Fair Assignment on Multi-Stage Graphs

Extended Abstract

Vibulan J
IIITDM Kancheepuram
Chennai, India
vibulanj@gmail.com

Swapnil Dhamal
Indian Institute of Technology Ropar
Rupnagar, India
swapnil.dhamal@iitrpr.ac.in

Shweta Jain
Indian Institute of Technology Ropar
Rupnagar, India
shwetajain@iitrpr.ac.in

Ojassvi Kumar
Indian Institute of Technology Ropar
Rupnagar, India
2020csb1187@iitrpr.ac.in

Aman Kumar
Indian Institute of Technology Ropar
Rupnagar, India
2020csb1153@iitrpr.ac.in

Harpreet Singh
Indian Institute of Technology Ropar
Rupnagar, India
2020mcb1237@iitrpr.ac.in

ABSTRACT

This paper explores the problem of fair assignment of disjoint paths to agents on *multi-stage graphs*. We motivate the problem by demonstrating that an assignment minimizing the overall cost of all the agents' paths may lead to significant envy among the agents. Showing NP-hardness of finding an envy-minimizing assignment, we propose algorithms that achieve a desired degree of envy while also providing a bound on the Cost of Fairness. Our algorithms run several orders of magnitude faster than a suitably formulated ILP.

CCS CONCEPTS

• **Theory of computation → Algorithmic game theory**; *Network optimization*; *Shortest paths.*

KEYWORDS

Fairness; Multi-stage graphs; Assignment; Routing

ACM Reference Format:

Vibulan J, Swapnil Dhamal, Shweta Jain, Ojassvi Kumar, Aman Kumar, and Harpreet Singh. 2025. Fair Assignment on Multi-Stage Graphs: Extended Abstract. In *Proc. of the 24th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2025), Detroit, Michigan, USA, May 19 – 23, 2025*, IFAAMAS, 3 pages.

1 INTRODUCTION

Resource assignment on graphs is essential for optimizing utilization in domains like manufacturing, project management, operating systems, parallel computing, and routing. In such graph representations, nodes represent tasks, resources, or processes, while edges capture dependencies. One such type of graph is the fully-connected multi-stage (FCMS) graph, where nodes are grouped into stages, with each node connected to all nodes in the next stage. The fully-connectedness is justifiable in applications where shortest path algorithms are applicable. FCMS graphs are widely used in supply chains [5], vehicle routing [24], project scheduling [41], and financial strategies [31]. While optimizing a utility objective, such as a function of cost and time, is well-studied for FCMS graphs, ensuring fair assignment (where resources or workloads are equitably assigned so that no agent is unfairly burdened or underutilized) is equally important. It is also critical to ensure that fairness is achieved without significantly increasing overall costs.

To understand the importance of fair assignment in FCMS graphs, consider a multi-product, multi-stage supply chain network [5]. A company like Bosch serves multiple clients (e.g., Toyota, Hyundai, Honda, Ford, BMW) by producing automobile parts through a manufacturing process using assembly lines available. Each stage represents a manufacturing phase, with nodes as workstations across locations. Every automobile part must pass through all stages, using one workstation per stage. Minimizing overall cost/time incurred due to transportation and manufacturing is crucial, but ensuring fair cost/time distribution across clients is equally important for fair market competition; this corresponds to minimizing envy in the assignment. Additionally, optimal production capacity utilization is necessary, preventing workstation overload by ensuring each node is assigned to not more than one agent. If a node has the capacity to accommodate multiple agents at a time, multiple replicas of this node can be created and treated as individual nodes.

This paper addresses the problem of fair assignment on FCMS graphs. For ease of exposition, we focus on a subclass, balanced FCMS (BFCMS) graphs, where the number of nodes in each stage is equal to the number of agents. The problem requires assigning each agent a non-overlapping path while mitigating envy (the difference in the costs of paths assigned to any two agents) while keeping the combined cost of all the paths within a certain bound.

