Diversity-seeking Swap Games in Networks

Extended Abstract

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ABSTRACT

Schelling games use a game-theoretic approach to study the phenomenon of residential segregation. We consider four *global* measures of diversity, and prove asymptotically tight or almost tight bounds on the price of anarchy with respect to these measures on both general graphs and common specific graphs. In addition we did simulations of our swap games.

KEYWORDS

Swap games; Schelling games; Diversity; Degree of Integration, Colorful Edges; Variety; Evenness; Tiling; Price of Anarchy

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1 INTRODUCTION AND NOTATION

Schelling [15, 16] showed that even small individual and local preferences for neighbours of the same type can lead to global segregation. In recent years, computer scientists have been studying related problems in the algorithmic game theory setting: strategic agents of different *types* are located at the vertices of a graph, and move to new positions to improve their own *utility*. Many variations of similarity seeking utility functions have been studied [1, 3–6, 11]. Two kinds of games have been considered: in a *jump* game, an agent can move to an unoccupied location to improve its utility, and in a *swap* game, two agents of different types swap locations if it would increase the utility of both of them.

Recently, diversity has been promoted as beneficial to society and institutions [7–9, 13]. This motivates the study of strategic games with agents that aim to increase the diversity in their neighbourhood [2, 10, 14]. Understanding what utility functions contribute to greater global diversity could help define appropriate incentives for businesses and governments. In this paper, we continue the study of *diversity-seeking* utility functions in strategic swap games.

The main measure of diversity considered previously is the *degree of integration*, i.e., the fraction of agents with at least one neighbour of a different type. In this paper, we introduce three measures of diversity: the number of *colorful edges*, the *neighborhood variety*, which is the average number of types of agents in an agent's neighborhood, and *evenness*, see the definition in 1.1

1.1. Notation, and preliminary results. In this paper $G = (V, E)$ denotes a connected graph where V, E is the set of vertices, edges, respectively. Let Δ and δ denote the maximum and minimum degree of G, respectively. Let $N(v)$ denote the set of neighbours of vertex $v \in V$. Let $\mathcal{T} = \{1, 2, \dots, t\}$ denote a set of *types*, and $\mathcal{X}$ denote a set of *agents* that is partitioned into t different types. For agent $A \in \mathcal{X}$, we use $\tau(A)$ to denote its type. Throughout the paper we assume $|V| = |\mathcal{X}| = n$. Let $k = \lfloor n/t \rfloor$. The partition of agents into types is called *equitable* if every type has either k or $k + 1$ agents. Clearly, an equitable partition always exists for every n and t.

An *assignment* of agents to vertices in G is a bijection $L : \mathcal{X} \to V$. We call $v = L(A)$ the *location* of agent A. Under an assignment L, we call agents A and B neighbors if $(L(A), L(B)) \in E(G)$, the edge $(L(A), L(B))$ is *monochromatic* if $\tau(A) = \tau(B)$, and *colorful* otherwise. We use $N(A) \subseteq \mathcal{X}$ to denote the set of neighboring agents of agent A, and $T(A) \subseteq \mathcal{T}$ to denote the set of types of agents in $N(A)$. An agent A is called *segregated* if $T(A) = \{\tau(A)\}$.

Given an assignment L for a graph G, a *utility function* U is a real-valued function that assigns to an agent its *utility* based on the agent's own type and the types of its neighboring agents. The *swap game under a utility function* U is defined as follows: Given an assignment L, a *move* in the game is a swap of locations of two agents A and B such that both A and B increase their utility. An assignment is in an *equilibrium* with respect to U if there is no move under U. We use $\mathrm{EQ}(G, U)$ to denote the set of all equilibrium assignments for the swap game on G under U.

A game on a graph G is called a *potential game* if and only if there is a non-negative real-valued function Φ on the set of assignments on G such that $\Phi(L') < \Phi(L)$ for any pair of assignments L' and L such that L' is obtained from L by a move in the game.

We define three utility functions for local diversity.

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- U_b, called *Binary*: $U_b(x) = 1$ if at least one neighbor of x is of a different type than x, and is 0 otherwise.
- $U_\#$, called *Difference-seeking* : $U_\#(x)$ is the number of neighbors of x whose type is different from $\tau(x)$.
- U_τ, called *Variety-seeking*: $U_\tau(x)$ is the number of types that are different from $\tau(x)$ in the neighborhood of x.

