Pure Nash Equilibrium and Strong Nash Equilibrium Computation in Additive Aggregate Games

Extended Abstract

Jared Soundy
The Beacom College of Computer & Cyber Sciences
Dakota State University
Madison, South Dakota, United States
jared.t.soundy@dsu.edu

Mohammad T. Irfan
Department of Computer Science
Bowdoin College
Brunswick, Maine, United States
mirfan@bowdoin.edu

Hau Chan
School of Computing
University of Nebraska-Lincoln
Lincoln, Nebraska, United States
hchan3@unl.edu

ABSTRACT

Aggregate games, first conceptualized by Nobel laureate Reinhard Selten in 1970, model the decision-making of interdependent agents where each agent's utility depends on their own action and the aggregation of everyone's actions. We consider computational questions on pure Nash equilibrium (PNE) and pure strong Nash equilibrium (SNE) for aggregate games. On the way, we define a new subclass of aggregate games we call additive aggregate games, which encompasses popular games like congestion games, anonymous games, Schelling games, etc. We show that PNE existence is NP-complete for very simple cases of additive aggregate games. We devise an efficient aggregate-space algorithm for determining the existence of a PNE and computing one (if exists) for bounded aggregate space. For SNE, we show that SNE recognition is co-NP-complete and SNE existence is Σ_2^P-complete, even for simple types of additive aggregate games. For large classes of aggregate games, we provide several novel and efficient aggregate-space algorithms for recognizing an SNE and deciding the existence of an SNE. Finally, we connect our results to several well-studied subclasses of aggregate games and show how our computational schemes can shed new light into these games.

KEYWORDS

Aggregate Games, Pure Nash Equilibrium, Strong Nash Equilibrium, Computational Complexity, Algorithms

ACM Reference Format:

Jared Soundy, Mohammad T. Irfan, and Hau Chan. 2025. Pure Nash Equilibrium and Strong Nash Equilibrium Computation in Additive Aggregate Games: Extended Abstract. In *Proc. of the 24th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2025), Detroit, Michigan, USA, May 19 – 23, 2025*, IFAAMAS, 3 pages.

1 AGGREGATE GAMES

We start by defining aggregate games. Let $N = \{1, \cdots, n\}$ be a set of n agents in a game. Each agent $i \in N$ has a set A_i of actions and selects an action $a_i \in A_i$. Let $m = \max_{i \in N} |A_i|$ be the maximum number of actions of any agent. Let $\boldsymbol{A} = A_1 \times A_2 \times \ldots \times A_n$ be the set of action profiles of all agents where an action profile $\boldsymbol{a} = (a_1, a_2, \cdots, a_n) \in \boldsymbol{A}$ consists of an action for each agent. Given an action profile $\boldsymbol{a} = (a_i, \boldsymbol{a}_{-i}) \in \boldsymbol{A}$, we use $\boldsymbol{a}_{-i}$ to refer to the actions of all agents except agent i. Given a subset of agents $I \subseteq N$, we use $\boldsymbol{a}_I$ to refer to the actions of each agent in I and $\boldsymbol{a}_{-I}$ to refer to the actions of agents not in I.

In an aggregate game, an agent's utility function depends on the agent's action and the aggregation of everyone's actions, including the actions different from the agent's own action [1, 3, 4, 10–13]. To capture this aggregation, we define an aggregator function $\phi : \boldsymbol{A} \longrightarrow Y$ that maps each action profile to an aggregate measure in the space $Y \subset \mathbb{Z}^d_{\geq 0}$. We denote agent i's utility function as $\pi_i : A_i \times Y \longrightarrow \mathbb{R}$, which maps i's action and an aggregate to a real number. We assume that the utility and aggregator functions can be evaluated efficiently.