Minimizing envy among agents has been studied in fair division of divisible [4, 15, 39] and indivisible goods/tasks [3, 6, 7, 26], as well as applications like group trip planning [38], vehicle routing [1, 29], bandwidth allocation [12], underwater network routing [14], ride-sharing cost allocation [27], load balancing [25], and fair delivery [22]. Several works have empirically studied fairness in delivery, without theoretical guarantees [17, 20, 28, 32]. Few works on the vehicular routing problem use Integer Linear Programming (ILP) for a fair solution [28]. Our focus is on efficient algorithmic solutions with provable bounds, linking the problem to fair route planning [21, 38]. Fairness in group trip planning has been studied [37, 38], but it focuses on selecting a common path for all agents, unlike our work. Supply chain optimization research has primarily aimed at efficiency improvements across different

Proc. of the 24th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2025), Y. Vorobeychik, S. Das, A. Nowé (eds.), May 19 – 23, 2025, Detroit, Michigan, USA. © 2025 International Foundation for Autonomous Agents and Multiagent Systems (www.ifaamas.org).

segments [8, 18, 19, 33, 34], with fairness efforts mainly addressing fair profit distribution among agents [11, 35, 36, 42], which differs from our work. Fairness in the FCMS graph problem is similar to fair allocation of items/tasks on graphs under 'connected and equal number of goods' constraints [9, 30], where nodes represent tasks. However, in FCMS, the cost of a task at one stage depends on the allocated task in the previous stage, unlike fixed valuations in a standard fair allocation. This also differentiates our problem from repeated allocation models [2, 10, 16, 23], which do not account for the dependency of the cost of a task allocated at a time on the earlier allocated task.

2 THE MODEL

Consider a weighted graph $\mathcal{G} = (V, E)$, where V is the set of nodes, E is the set of edges, and w_e is the weight of edge e. Set V is partitioned into K disjoint subsets, each containing n nodes. A balanced fully-connected multi-stage (BFCMS) graph is defined as a sequence of K such stages where a node from stage j is linked to every node in stage $j+1$. Let $M = \max_{e \in E} w_e$ denote the maximum edge weight in this graph. A valid solution S to the assignment problem in BFCMS graph comprises n disjoint paths, each originating from a node in stage 1 and terminating at a node in stage K. The solution is expressed as $S = (P_1, \ldots, P_n)$, where each $P_i = (p_i^1, \ldots, p_i^{K-1})$ signifies an individual path with edges p_i^j from stage j to $j+1$ for an agent i. Let us denote the set of all valid solutions by $\mathcal{F}$. Since each agent i is assigned a path P_i, the path's cost, say $C(P_i) = \sum_{e \in P_i} w_e$, represents the cost incurred by agent i.

Definition 2.1 (Cost of a Solution). The cost of a solution $S \in \mathcal{F}$, where $S = (P_1, \ldots, P_n)$, is defined as the sum of the weights of all the edges in the solution, expressed as $C(S) = \sum_{i=1}^{n} \sum_{e \in P_i} w_e$.

A minimum-cost solution can be obtained using a simple adaptation of Suurballe's algorithm [40], which finds n node-disjoint paths of the minimum total length from a source to a terminal node.

Definition 2.2 (Envy of a Solution). The envy of a solution $S \in \mathcal{F}$ is defined as the maximum difference between the costs of any two agents in the solution, given by $\mathcal{E}(S) = \max_{P_i, P_j \in S} \left(C(P_i) - C(P_j) \right)$.

The envy of a minimum-cost solution could be significantly high. Figure 1a depicts such a scenario with $n = 2$ agents and K stages; the unique minimum-cost assignment is marked with blue and red paths. Here, the total cost, as well as envy of the solution, is $(K-1)(M-\delta)$. With an arbitrarily high value of K and especially if $\delta << M$, the envy is arbitrarily high. Now, if we swap the nodes allocated to the two agents from stage $\lfloor K/2 \rfloor + 1$ onwards, we obtain a new solution (Figure 1b), which has an envy that is lower than M. However, additional cost is incurred as a result of this swap due to considering suboptimal edges. Here, the total cost rises to $(K-2)(M-\delta) + 2M$.

