On the Existence of EFX Allocations in Multigraphs

Extended Abstract

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ABSTRACT

We study the problem of "fairly" dividing indivisible goods to several agents that have valuation set functions over the sets of goods. As fair we consider the allocations that are envy-free up to any good (EFX), i.e., no agent envies any proper subset of the goods given to any other agent. The existence or not of EFX allocations is a major open problem in Fair Division, and there are only positive results for special cases.

Christodoulou et al. [19] introduced a restriction on the agents' valuations according to a graph structure: the vertices correspond to agents and the edges to goods, and each vertex/agent has zero marginal value (or in other words, they are indifferent) for the edges/goods that are not adjacent to them. The existence of EFX allocations has been shown for simple graphs with general monotone valuations [19], and for multigraphs for restricted additive valuations [28].

In this work, we push the state-of-the-art further, and show that the EFX allocations always exists in *multigraphs* and *general monotone valuations* if any of the following two conditions hold: either (a) each agent has at most $\lceil \frac{n}{4} \rceil - 1$ neighbors, where n is the total number of agents, or (b) the shortest cycle with non-parallel edges has length at least 6.

KEYWORDS

Fair Division; EFX Allocations; Algorithm Design

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1 INTRODUCTION

We study a problem of "fairly" dividing indivisible goods to many agents. The question of how to divide resources to several agents in a fair way dates back to the ancient times, e.g., dividing land, and it raised important research questions since the late 40's [34]. One prominent notion in fair division is *envy-free allocations*, where nobody envies what is allocated to any other agent, which was formally introduced a bit later [22, 23, 36]. Initially, the problem was studied under the scope of divisible resources, where envy-free allocations are known to always exist [7, 35, 37].

The focus of this work is on indivisible goods, with multiple applications, such as dividing inheritance, and assigning courses to students [13]. The non-profit website Spliddit (http://www.spliddit.org/) provides mechanisms for several such applications. It is easy to see that envy-free allocations are not guaranteed to exist; for instance consider two agents and one indivisible good, then whoever gets the good is envied by the other agent.

This example demonstrates how strong the requirement of completely envy-freeness is for the scenario of indivisible goods.

This has led to the study of two basic relaxations of envy-freeness, namely envy-freeness up to one good (EF1) [12] and envy-freeness up to any good (EFX) [15]. EF1 is a weaker notion than EFX, and it is guaranteed to always exist and can be found in polynomial time [29]. On the other hand, it is not known if EFX allocations are guaranteed to exist in general, and it has been characterized as "Fair Division's Most Enigmatic Question" [33]. EFX allocations are known to exist for special cases: e.g., for 2 agents with general monotone valuations [32], for 3 agents with additive valuations or a slightly more general class [2, 16], and for many agents with identical monotone valuations [32], or with additive valuations where each agent is restricted to have one of the two fixed values for each good [4].

Surprisingly, it was recently shown that EFX allocations need not exists in the case with chores, i.e., negatively valued items [20]. This is the first result of non-existence of EFX for monotone valuation functions, and the construction requires only 3 agents and 6 goods. This is an interesting separation between goods and chores, as for the case of goods it is known that EFX allocations are guaranteed to exist when the number of goods are at most 3 more than the number of agents [30].

Unfortunately, little is known for the case with multiple agents and multiple goods; additionally to the works that have been already mentioned [4, 32], EFX allocations are known to exist when agents' preference follow a lexicographic order defined by their preference over singletons [25], and when the valuations have dichotomous marginals, i.e., the marginal value when a good is added to a set is either 0 or 1 [8]. All those works consider high restrictions and resemblance on the agents valuations. Towards broadening our understanding for the case of multiple agents and goods, Christodoulou et al. [19] introduced a setting that is related to our work, where the valuations are defined based on a graph: given a graph, the agents correspond to the vertices of the graph, and the goods to the edges. Then, each agent is indifferent for the

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goods/edges that are not adjacent to them. In [19], they showed that EFX allocations always exist on graphs.

In this work we consider a multigraph, which can be interpreted as follows: *each good is of interest for at most two agents.* The motivation in the multigraph setting is, similarly to [19], the division of territories between nations, areas of interest between neighboring countries and more generally division of geographic settings. Another application is to allocate available space for research teams and collaborators, which is always a challenging task, and becomes even more difficult when there are multiple conflicts for available areas. It was recently showed that EFX allocation always exists in multigraphs when all agents have restricted additive valuations [28], i.e., each good g has a fixed value v_g, and each agent may value that good by v_g or not value it at all, in which case he has value 0. We generalize this result with respect to the valuation functions, where we use general monotone valuations, on the expense of having restrictions on the multigraph, where each agent has at most $\lceil \frac{n}{4} \rceil - 1$ neighbors, where n is the total number of agents, or the shortest cycle with no parallel edges has length at least 6. Two other papers independently and in parallel showed EFX allocations involving multigraphs [1, 11].