Definition 1 (Aggregate Game). *The tuple $(N, \{A_i, \pi_i\}_{i \in N}, \phi)$ defines an aggregate game. It consists of a set N of agents, a set A_i of actions for each agent i, and a utility function $\pi_i(a_i, \phi(\boldsymbol{a}))$ for each agent $i \in N$ and $\boldsymbol{a} \in \boldsymbol{A}$, where π_i is a function of i's actions in A_i and the aggregator function ϕ's outputs in Y.*

We next define an additive aggregate game.

Definition 2 (Additive Aggregate Game). *An additive aggregate game $(N, \{A_i, \pi_i\}_{i \in N}, \phi)$ is an aggregate game where for any $\boldsymbol{a} = (a_1, \cdots, a_n) \in \boldsymbol{A}$, the aggregator function is additively separable: $\phi(\boldsymbol{a}) = \phi_1(a_1) + \cdots + \phi_n(a_n)$ for some function $\phi_i : A_i \longrightarrow Y$ for each $i \in N$.*

Given an instance of an additive aggregate game, we are interested in pure Nash equilibrium (PNE) and pure strong Nash equilibrium (SNE) concepts [7], where agents act deterministically.

2 PNE COMPUTATION

We first establish the hardness of PNE existence in very simple instances of additive aggregate games.

Theorem 1. *It is NP-complete to decide PNE existence in additive 2-action aggregate games, even when the dimension of the aggregate space is constant and the utility function of each agent returns two integer values.*

We next utilize the structure and parameters of additive aggregate games to design efficient general-purpose algorithms for determining the existence of a PNE. Our algorithmic approach systematically explores the aggregate space and determines whether

Proc. of the 24th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2025), Y. Vorobeychik, S. Das, A. Nowé (eds.), May 19 – 23, 2025, Detroit, Michigan, USA.

an aggregate is consistent with a PNE (i.e., for a given $y \in Y$, is there a PNE $\boldsymbol{a}^*$ such that $\phi(\boldsymbol{a}^*) = y$?) by solving three subproblems stated below. The fundamental algorithmic approach was derived from a CSP formulation in the context of congestion games in [9]. Here, we extend it to general additive aggregate games and also consider SNE, which did not get any attention in [9].

Problem 1 (Deviation). *Given an additive aggregate game instance and $y \in Y$, for each agent $i \in N$ and any $a_i, a_i' \in A_i$, compute* $\texttt{deviate}(a_i, y, a_i') = \hat{y}$ *which returns an aggregate* $\hat{y} = (y - \phi_i(a_i)) + \phi_i(a_i')$ ($\in Y$ *or* $\notin Y$) *when agent i deviates from a_i to a_i' under y.*

Problem 2 (Aggregate Best Response). *Given an additive aggregate game instance and $y \in Y$, for each agent $i \in N$, find the set of best-response actions given aggregate y. That is,*

$$BR_i(y) = \{a_i \in A_i \mid \pi_i(a_i, y) \geq \pi_i(a_i', \hat{y}) \text{ for each } a_i' \in A_i \text{ such that } \texttt{deviate}(a_i, y, a_i') = \hat{y} \in Y\}.$$

Problem 3 (Construction). *Given an additive aggregate game instance and a subset $\tilde{A}_i \subseteq A_i$ of actions for each agent $i \in N$, determine if there exists $\tilde{\boldsymbol{a}} = (\tilde{a}_1, \ldots, \tilde{a}_n) \in \prod_{i \in N} \tilde{A}_i$ such that $\phi(\tilde{\boldsymbol{a}}) = y$.*

We show that Problem 3 is strongly NP-complete. Despite this, we present an efficient graph-based dynamic programming algorithm that is polynomial in the size of the aggregate space Y.

Theorem 2. *There is an $O(|Y|(nm^2 + |Y|nm))$ algorithm for determining the existence of a PNE and returning a PNE (if it exists) for additive aggregate games.*

3 SNE COMPUTATION

We investigate two problems on pure strong Nash equilibrium (SNE): recognizing whether an action profile is an SNE and computing an SNE (if it exists).