Definition 2.3 (Cost of Fairness). The Cost of Fairness (CoF) for a fair algorithm $\mathcal{A}$ is defined as the ratio of the cost of the solution $S_{\mathcal{A}}$ produced by $\mathcal{A}$ to the cost of the minimum-cost solution $S^* = \arg\min_{S \in \mathcal{F}} C(S)$, given by $CoF(\mathcal{A}) = \frac{C(S_{\mathcal{A}})}{C(S^*)}$.

This definition of CoF is in line with [13]. Note that it differs from Price of Fairness (PoF), which quantifies the worst-case ratio of the optimal 'fair' solution to that of the optimal solution.

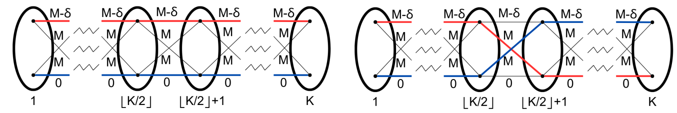

Figure 1: Illustration of (a) a minimum-cost solution being highly unfair and (b) a possible workaround.

3 FAIR ASSIGNMENT

Our work provides an algorithmic solution to the assignment problem on BFCMS, that results in low envy with a bound on CoF.

THEOREM 3.1. *Finding an envy-minimizing assignment on an arbitrary BFCMS graph with a given $K \geq 4$ is NP-hard, and so is that with a given $n \geq 2$.*

For the case of $n = 2$ agents, if $\mathcal{E}(S^*) > 2M$, we identify a stage i in S^* as the minimum stage where the cost difference between the agents' subpaths becomes greater than $\frac{\mathcal{E}(S^*)}{2}$, and swap the nodes allocated to the agents from this stage onwards. We term this, the Cost-Balance (*C-Balance*) algorithm.

THEOREM 3.2. *Consider a BFCMS graph with 2 agents, K stages, and the maximum edge weight M. Then, C-Balance achieves an envy of at most $2M$, and its CoF is bounded by 2. Further, the envy bound of $2M$ is tight, i.e., there exists an instance where envy is $2M$.*

For $n > 2$ agents, we iteratively choose agents with the minimum and maximum path costs and perform a swap of assignments between them using the *C-Balance* algorithm. This process is repeated until the overall envy is bounded by $(2+\alpha)M$, for any given $\alpha > 0$. We term this, the Dynamic Cost-Balance (*DC-Balance*) algorithm.

THEOREM 3.3. *Consider a BFCMS graph with $n > 2$ agents, K stages, and the maximum edge weight M, such that the minimum-cost solution results in an envy of $\mathcal{E}(S^*)$. Then, for achieving an envy of at most $(2+\alpha)M$ for any $\alpha > 0$, the CoF of DC-Balance is bounded by* $1 + \frac{2M\lfloor \frac{n}{2} \rfloor \left\lceil \log_2 \left(\frac{\mathcal{E}(S^*) - 2M}{\alpha M} \right) \right\rceil}{C(S^*)}$, *which is* $O\left(n \log\left(\frac{K}{\alpha}\right)\right)$.

With the help of example instances, it can be shown that for minimizing envy, it is infeasible to provide a constant instance-independent bound on CoF. It is also easy to see that our approach and results are directly applicable to general FCMS graphs wherein there may exist stages with more nodes than the number of agents, by applying our algorithm on the BFCMS graph that is induced by the n node-disjoint paths in the minimum-cost solution.

