Balancing Fairness and Efficiency in the Allocation of Indivisible Goods

Doctoral Consortium

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ABSTRACT

The allocation of indivisible goods is a ubiquitous problem, and two main objectives are fairness and efficiency. Depending on the fairness and efficiency benchmarks used, these objectives can be aligned or conflicting with each other. We study the degree of compatibility between certain fairness and efficiency notions in two different ways. First, we investigate the egalitarian prices of several fairness properties and compare them to their utilitarian prices. Secondly, we characterise all additive welfarist rules guaranteeing envy-freeness up to one good (EF1) for different classes of instances.

KEYWORDS

Fair division; Indivisible goods; Social welfare

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1 INTRODUCTION

Fair allocation of limited resources is a fundamental problem for people living together in a community. We face it every day, for instance, when sharing food at a dinner table. On a larger scale, there are other examples such as the problems of dividing an inheritance and allocating a budget across multiple departments. Due to the ubiquity and complexity of such problems, they have been studied by researchers in the area of *fair division* [4, 12, 13].

There is a trivial solution to the problem of allocating resources fairly: by giving all agents nothing. This solution is fair in the sense that all agents get the same amount of resources, or rather, lack thereof. However, this is undesirable since all agents are unhappy and there are resources which are unused. Hence, in the problem of fair division, an important goal besides fairness is efficiency.

Efficiency is usually measured by a welfare function. The simplest and most common welfare function is utilitarian welfare, which is the sum of utilities across all agents. However, a welfare function could technically be any non-decreasing n-ary function which is defined on any n-tuple of non-negative reals. Hence, there are infinitely many welfare functions, some of which are well-studied, but most are not. The problem is then to choose which welfare function to use as a measure of efficiency.

Depending on which welfare function we use, there might be a conflict between the two main goals in resource allocation—fairness and efficiency. Consider an instance with two agents and two goods, where agent 1 values each good at 100 and agent 2 values each good at 1. The allocation that maximises utilitarian welfare would allocate both goods to agent 1, since she values them more than agent 2. However, this is not so fair to agent 2. On the other hand, a different welfare function—Nash welfare—is known to be fairer, as maximising Nash welfare implies envy-freeness up to one good (EF1) [6].

We analyse the the degree of compatibility between certain fairness and efficiency notions in two different ways. The first method is by looking at the *price of fairness*, which quantifies the loss of efficiency due to fairness constraints. The price of fairness can be defined with respect to different efficiency measures. We study the price of fairness with respect to egalitarian welfare [7] and compare our findings to existing results on the price of fairness with respect to utilitarian welfare. The second approach is by examining the contexts in which certain efficiency and fairness criteria are perfectly compatible. Here we focus only on allocations that maximise a welfare function. We call a rule that chooses such an allocation *a welfarist rule*. We consider in particular *additive welfarist rules* defined by additive welfare functions—n-ary functions that can be expressed as the sum of some function of an element of the input tuple. Our contribution is to characterise all the additive welfarist rules that guarantee EF1 for certain subclasses of instances [8].

2 THE EGALITARIAN PRICE OF FAIRNESS

As mentioned earlier, for some welfare functions, there is a conflict between maximising welfare and achieving fairness. The price of fairness was introduced as a way to measure this conflict [3, 5], by comparing the maximum welfare overall with the maximum welfare under the constraint of some fairness property. Formally, we define the price of fairness POF_P of a fairness property P as

$$\mathrm{POF}_P \coloneqq \sup_{\mathcal{I}} \frac{\max_{\mathcal{A}} \mathrm{SW}(\mathcal{A})}{\max_{\mathcal{A} \in P(\mathcal{I})} \mathrm{SW}(\mathcal{A})},$$

where SW is the welfare function, $\mathcal{I}$ is an instance, $\mathcal{A}$ is an allocation of the given instance and $P(\mathcal{I})$ is the set of all allocations of $\mathcal{I}$ satisfying property P. The price of fairness is most often defined using utilitarian welfare as the welfare function. The utilitarian price of fairness has been studied in various settings, including both divisible and indivisible goods and chores [1–3, 5].