U_b is the simplest diversity-seeking utility function, while $U_\#$ and U_τ also consider the structure of types in the neighbourhood. Note that segregated agents have utility 0 in all three utility functions.

Observation 1. *(1) if $t = 2$, then $U_b = U_\tau$;*
(2) every equilibrium assignment under any of $U \in \{U_b, U_\#, U_\tau\}$ has at most one type whose agents are segregated;
(3) $\mathrm{EQ}(G, U_\#), \mathrm{EQ}(G, U_\tau) \subseteq \mathrm{EQ}(G, U_b)$.

Given an assignment L, every utility function U induces a *social welfare* $\mathrm{SW}(L, U) = \sum_{A \in \mathcal{X}} U(A)$. We also consider the following four *global* diversity measures.

- *Degree of integration*: $\mathrm{DOI}(L)$ is the fraction of agents A with at least one neighbor of a type different from A.
- *Colorful edges*: $\mathrm{CE}(L)$ is the number of colorful edges in L.
- *Neighborhood variety:* $\mathrm{NV}(L)$ is the average number of types in the neighborhood of agents, where we only count types in the neighborhood that are different from the agent's type.
- *Evenness*: $\mathrm{EV}(L) = \frac{1}{\sum_{A \in \mathcal{X}} \|\Pi_A\|_2^2}$.

Note that $\mathrm{DOI}(L) = \mathrm{SW}(L, U_b)/n$, $\mathrm{CE}(L) = \mathrm{SW}(L, U_\#)/2$, and $\mathrm{NV}(L) = \mathrm{SW}(L, U_\tau)/n$. The L^2 norm $\|\Pi_A\|_2^2$ is used to measure local evenness of neighbours distribution of agent A.

The diversity measures above have the following trivial bounds: (1) $\mathrm{DOI}(L) \leq 1$, $\mathrm{CE}(L) \leq |E(G)|$, $\mathrm{NV}(L) \leq \min\{t-1, \Delta\}$; (2) for δ-regular graphs, $\mathrm{EV}(L) \leq t/(n \cdot \delta^2)$, since by the Cauchy-Schwarz inequality $\|\Pi_A\|_2^2 = \sum_{i=1}^t \Pi_A(i)^2 \geq (\sum_{i=1}^t \Pi_A(i))^2/t = \delta^2/t$.

Consider a swap game on graph G under utility function U. Let μ denote a diversity measure, L^* be an assignment that *maximizes* μ, and L^e and L^m be *equilibrium* assignments that *minimize* and *maximize* μ respectively. The *price of anarchy (PoA)* is defined to be $PoA(\mu, U, G) = \mu(L^*)/\mu(L^e)$, and the *price of stability* is defined to be $PoS(\mu, U, G) = \mu(L^*)/\mu(L^m)$

2 OUR RESULTS

Full proofs of our results are in [12]. Observation 1 and the note after defining diversity measures imply the following.

Corollary 2.1. *(1) If $t = 2$, then $PoA(\mu, U_b, G) = PoA(\mu, U_\tau, G)$;*
(2) $PoA(\mu, U_\#, G), PoA(\mu, U_\tau, G) \leq PoA(\mu, U_b, G)$;
(3) $PoA(\mathrm{SW}, U_b, G) = PoA(\mathrm{DOI}, U_b, G)$,
$PoA(\mathrm{SW}, U_\#, G) = PoA(\mathrm{CE}, U_\#, G)$, and
$PoA(\mathrm{SW}, U_\tau, G) = PoA(\mathrm{NV}, U_\tau, G)$.

2.1. Existence of equilibrium. By determining for each utility function the conditions on when a swap can occur, and the number of colorful edges that can exist in an equilibrium assignment, we obtain the following theorem.

Theorem 2.2. *On every graph G, the swap games under $U_b, U_\tau, U_\#$ are all potential games that reach their respective equilibria after at most $|E|/2$ moves.*

2.2. Efficiency at equilibrium. The theorem below shows that the simple upper bounds on DOI, NV and EV given above are all tight for equitable agents.