We observe that our algorithms run several orders of magnitude faster than an ILP formulated to minimize the total cost with the constraint that envy is bounded by $2M$. For instance, for 20 agents and 40 stages, ILP (solved using Gurobi Optimizer v11) is observed to be slower by a factor of $\sim 3 \times 10^5$ as compared to *DC-Balance*. With the ILP approach being computationally exorbitant, a primary advantage of our approach is its low computational cost while still providing bounds on relevant measures such as envy and CoF.

ACKNOWLEDGMENTS

Vibulan J was supported by Uplink Internship Program of India Chapter of ACM SIGKDD.

REFERENCES

[1] Martin Aleksandrov. 2022. Driver Equitability and Customer Optimality in Intelligent Vehicle Applications. In *Proceedings of the 21st EPIA Conference on Artificial Intelligence*. Springer, 309–321.

[2] Martin Aleksandrov. 2023. Limiting Inequalities in Repeated House and Task Allocation. In *Proceedings of the 36th Australasian Joint Conference on Artificial Intelligence*. Springer, 338–349.

[3] Georgios Amanatidis, Georgios Birmpas, Aris Filos-Ratsikas, and Alexandros A Voudouris. 2022. Fair Division of Indivisible Goods: A Survey. In *Proceedings of the 31st International Joint Conference on Artificial Intelligence*. IJCAI, 5385–5393.

[4] Haris Aziz and Simon Mackenzie. 2016. A Discrete and Bounded Envy-Free Cake Cutting Protocol for Any Number of Agents. In *Proceedings of the 57th Annual IEEE Symposium on Foundations of Computer Science*. IEEE, 416–427.

[5] Peyman Bahrampour, Mansoureh Safari, and Mahmood Baghban Taraghdari. 2016. Modeling Multi-Product Multi-Stage Supply Chain Network Design. *Procedia Economics and Finance* 36 (2016), 70–80.

[6] Xiaohui Bei, Zihao Li, Jinyan Liu, Shengxin Liu, and Xinhang Lu. 2021. Fair Division of Mixed Divisible and Indivisible Goods. *Artificial Intelligence* 293 (2021), 103436.

[7] Umang Bhaskar, A. R. Sricharan, and Rohit Vaish. 2021. On Approximate Envy-Freeness for Indivisible Chores and Mixed Resources. In *Proceedings of the 24th International Conference on Approximation Algorithms for Combinatorial Optimization Problems*. Springer, 1:1–1:23.

[8] Shantanu Biswas and Yadati Narahari. 2004. Object-Oriented Modeling and Decision Support for Supply Chains. *European Journal of Operational Research* 153, 3 (2004), 704–726.

[9] Sylvain Bouveret, Katarína Cechlárová, Edith Elkind, Ayumi Igarashi, and Dominik Peters. 2017. Fair Division of a Graph. In *Proceedings of the 26th International Joint Conference on Artificial Intelligence*. IJCAI, 135–141.

[10] Ioannis Caragiannis and Shivika Narang. 2023. Repeatedly Matching Items to Agents Fairly and Efficiently. In *Proceedings of the 16th International Symposium on Algorithmic Game Theory*. Springer, 347–364.

[11] Cheng-Liang Chen, Bin-Wei Wang, and Wen-Cheng Lee. 2003. Multiobjective Optimization for a Multienterprise Supply Chain Network. *Industrial & Engineering Chemistry Research* 42, 9 (2003), 1879–1889.

[12] Stefano Coniglio, Luca G. Gianoli, Edoardo Amaldi, and Antonio Capone. 2020. Elastic Traffic Engineering Subject to a Fair Bandwidth Allocation via Bilevel Programming. *IEEE/ACM Transactions on Networking* 28, 6 (2020), 2407–2420.

[13] Shantanu Das, Swapnil Dhamal, Ganesh Ghalme, Shweta Jain, and Sujit Gujar. 2022. Individual Fairness in Feature-based Pricing for Monopoly Markets. In *Proceedings of the 38th Conference on Uncertainty in Artificial Intelligence*. PMLR, 486–495.

