to receive at least the same "value per share" as others. For example, if agent i has twice the entitlement of agent j, i expects a bundle worth at least twice as much as j's.

Formally, an allocation is called *weighted envy-free (WEF)* (see e.g. Chakraborty et al. [9], Robertson and Webb [24], Zeng [37]) if for every two agents i and j, $\frac{1}{w_i}$ times the utility that i assigns to his own bundle is at least as high as $\frac{1}{w_j}$ times the utility that i assigns to the bundle of j, where w_i is i's entitlement and w_j is j's entitlement.

Now, we define *weighted envy-freeability (WEF-ablity)*, the key concept we propose, analogously to the unweighted case: an allocation is WEF-able if it can be made WEF with subsides. More precisely, an allocation is WEF-able if for every two agents i and j, $\frac{1}{w_i}$ times the sum of the utility that i assigns to his own bundle and the subsidy he receives is at least as high as $\frac{1}{w_j}$ times the sum of the utility that i assigns to the bundle of j and the subsidy j receives. Here, we assume quasi-linear utilities.

To illustrate the difficulty in this generalized setting, we show that the results from Brustle et al. [7], Halpern and Shah [18] fail when agents have different entitlements.

Example 1.1. There are two items o_1, o_2 and two agents i_1, i_2, with weights $w_1 = 1, w_2 = 10$ and valuation functions

$$\begin{bmatrix} & o_1 & o_2 \\ i_1 & 5 & 7 \\ i_2 & 10 & 8 \end{bmatrix}$$

We will show that, contrary to the result of Halpern and Shah [18], there exists a division of items where no permutation satisfies WEF.

Consider the bundles $A_1 = \{o_1\}$ and $A_2 = \{o_2\}$, where i_1 receives A_1 and i_2 receives A_2. Let s_1 and s_2 represent the subsidies for i_1 and i_2, respectively. The utility of i_1 for their own bundle is $5 + s_1$, and for i_2's bundle, it is $7 + s_2$. To satisfy WEF, we need: $\frac{5+s_1}{1} \geq \frac{7+s_2}{10}$, which implies $s_2 \leq 43 + 10s_1$. Similarly, for agent i_2, WEF requires: $\frac{8+s_2}{10} \geq \frac{10+s_1}{1}$, which implies $s_2 \geq 92 + 10s_1$. These two conditions are contradictory, so no subsidies can make this allocation WEF. Next, consider the permutation where i_1 receives A_2 and i_2 receives A_1. In this case, WEF requires: $\frac{7+s_1}{1} \geq \frac{5+s_2}{10}$, which implies $s_2 \leq 65+10s_1$, and for i_2: $\frac{10+s_2}{10} \geq \frac{8+s_1}{1}$, which implies $s_2 \geq 70 + 10s_1$. Again, these conditions are contradictory, proving that no permutation of bundles satisfies WEF. This example also shows that the Iterated Maximum Matching algorithm of Brustle et al. [7] does not guarantee WEF. The algorithm yields an allocation where all agents receive the same number of items, but as shown, no such allocation can be made WEF.

Of course, since the unweighted case is equivalent to the weighted case where each weight $w_i = 1/n$, all negative results from the unweighted setting extend to the weighted case. In particular, it is NP-hard to compute the minimum subsidy required to achieve (weighted) envy-freeness, even in the binary additive case (as shown in [18, Corollary 1]). Thus, following previous work, we develop polynomial-time algorithms that, while not necessarily optimal, guarantee an upper bound on the total subsidy.

1.1 Related Work

Our work integrates two lines of research: fair allocation with monetary transfers and fair allocation with different entitlements. We survey each line separately in [20].

Few works address both entitlements and subsidies. Wu et al. [34] presented a polynomial-time algorithm for computing a weighted *proportional* allocation of *chores* among agents with additive valuations, with total subsidy at most $\frac{(n-1)V}{2}$.

In a subsequent work (Wu et al. [36]), they further improved this bound to $(\frac{n}{3} - \frac{1}{6})V$. As far as we know, weighted envy-freeness with subsides has not been studied yet. Our paper aims to fill this gap.

1.2 Our Results

We derive bounds on the amount of subsidy required in order to attain a WEF allocation, in several different settings. We assume, without loss of generality, that the entitlements are ordered in increasing order: $w_1 \leq w_2 \leq \ldots \leq w_n$. We denote $W := \sum_{i=1}^{n} w_i$.

First, we assume a given allocation. As shown in Example 1.1, there are instances in which no rearrangement of bundles yields a WEF-able allocation. We prove a necessary and sufficient condition under which the allocation is WEF-able. We show that, when the allocation is WEF-able, a total subsidy of $(\frac{W}{w_1} - 1)mV$ is sufficient to make it WEF, and prove that this bound is tight in the worst case.

This raises the question of whether a weighted-envy-free allocation with subsidy always exists? We answer this affirmatively. For additive valuations and integer weights, we show an m-independent upper bound: $\frac{W-w_1}{\gcd(\mathbf{w})}V$, where $\gcd(\mathbf{w})$ is the greatest common divisor of all weights — largest number d such that w_i/d is an integer for all $i \in N$. Our algorithm extends Brustle et al. [7] algorithm, which, in the unweighted setting, ensures a total subsidy of at most $(n-1)V$. For equal entitlements, normalizing weights to 1 preserves the same bounds as in the unweighted case.

Following Halpern and Shah [18], we study not only the setting of general additive valuations but also the settings where agents have *identical* and *binary* additive valuations.

For identical additive valuations, we compute a WEF-able and WEF(0, 1) allocation with total subsidy at most $(n-1)V$, which is tight even in the unweighted case. Interestingly, in this special case (in contrast with the general case), the bound on the subsidy does not depend on the weights. For binary-additive valuations, we adapt the General Yankee Swap algorithm (Viswanathan and Zick [30]) to compute a WEF-able and WEF(0, 1) allocation with total subsidy at most $\frac{W}{w_1} - 1$, reducing to $n-1$ for equal weight.

Our findings and contributions are briefly summarized in Table 1.

Table 1: Distinctions between outcomes established in prior research, and those newly established in the present study, highlighted in bold. All subsidy upper bounds are attainable by polynomial-time algorithms.

	Unweighted Setup	Weighted Setup		
		General Valuations	Identical Valuations	Binary Valuations
Characterization of WEF Allocation	(1) No positive cost cycles, (2) USW maximization [18]	No positive-cost cycles. **[20]**		
Permutation of a given allocation, that maximizes sum of values	Always EF-able [18]	Not necessarily WEF-able **[20]**	Always WEF-able. **[20]**	For non-redundant allocation: Always WEF-able **[20]**
Total subsidy upper bound	$(n-1)V$ [18]	$\frac{W-w_1}{\gcd(\mathbf{w})}V$ **[20]**	$(n-1)V$ **[20]**	$\frac{W}{w_1}-1$ **[20]**
Subsidy bound for a given allocation	$(n-1)mV$ [18]	$\left(\frac{W}{w_1}-1\right)mV$ **[20]**		
Total subsidy lower bound	$(n-1)V$ [18]	$\left(\frac{W}{w_1}-1\right)V$ **[20]**	$(n-1)V$ **[20]**	$\frac{W}{w_2}-1$ **[20]**

Remark 1.2. Practical fair allocation cases use budget-balanced payments rather than subsidies. We use the subsidies terminology for consistency with previous works.

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