Theorem 2.3. *(1) For equitable agents, for every $U \in \{U_b, U_\#, U_\tau\}$, for every graph G, $PoA(\mathrm{DOI}, U, G), PoA(\mathrm{SW}, U_b, G) \lesssim t/(t-1)$, $PoA(\mathrm{NV}, U, G), PoA(\mathrm{SW}, U_\tau, G) \leq t$, and for every δ-regular graph G, $PoA(\mathrm{EV}, U, G) \leq t$.*
(2) There exists graph G^* such that for equitable agents, and for every $U \in \{U_b, U_\#, U_\tau\}$,
$PoS(\mathrm{DOI}, U, G^*) = PoS(\mathrm{NV}, U, G^*) = PoS(\mathrm{EV}, U, G^*) = 1$.

Theorem 2.4. *For every graph G and equitable agents,*
(1) $PoA(\mathrm{CE}, U_b, G) \lesssim \Delta t/(t-1)$, and there exists graph G^* on which this is tight; also $PoS(\mathrm{CE}, U_b, G^*) = 1$.
(2) $PoA(\mathrm{CE}, U_\#, G) = PoA(\mathrm{SW}, U_\#, G) \lesssim 2\Delta/\delta$, furthermore: (i) $PoA(\mathrm{CE}, U_\#, G) \to \Delta/\delta$, for $t \geq \omega(\delta \log \delta)$, (ii) $PoA(\mathrm{CE}, U_\#, G) \lesssim 2/(\delta/\Delta + \Omega(t^2/\delta\Delta - 1/\Delta))$, for $\Omega(\sqrt{\delta}) \leq t \leq O(\delta)$.
(3) $PoA(\mathrm{CE}, U_\tau, G) \lesssim \Delta t/(t-1)$, furthermore: (i) $PoA(\mathrm{CE}, U_\tau, G) \to \Delta/\delta$, for $t \geq \omega(\delta^3)$, (ii) $PoA(\mathrm{CE}, U_\tau, G) \leq O(\Delta/t^{1/3})$, for $t \leq O(\delta^3)$.
(4) $PoS(\mathrm{CE}, U_\#, G) = 1$ when $t \geq \Delta + 1$.

2.2. Cycles, cylinders, tori. We studied the PoA of SW and CE for cycles, cylinders, and grids and obtained tight or better bounds than those implied for general graphs. Table 1 summarizes our results.

Table 1: Except the PoA of CE for $U_\#$ on cylinders and tori, all other bounds are tight. A bold font represents a better bound than that for general graph,the non-bold matches the corresponding bounds. The PoA of CE for $U_\#$ on T_n lies in-between $t/(t-1)$ and $\min\{t/(t-13/4), 4t/(3(t-1))\}$.

t			Cycles	Cylinders	Tori
2	SW	U_b, U_τ	4/3	3/2	8/5
		$U_\#$	**3/2**	**3/2**	**5/3**
	CE	U_b, U_τ	2	3	4
		$U_\#$	**3/2**	**3/2**	**5/3**
≥ 3	SW	U_b	$\frac{t}{t-1}$	$\frac{t}{t-1}$	$\frac{t}{t-1}$
		$U_\#$	$\frac{t}{t-1}$	$\left[\frac{t}{t-1}, \frac{t}{t-7/3}\right]$	**see caption**
		U_τ	$\frac{2t}{t-1}$	$\min\{t, \frac{3t}{t-1}\}$	$\min\{t, \frac{4t}{t-1}\}$
	CE	U_b	$\frac{2t}{t-1}$	$\frac{3t}{t-1}$	$\frac{4t}{t-1}$
		$U_\#$	$\frac{t}{t-1}$	$\left[\frac{t}{t-1}, \frac{t}{t-7/3}\right]$	**see caption**
		U_τ	$\frac{2t}{2t-3}$	**27/10** ($t = 3$)	**24/7** ($t = 3$)

2.3. Experiments. For every $U \in \{U_b, U_\#, U_\tau\}$, and types $t = 2, \ldots, 9$, we ran the swap game on a 4-regular torus with two different initializations: the *random input* and the *Schelling input*, obtained by running a swap Schelling game using the similarity-seeking utility function of [1] on the random input. See [12] for detailed experimental results. The main findings are:
(1) Segregation is effectively removed when agents are diversity-seeking; however, strong diversity such as measured by evenness is still hard to achieve via $U_b, U_\#$ or U_τ.
(2) Regardless of the starting input, the swap game under $U_\#$ performs better than the game under U_τ for DOI and CE, while U_τ is better for NV and EV.

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