[14] Roee Diamant, Paolo Casari, Filippo Campagnaro, Oleksiy Kebkal, Veronika Kebkal, and Michele Zorzi. 2018. Fair and Throughput-Optimal Routing in Multimodal Underwater Networks. *IEEE Transactions on Wireless Communications* 17, 3 (2018), 1738–1754.

[15] Lester E. Dubins and Edwin H. Spanier. 1961. How to Cut a Cake Fairly. *The American Mathematical Monthly* 68, 1P1 (1961), 1–17.

[16] Ulle Endriss. 2013. Reduction of Economic Inequality in Combinatorial Domains. In *Proceedings of the 12th International Conference on Autonomous Agents and Multi-Agent Systems*. IFAAMAS, 175–182.

[17] Seyed Esmaeili, Sharmila Duppala, Davidson Cheng, Vedant Nanda, Aravind Srinivasan, and John P. Dickerson. 2023. Rawlsian Fairness in Online Bipartite Matching: Two-Sided, Group, and Individual. In *Proceedings of the 37th AAAI Conference on Artificial Intelligence*. AAAI, 5624–5632.

[18] Dinesh Garg, Yadati Narahari, Earnest Foster, Devadatta Kulkarni, and Jeffrey D Tew. 2005. A Groves Mechanism Approach to Decentralized Design of Supply Chains. In *Proceedings of the 7th IEEE International Conference on E-Commerce Technology*. IEEE, 330–337.

[19] Joseph Geunes and Panos M. Pardalos. 2006. *Supply Chain Optimization*. Vol. 98. Springer Science & Business Media.

[20] Anjali Gupta, Rahul Yadav, Ashish Nair, Abhijnan Chakraborty, Sayan Ranu, and Amitabha Bagchi. 2022. FairFoody: Bringing in Fairness in Food Delivery. In *Proceedings of the 36th AAAI Conference on Artificial Intelligence*. AAAI, 11900–11907.

[21] Tanzima Hashem, Sukarna Barua, Mohammed Eunus Ali, Lars Kulik, and Egemen Tanin. 2015. Efficient Computation of Trips with Friends and Families. In *Proceedings of the 24th ACM International Conference on Information and Knowledge Management*. ACM, 931–940.

[22] Hadi Hosseini, Shivika Narang, and Tomasz Was. 2024. Fair Distribution of Delivery Orders. In *Proceedings of the 33rd International Joint Conference on Artificial Intelligence*. IJCAI, 2825–2833.

[23] Ayumi Igarashi, Martin Lackner, Oliviero Nardi, and Arianna Novaro. 2024. Repeated Fair Allocation of Indivisible Items. In *Proceedings of the 38th AAAI Conference on Artificial Intelligence*. AAAI, 9781–9789.

[24] Byung-In Kim, Jeongin Koo, and Junhyuk Park. 2010. The Combined Manpower-Vehicle Routing Problem for Multi-Staged Services. *Expert Systems with Applications* 37, 12 (2010), 8424–8431.

[25] Avadh Kishor, Rajdeep Niyogi, and Bharadwaj Veeravalli. 2020. Fairness-Aware Mechanism for Load Balancing in Distributed Systems. *IEEE Transactions on Services Computing* 15, 4 (2020), 2275–2288.

[26] Richard J. Lipton, Evangelos Markakis, Elchanan Mossel, and Amin Saberi. 2004. On Approximately Fair Allocations of Indivisible Goods. In *Proceedings of the 5th ACM Conference on Electronic Commerce*. ACM, 125–131.

[27] Wei Lu and Luca Quadrifoglio. 2019. Fair Cost Allocation for Ridesharing Services – Modeling, Mathematical Programming, and an Algorithm to Find the Nucleolus. *Transportation Research Part B: Methodological* 121 (2019), 41–55.