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Another well-studied welfare function is egalitarian welfare, which is the minimum utility across all agents. Intuitively, egalitarian welfare places the highest importance on the poorest agent; this makes egalitarian welfare arguably fairer than utilitarian welfare. A natural direction is therefore to compare the two welfare functions by comparing the utilitarian and the egalitarian prices of fairness.

In our paper [7], we examine the egalitarian prices of fairness in the context of indivisible goods. In particular, we provide bounds on the egalitarian prices of three fairness properties: envy-freeness up to one good (EF1), balancedness and round-robin. Moreover, we also study the egalitarian prices of two efficiency notions: maximum Nash welfare (MNW) and maximum utilitarian welfare (MUW). Even though these are technically not prices of fairness, they still demonstrate the trade-off between the corresponding efficiency notion and egalitarian welfare.

We compare the egalitarian prices from our paper with the corresponding utilitarian prices from the papers of Barman et al. [1] and Bei et al. [2], and present this comparison in Table 1. According to our intuition that egalitarian welfare is seemingly fairer than utilitarian welfare, the egalitarian prices of fairness should be *lower* than their utilitarian counterparts. However, the results we found are contrary to our prediction.

Table 1: Comparison between egalitarian and utilitarian prices of fairness for indivisible goods

Property		Price of fairness	
		Egalitarian [7]	Utilitarian [1, 2]
EF1		$\Theta(n)$	$\Theta(\sqrt{n})$
Balanced		n	$\Theta(\sqrt{n})$
Round-robin		$\Theta(n)$	n
MNW	$(n = 2)$	≈ 2	≈ 1.2
	$(n \geq 3)$	∞	$\Theta(n)$
MUW		∞	1
MEW		1	$\Theta(n)$

For all of the fairness properties studied in our paper, the egalitarian prices are asymptotically as high as the utilitarian prices, or higher. This shows that, against our intuition, those fairness properties are less compatible with egalitarian welfare than with utilitarian welfare. Comparing the egalitarian and the utilitarian prices of MNW also reveals that egalitarian welfare is indeed less aligned with Nash welfare than utilitarian welfare is. For completeness, we also show that the egalitarian price of MUW is greater than the utilitarian price of MEW.

3 THE FAIRNESS OF ADDITIVE WELFARIST RULES

In the previous section, we established the price of fairness which measures the trade-off between fairness and efficiency. On the other hand, for some welfare functions and fairness notions, there is no such trade-off as maximising the welfare leads to fairness. Indeed, if we consider all instances, then MNW is the only welfarist rule guaranteeing EF1 [15]. However, if we restrict to a smaller subclass of instances, there might be other welfarist rules that always choose EF1 allocations. For example, maximum harmonic welfare (MHW) implies EF1 for all integer-valued instances [11]. Furthermore, for any normalised instance with 2 agents, allocations that maximise p-mean with $p \leq 0$ must be EF1 [9]. Having said that, not much else was known about other welfarist rules and whether they ensure EF1 for certain subclasses of instances.

Our contribution is to characterise additive welfarist rules that yield EF1 for certain subclasses of instances [8]. Our paper focuses on the setting of indivisible goods, with EF1 as the fairness measure. Unlike envy-freeness, EF1 is always satisfiable for any instance of indivisible goods. As such, EF1 is the most commonly used relaxation of envy-freeness.

In the paper, we consider mainly two classes of instances: real- and integer-valued instances. Integer-valued instances is an important subclass in terms of real-world applications. For example, Spliddit [10], a popular fair division website, only allows users to specify integer values for goods. For each of the two subclasses, we further consider the subclass where each agent values all goods identically, the subclass with binary valuations and the subclass of two-value instances.

For real-valued instances, we strengthen the results of Suksompong [14] and show that the only additive welfarist rules that guarantee EF1 are equivalent to MNW, even if we restrict the class of instances to only those with identical goods or to only two-value instances. Since any concave function ensures EF1 for any normalised instances with identical goods, this suggests that MNW's unique fairness stems from its scale-invariance.