Our Contribution. Our results are summarized in the following two theorems which hold for general monotone valuations:

Theorem 1.1. *In multigraphs of n vertices, with at most $\lceil \frac{n}{4} \rceil - 1$ neighbors per agent, an EFX allocation always exists.*

Theorem 1.2. *In multigraphs, where the shortest cycle with non-parallel edges has length at least 6, an EFX allocation always exists.*

Our Techniques. Here we discuss our main techniques in order to construct EFX allocations in multigraphs. The construction of EFX allocations follows the same skeleton for both our results: we produce an initial partial allocation to satisfy certain properties, then we reduce envy by making local reassignments of the allocation, and finally we allocate the rest of the unallocated edges by preserving EFX (relying on the initial properties).

We make use of the cut-and-choose-based protocol of [32] for two agents: one agent cuts the set of goods into two bundles where he is EFX-satisfied with each of them (i.e., no matter which of the two bundles he receives, he does not envy the other bundle up to any good), and the other agent chooses his favorite bundle among those two. This simple protocol results in an EFX allocation for two agents, even if they have general monotone valuations.

Remark 1. We remark that according to the original definition of EFX in [15], where each agent i is not envious against any other agent j after the hypothetical removal of a *positive valued* good for i from the j's bundle, the cut-and-choose protocol provides a simple EFX orientation (i.e., each edge is given to one of its endpoints) for multigraphs; we remark though that finding such an allocation may be computationally hard [24, 32]. On the other hand, following the traditional definition of EFX, the good that is hypothetically removed from j's bundle may be indifferent for i, and the local allocation of the cut-and-choose protocol is insufficient.

Following the above remark, we make use of the cut-and-choose protocol in order to partition the set E_{ij} into two bundles, but we may consider two different partitions depending on which endpoint "cuts". Note that there may be two different EFX allocations derived by the cut-and-choose protocol, depending on who "cuts", and moreover, only the agent who "cuts" might be envious of the other agent. Therefore, by controlling which endpoint "cuts", we in fact control the direction of the envy, and manage to generalize the ideas of [19]. However, we also put the cut-and-choose protocol in use in a different way: if two agents do not agree on having the same cut, we use the EFX-cut of one of them in order to create a partition of *three* bundles where the two agents have different most valued set. This tool was proven to be very useful for constructing the EFX allocation for Theorem 1.1, where we want to minimize the number of envied agents. In both approaches, one crucial condition that we always upkeep is that *we never allocate more than one bundle of the partition of E_{ij} to the same vertex.*

Our approach can be seen as a three-step procedure: i) We define an initial allocation where each agent receives exactly one bundle (derived from carefully constructed partitions) from the common edges with exactly one of his neighbors. In this step we guarantee some ground properties on the allocation. ii) We perform an algorithm, that can be seen as a generalization of Algorithm 2 of [19], to satisfy extra properties by preserving an EFX orientation, while ensuring that any non-envied agent has received exactly one adjacent bundle associated with *each* of their neighbors. At this step we have finalized any orientation of the edges, whose allocation will not change in the next step. iii) We appropriately allocate all the unallocated edges to non-envied vertices that are *not endpoints* of the edges, while preserving the EFX guarantee.

We refer the reader to our full version paper for more details.

2 FURTHER RELATED WORK

We focus on references related to EFX, and we defer the reader to a recent survey [3] that discusses other notions of fairness, as well.

The existence of EFX allocations in simple graphs, has been studied for goods [19], and for mixed manna settings [39]. In [19] they further showed that EFX orientations need not exist, and even deciding if there exists an EFX orientation is NP-complete. Following that, several works studied the existence of EFX orientations and its hardness [21, 26, 38].

Approximate EFX, α-EFX, has also been studied: $\frac{1}{2}$-EFX allocations are known to exist for subadditive valuations [32], $(\phi - 1) - EFX$ allocations for additive valuations [6], and $\frac{2}{3}$-EFX allocations for additive valuations under several restrictions [5, 31]. Regarding graph settings, in [5] they showed $\frac{2}{3}$-EFX for additive valuations in multigraphs, and in [28] they showed $\frac{\sqrt{2}}{2}$-EFX for subadditive valuations in the case of hypergraphs under the restriction that any two agents share at most one edge. Another relaxation of EFX considers partial EFX allocations, also known as EFX with charity [14]. There exists a line of works towards reducing the number of unallocated goods [10, 18, 28], which in some cases is done on the expense of satisfying the exact EFX condition [2, 9, 17, 27].

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