[28] Aitor López Sánchez, Marin Lujak, Frederic Semet, and Holger Billhardt. 2022. On Balancing Fairness and Efficiency in Routing of Cooperative Vehicle Fleets. In *Proceedings of the 12th International Workshop on Agents in Traffic and Transportation (co-located with IJCAI-ECAI 2022)*, Vol. 3173. CEUR Workshop Proceedings series, 62–76.

[29] Piotr Matl, Richard F. Hartl, and Thibaut Vidal. 2018. Workload Equity in Vehicle Routing Problems: A Survey and Analysis. *Transportation Science* 52, 2 (2018), 239–260.

[30] Neeldhara Misra and Debanuj Nayak. 2022. On Fair Division with Binary Valuations Respecting Social Networks. In *Proceedings of the 8th Annual International Conference on Algorithms and Discrete Applied Mathematics*. Springer, 265–278.

[31] John M. Mulvey and Bala Shetty. 2004. Financial Planning via Multi-Stage Stochastic Optimization. *Computers & Operations Research* 31, 1 (2004), 1–20.

[32] Ashish Nair, Rahul Yadav, Anjali Gupta, Abhijnan Chakraborty, Sayan Ranu, and Amitabha Bagchi. 2022. Gigs with Guarantees: Achieving Fair Wage for Food Delivery Workers. In *Proceedings of the 31st International Joint Conference on Artificial Intelligence*. IJCAI, 5122–5128. Special Track on AI for Good.

[33] Yadati Narahari and Shantanu Biswas. 2007. Performance Measures and Performance Models for Supply Chain Decision Making. In *Measuring Supply Chain Performance*. ICFAI (Institute of Chartered Financial Analysts of India) University Press, Chapter 7, 97–122.

[34] Yadati Narahari and Nikesh Kumar Srivastava. 2007. A Bayesian Incentive Compatible Mechanism for Decentralized Supply Chain Formation. In *Proceedings of the 9th IEEE International Conference on E-Commerce Technology and the 4th IEEE International Conference on Enterprise Computing, E-Commerce and E-Services*. IEEE, 315–322.

[35] Rui Qiu, Qi Liao, Renfu Tu, Yingqi Jiao, An Yang, Zhichao Guo, and Yongtu Liang. 2023. Pipeline Pricing and Logistics Planning in the Refined Product Supply Chain based on Fair Profit Distribution. *Computers & Industrial Engineering* 175 (2023), 108840.

[36] Tadeusz Sawik. 2015. On the Fair Optimization of Cost and Customer Service Level in a Supply Chain under Disruption Risks. *Omega* 53 (2015), 58–66.

[37] Mayank Singhal and Suman Banerjee. 2022. Envy-Free Trip Planning in Group Trip Planning Query Problem. In *Proceedings of the 25th International Conference on Network-Based Information Systems*. Springer, 212–223.

[38] Napendra Solanki, Shweta Jain, Suman Banerjee, and Yayathi Pavan Kumar S. 2023. Fairness Driven Efficient Algorithms for Sequenced Group Trip Planning Query Problem. In *Proceedings of the 22nd International Conference on Autonomous Agents and Multiagent Systems*. IFAAMAS, 86–94.

[39] Hugo Steinhaus. 1949. Sur la Division Pragmatique. *Econometrica: Journal of the Econometric Society* 17 (1949), 315–319.

[40] John W. Suurballe. 1974. Disjoint Paths in a Network. *Networks* 4, 2 (1974), 125–145.

[41] L. Valadares Tavares. 1990. A Multi-Stage Non-Deterministic Model for Project Scheduling under Resources Constraints. *European Journal of Operational Research* 49, 1 (1990), 92–101.

[42] Dajun Yue and Fengqi You. 2014. Fair Profit Allocation in Supply Chain Optimization with Transfer Price and Revenue Sharing: MINLP Model and Algorithm for Cellulosic Biofuel Supply Chains. *AIChE Journal* 60, 9 (2014), 3211–3229.