Matching Markets with Chores

Extended Abstract

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ABSTRACT

The fair division of chores, as well as mixed manna (goods and chores), has received substantial recent attention in the fair division literature; however, ours is the first paper to extend this research to matching markets. Indeed, our contention is that matching markets are a natural setting for this purpose, since the manna that fit into the limited number of hours available in a day can be viewed as one unit of allocation. We extend several well-known results that hold for goods to the settings of chores and mixed manna. In addition, we show that the natural notion of an earnings-based equilibrium, which is more natural in the case of all chores, is equivalent to the pricing-based equilibrium given by Hylland and Zeckhauser for the case of goods.

KEYWORDS

Matching markets; market design; chores; fair division

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1 INTRODUCTION

In a *one-sided matching market*, we are given a set A of *agents* and a set G of *items* (traditionally goods). Each agent has preferences over the items. We assume that $|A| = |G| = n$ and the goal is then to find a perfect matching between items and agents which satisfies certain desirable properties including fairness and efficiency.

Markets of this kind arise in various situations in which we want to fairly allocate items/entities among people but in which payments would be considered immoral or impractical. For example, consider assigning students to schools (or to individual courses), assigning organ donations to recipients, or doctors to hospitals.

In this paper, we focus on cardinal utilities, i.e. each agent i submits numerical utilities $(u_{ij})_{j\in G}$ for the items. Whereas ordinal preferences are easier to elicit, cardinal preferences are more expressive, thereby producing higher quality allocations and leading to significant gain in efficiency; see [18].

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Integral allocations in matching markets cannot achieve any reasonable measure of fairness. Hence, it is customary to allow lotteries over matchings, i.e. fractional perfect matchings. In this setting, the classic mechanism is due to Hylland and Zeckhauser (HZ) [17] based on competitive equilibrium. It finds lotteries which are Pareto-optimal (PO) and envy-free (EF). Moreover, it is asymptotically incentive compatible [15]. However, the problem of approximating the HZ equilibrium is PPAD-complete [9, 21], and so is the more general problem of finding any EF+PO allocation [7, 20].

In the HZ mechanism, all utilities are assumed to be *non-negative*, i.e. G consists exclusively of *goods*. While considering chores as well, three types of settings need to be studied:

(1) In the *goods* setting, we have $u_{ij} \geq 0$ for all i, j.
(2) In the *chores* setting, we have $u_{ij} \leq 0$ for all i, j.
(3) The *mixed* setting, also called *mixed manna*.

For a motivating example, consider the division of various tasks or activities among people. For any given agent, an activity might be enjoyable ($u_{ij} > 0$) or it might be a chore ($u_{ij} < 0$). The matching constraints on the agents enforce the fact that each agent has only a limited number of hours available in a day to do both enjoyable and displeasing activities.

The fair division of chores as well as mixed manna has received substantial recent attention in the fair division literature [2, 14, 19]. Ours is the first paper to extend this research to the area of matching markets. In the rest of this extended abstract we will outline the core contributions of our paper.

2 EQUILIBRIUM NOTIONS

The core idea behind the HZ mechanism is to implement a *pseudo-market*, i.e. to introduce some amount of fake money to create a market with money and then to use a market equilibrium in order to find our desirable allocations. In order for this to be fair, in the HZ mechanism, each agent gets exactly one unit of fake money. The corresponding equilibrium notion (HZ equilibrium) was shown by Hylland and Zeckhauser to always exist [17]. Moreover, HZ equilibria are Pareto-optimal and envy-free.

Inspired by recent work on fair division and market equilibria with chores [6–8], we define an analogous equilibrium notion (HZ earnings equilibrium) in which agents need to *earn* money rather than *spend* it. This is a more natural equilibrium in the chores setting since we would want to compensate agents for doing chores.

3 SHIFTING UTILITIES

Our first observation is that due to the fact that we are considering only fractional perfect matchings, most interesting properties will be preserved by a simple shifting operation.

Definition 3.1. Let $(u_{ij})_{i\in A, j\in G}$ and $(c_i)_{i\in A}$ be rational numbers. Then we define $(u+c)_{ij} \coloneqq u_{ij} + c_i$.

LEMMA 3.2. *Let x be a fractional perfect matching which is envy-free wrt. utilities u. Then x is also envy-free wrt. utilities $u + c$.*

LEMMA 3.3. *Let x be a fractional perfect matching which is Pareto-optimal wrt. utilities u. Then x is also Pareto-optimal wrt. utilities $u + c$.*

LEMMA 3.4. *Let (x, p) be an HZ equilibrium wrt. utilities u. Then (x, p) is also an HZ equilibrium wrt. utilities $u + c$.*

Thus, we need not restrict HZ equilibria to the all goods setting as is commonly done. In order to show existence of HZ equilibria for arbitrary utilities, simply shift everything into the non-negative, get an HZ equilibrium, and then use Lemma 3.4 to show that said equilibrium is also an HZ equilibrium under the original, mixed utilities. The same observation holds for HZ earnings equilibria.