For the class of integer-valued instances, we show that there exist other additive welfarist rules besides MNW and MHW that give EF1 allocations. Some examples include additive welfarist rules defined by the function $\log(x + c)$ with $c \in [0, 1]$. If we further restrict the class of instances, then there are even more additive welfarist rules that ensure EF1. The relationships between the different classes of instances are represented by the following series of subset relations:

$$SW_{\text{int, any}} \subseteq SW_{\text{int, two-value}} \subseteq SW_{\text{int, identical goods}} \subseteq SW_{\text{int, binary}},$$

where $SW_{\text{int, }P}$ denotes the set of additive welfarist rules that ensure EF1 for the classes of integer-valued instances satisfying property P. In particular, the additive welfarist rules that yield EF1 for all binary instances are precisely those whose defining functions are concave in the non-negative integer domain.

4 CONCLUSION AND FUTURE WORK

In general, it can be said that fairness and efficiency are two orthogonal aims in fair division. However, depending on the specific definitions of fairness and efficiency used, the two notions may interact in completely opposite ways: some fairness and efficiency criteria are perfectly aligned, while others are incompatible. We analyse the interaction between fairness and efficiency through two different methods: by comparing the prices of fairness and by characterising additive welfarist rules which are fair.

For future work, I plan to investigate more problems related to welfare functions as well as other problems in fair division involving similar concepts such as the price of fairness.

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REFERENCES

[1] Siddharth Barman, Umang Bhaskar, and Nisarg Shah. 2020. Optimal Bounds on the Price of Fairness for Indivisible Goods. In *Proceedings of the 16th International Conference on Web and Internet Economics (WINE)*. 356–369.

[2] Xiaohui Bei, Xinhang Lu, Pasin Manurangsi, and Warut Suksompong. 2019. The Price of Fairness for Indivisible Goods. In *Proceedings of the 28th International Joint Conference on Artificial Intelligence (IJCAI)*. 81–87.

[3] Dimitris Bertsimas, Vivek Farias, and Nikolaos Trichakis. 2011. The Price of Fairness. *Operations Research* 59, 1 (2011), 17–31.

[4] Steven J. Brams and Alan D. Taylor. 1996. *Fair Division: From Cake-Cutting to Dispute Resolution.* Cambridge University Press.

[5] Ioannis Caragiannis, Christos Kaklamanis, Panagiotis Kanellopoulos, and Maria Kyropoulou. 2012. The Efficiency of Fair Division. *Theory of Computing Systems* 50, 4 (2012), 589–610.

[6] Ioannis Caragiannis, David Kurokawa, Hervé Moulin, Ariel D. Procaccia, Nisarg Shah, and Junxing Wang. 2019. The unreasonable fairness of maximum Nash welfare. *ACM Transactions on Economics and Computation* 7, 3 (2019), 12:1–12:32.

[7] Karen Frilya Celine, Muhammad Ayaz Dzulfikar, and Ivan Adrian Koswara. 2023. Egalitarian price of fairness for indivisible goods. In *Proceedings of the 20th Pacific Rim International Conference on Artificial Intelligence (PRICAI)*. 23–28.

[8] Karen Frilya Celine, Warut Suksompong, and Sheung Man Yuen. 2025. On the fairness of additive welfarist rules. In *Proceedings of the 24th International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*. Forthcoming.

[9] Owen Eckart, Alexandros Psomas, and Paritosh Verma. 2024. On the fairness of normalized p-means for allocating goods and chores. In *Proceedings of the 25th ACM Conference on Economics and Computation (EC)*. 1267.

[10] Jonathan Goldman and Ariel D. Procaccia. 2014. Spliddit: Unleashing fair division algorithms. *ACM SIGecom Exchanges* 13, 2 (2014), 41–46.

[11] Luisa Montanari, Ulrike Schmidt-Kraepelin, Warut Suksompong, and Nicholas Teh. 2024. Weighted envy-freeness for submodular valuations. In *Proceedings of the 38th AAAI Conference on Artificial Intelligence (AAAI)*. 9865–9873.

[12] Hervé Moulin. 2003. *Fair Division and Collective Welfare.* MIT Press.

[13] Jack Robertson and William Webb. 1998. *Cake-Cutting Algorithms: Be Fair if You Can.* Peters/CRC Press.

[14] Warut Suksompong. 2023. A characterization of maximum Nash welfare for indivisible goods. *Economics Letters* 222 (2023), 110956.

[15] Sheung Man Yuen and Warut Suksompong. 2023. Extending the characterization of maximum Nash welfare. *Economics Letters* 224 (2023), 111030.