3.1 Recognizing an SNE

We show that recognizing whether a given action profile is an SNE is co-NP-complete for special types of additive aggregate games. To put this result in context, this problem is polynomial-time solvable for anonymous games, a subclass of additive aggregate games [8].

Theorem 3. *It is co-NP-complete to recognize whether a given action profile is an SNE for an additive aggregate game with a constant number of actions for each player.*

We present another aggregate-space algorithm to determine whether a given action profile is an SNE. The algorithm is efficient when the size of the aggregate space is bounded.

Theorem 4. *There is an $O(|Y|^2 n^2 m)$ algorithm for determining whether a given action profile is an SNE in additive aggregate games.*

3.2 Computing an SNE

Given that the SNE recognition problem is already hard, determining the existence of an SNE is likely to be hard. As we show below, the SNE existence problem is indeed Σ_2^P-complete.

Theorem 5. *It is Σ_2^P-complete to determine the existence of an SNE in additive aggregate games, even when the agents have a constant number of actions.*

Given the above Σ_2^P-completeness, we consider additional properties and develop efficient algorithms for bounded aggregate space.

Symmetric Additive Aggregate Games. Given an additive aggregate game $(N, \{A_i, \pi_i\}_{i \in N}, \phi)$, it is *symmetric* if and only if for each agent $i \in N$, $A_i = A$, and $\pi_i = \pi$, and $\phi_i = \phi_0$. Let the number of dimensions of $Y \subseteq \mathbb{Z}^d_{>0}$ be d. We define the *support* $(\phi_0(a)) = \{k \in \{1, \ldots, d\} \mid \phi_0(a)_k > 0\}$ to be the set of dimensions that action $a \in A$ contributes to. For $d \geq |A|$ and $support(\phi_0(a)) \cap support(\phi_0(a')) = \emptyset$ for all distinct $a, a' \in A$, we present an $O(|Y|^3 n^2 m)$ algorithm for SNE computation (if it exists).

Non-Increasing Additive Aggregate Games. Next, we consider additive aggregate games with non-increasing utility functions, where an agent's utility is only affected by the elements of aggregate that they affect. Given an additive aggregate game $(N, \{A_i, \pi_i\}_{i \in N}, \phi)$, let the dimension of $Y \subseteq \mathbb{Z}^d_{>0}$ be $d = |A|$ and $A = A_i = \{1, 2, \ldots, d\}$ for all $i \in N$. This game is non-increasing if and only if for any i and any a_i, $\pi_i(a_i, y') \leq \pi_i(a_i, y)$ whenever $y \leq y' \in Y$ (i.e., $y_j \leq y_j'$ for $j = 1, \ldots, d$). We give an $O(|Y|(nm^2 + |Y|nm + |Y|n))$ constructive algorithm for SNE existence when $support(\phi_i(a_i)) = \{a_i\}$ for all a_i and i, and $\pi_i(a_i, y) = \pi_i(a_i, y')$ for any $y, y' \in Y$ in which $y_{a_i} = y'_{a_i}$.

4 CONNECTION TO OTHER GAMES

We establish connections between additive aggregate games and various popular classes of games, namely, congestion games, anonymous games, Schelling games, and Cournot games. First, we show congestion games [14, 15] to be a type of additive aggregate games and present the following results for n agents and m actions.

Theorem 6. *There is an $O(n^{|R|}(nm^2 + n^{|R|}nm))$ algorithm for computing a PNE for congestion games. The algorithm runs in polynomial time for bounded number of resources.*

Theorem 7. *There is an $O(n^{2|R|}n^2 m)$ algorithm for recognizing whether a given action profile is an SNE in congestion games. The algorithm runs in polynomial time for bounded number of resources.*

We show that anonymous games [5, 8] are a subclass of additive aggregate games and further explore this connection. In particular, we have shown SNE recognition and computation problems are co-NP-complete and Σ_2^P-complete for additive aggregate games, respectively. In contrast, these problems are in P and NP-complete, respectively, for anonymous games [8].