4 EQUIVALENCE OF EQUILIBRIA

Next, we show that HZ equilibria and HZ earnings equilibria are, rather surprisingly, the same thing.

THEOREM 4.1. *Let x be some fractional perfect matching. If there exist prices $(p_j)_{j\in G}$ making (x, p) an HZ equilibrium, then there also exist earnings $(q_j)_{j\in G}$ making (x, q) an HZ earnings equilibrium and vice versa.*

As a corollary, we of course get that HZ earnings equilibria always exist.

THEOREM 4.2. *An HZ earnings equilibrium always exists.*

5 BI-VALUED UTILITIES

An interesting special case in the all-goods setting is that in which $u_{ij} \in \{0, 1\}$ for all i, j. Utilities of this type represent the only special case (with unbounded agents / goods) in which HZ equilibria are known to be polynomial time computable [21]. Moreover, due to a result of Bogomolnaia and Moulin [5], the HZ mechanism is also incentive compatible in the dichotomous setting.

We can easily extend these results to more generalized bi-valued utilities, including chores. Such utilities have been extensively studied (see, e.g., [4, 5, 10, 11, 21]) due to their practical relevance.

THEOREM 5.1. *If the utilities are of the form $u_{ij} \in \{a_i, b_i\}$ with $(a_i)_{i\in A}$ and $(b_i)_{i\in B}$ rational, we can compute HZ equilibria in polynomial time. Moreover, the resulting mechanism is strategy-proof.*

Due to Theorem 4.1, we can also compute HZ earnings equilibria in polynomial time for bi-valued instances.

6 CONSTANTLY MANY AGENT TYPES

Another interesting special case is the setting in which there are constantly many types of agents. Here, we are given a small set A of agents and a set G of goods or chores. Each agent $i \in A$ has some demand d_i such that $\sum_{i\in A} d_i = |G| = n$. The goal is to fractionally assign G to A such that every agent i gets exactly d_i units of goods and chores.

The setting with constantly many agents can be approached in two different ways. First, there are algorithms that compute (approximate) HZ equilibria with a constant number of agents. [1, 13] However, these algorithms are intractable in practice for all but the smallest instances.

On the other hand, if the entire instance has constant size, i.e. both the types of agents *and* of goods are constant, then we show how to use a polyhedral approach [20] to efficiently find EF+PO allocations.

Lastly, let us consider an even more special case in which there are only two types of agents. This scenario has been extensively studied across various related contexts (see, e.g., [3, 12]) In this setting, we show that is possible to find an EF+PO allocation in polynomial time via linear programming and the following lemma.

LEMMA 6.1. *Assume there are only two types of agents with different demands, i.e. $|A| = 2$, and let x be an envy-free fractional perfect matching. Moreover, let y be another fractional perfect matching which is Pareto-better than x. Then y is also envy-free.*

7 NASH BARGAINING

Tröbst and Vazirani [20] recently showed that in fact finding any allocation which is envy-free and Pareto-optimal is already PPAD hard. However, they also show that the alternative, Nash-bargaining-based mechanism proposed by Hosseini and Vazirani [16] is 2-approximately envy-free and 2-approximately incentive compatible. The idea behind this mechanism is to maximize Nash welfare, i.e. the product of agents' utilities.

The attractive game theoretic properties together with the polynomial time computability make Nash bargaining a promising alternative to HZ for the all goods setting. We therefore pose the question: is there an analogous mechanism for the chores or even mixed settings?

There are two natural ways in which the Nash-bargaining-based mechanism can be generalized to the chores setting:

(1) We can minimize the product of agents' *disutilities*, i.e. the negation of their utilities.
(2) We can maximize the product of agent's disutilities over the set of Pareto-optimal fractional perfect matchings. This generalization is inspired by the work of Bogomolnaia et al. [6] who showed that this is the right generalization of Nash bargaining to the chores setting *without* matching constraints.

In both cases, the resulting allocations are obviously Pareto-optimal. We provide counter-examples to show that neither generalization results in bounded envy.

8 DISCUSSION

In this paper, we initiated the study of matching markets involving chores. Many results from the goods setting can be extended to mixed or chores settings via utility shifting. However, our work also leads to a very natural and exciting open problem:

QUESTION 8.1. *Is there a polynomial time algorithm that finds approximately fair and efficient allocations in a cardinal-utility matching market with chores?*

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