We further connect Schelling games [2, 6] to additive aggregate games as a subclass. For n agents, $k \geq 2$ agent types, and m location choices, we have the following results. Notably, the algorithms run in polynomial time for bounded number of locations and types.

Theorem 8. *There is an $O(n^{mk}(nm^2 + n^{mk+1}m))$ constructive algorithm for PNE existence in Schelling games.*

Theorem 9. *There is an $O(n^{2mk+2}m)$ algorithm for recognizing whether a given action profile is an SNE in Schelling games.*

Above is the *first SNE result on SGs* to our knowledge. This highlights the broad applicability of our technical results.

ACKNOWLEDGMENTS

MTI is grateful for support from NSF Award IIS-1910203. HC thanks NIH for Award P20GM130461 and NSF for Award IIS-2302999.

REFERENCES

[1] Daron Acemoglu and Martin Kaae Jensen. 2013. Aggregate comparative statics. *Games and Economic Behavior* 81 (2013), 27–49.

[2] Hau Chan, Mohammad T. Irfan, and Cuong Viet Than. 2020. Schelling Models with Localized Social Influence: A Game-Theoretic Framework. In *Proceedings of the 19th International Conference on Autonomous Agents and MultiAgent Systems*. IFAAMAS, Online, 240–248.

[3] Luis C Corchón. 1994. Comparative statics for aggregative games the strong concavity case. *Mathematical Social Sciences* 28, 3 (1994), 151–165.

[4] Richard Cornes and Roger Hartley. 2012. Fully aggregative games. *Economics Letters* 116, 3 (2012), 631–633.

[5] Constantinos Daskalakis and Christos H Papadimitriou. 2015. Approximate Nash equilibria in anonymous games. *Journal of Economic Theory* 156 (2015), 207–245.

[6] Edith Elkind, Jiarui Gan, Ayumi Igarashi, Warut Suksompong, and Alexandros A. Voudouris. 2019. Schelling Games on Graphs. In *Proceedings of the Twenty-Eighth International Joint Conference on Artificial Intelligence, IJCAI-19*. International Joint Conferences on Artificial Intelligence Organization, Macao, China, 266–272. https://doi.org/10.24963/ijcai.2019/38

[7] Drew Fudenberg. 1991. *Game theory.* MIT press, Cambridge, MA.

[8] Martin Hoefer and Alexander Skopalik. 2013. On the complexity of Pareto-optimal Nash and strong equilibria. *Theory of computing systems* 53, 3 (2013), 441–453.

[9] Mohammad T. Irfan, Hau Chan, and Jared Soundy. 2024. Computing Nash Equilibria in Multidimensional Congestion Games. In *Proceedings of the 23rd International Conference on Autonomous Agents and Multiagent Systems*. IFAAMAS, Auckland, New Zealand, 2309–2311.

[10] Martin Kaae Jensen. 2010. Aggregative games and best-reply potentials. *Economic theory* 43, 1 (2010), 45–66.

[11] Jayash Koshal, Angelia Nedić, and Uday V Shanbhag. 2016. Distributed algorithms for aggregative games on graphs. *Operations Research* 64, 3 (2016), 680–704.

[12] David Martimort and Lars Stole. 2011. *Aggregate representations of aggregate games*. Working paper. Universität München.

[13] David Martimort and Lars Stole. 2012. Representing equilibrium aggregates in aggregate games with applications to common agency. *Games and Economic Behavior* 76, 2 (2012), 753–772.

[14] Dov Monderer and Lloyd S Shapley. 1996. Potential games. *Games and economic behavior* 14, 1 (1996), 124–143.

[15] Robert W. Rosenthal. 1973. A class of games possessing pure-strategy Nash equilibria. *International Journal of Game Theory* 2, 1 (Dec. 1973), 65–67. https://doi.org/10.1007/BF01737559