# EFX Allocations and Orientations on Bipartite Multi-graphs: A Complete Picture

Mahyar Afshinmehr
Sharif University of Technology
Tehran, Iran
mahyarafshinmehr@gmail.com

Alireza Danaei
Sharif University of Technology
Tehran, Iran
alireza.danaei2002@gmail.com

Mehrafarin Kazemi
Sharif University of Technology
Tehran, Iran
mehr.kzm@gmail.com

Kurt Mehlhorn
Max Planck Institute for Informatics,
Saarland Informatics Campus
Saarbrücken, Germany
mehlhorn@mpi-inf.mpg.de

Nidhi Rathi
Max Planck Institute for Informatics,
Saarland Informatics Campus
Saarbrücken, Germany
nrathi@mpi-inf.mpg.de

## ABSTRACT

We consider the fundamental problem of *fairly* allocating a set of indivisible items among agents having valuations that are represented by a *multi-graph* – here, agents appear as the vertices and items as the edges between them and each vertex (agent) only values the set of its incident edges (items). The goal is to find a fair, i.e., *envy-free up to any item* (EFX) allocation. This model has recently been introduced by Christodoulou et al. [19] where they show that EFX allocations always exist on simple graphs for *monotone* valuations, i.e., where any two agents can share at most one edge (item). A natural question arises as to what happens when we go beyond simple graphs and study various classes of multi-graphs?

We answer the above question affirmatively for the valuation class of *bipartite multi-graphs* and *multi-cycles*. Our main positive result is that EFX allocations always exist on bipartite multi-graphs for agents with *additive* valuations and can be computed in polynomial time, thereby joining in the few sets of scenarios where EFX allocations are known to exist for an arbitrary number of agents.

Next, we study EFX *orientations* (i.e., allocations where every item is allocated to one of its two endpoint agents) and give a complete picture of when they exist for bipartite multi-graphs dependent on two parameters—the number of edges shared between any two agents and the diameter of the graph. Finally, we prove that it is NP-complete to determine whether a given fair division instance on a bipartite multi-graph admits an EFX orientation.

## KEYWORDS

Fair Division; EFX; Graphs; Orientation; NP-complete

**ACM Reference Format:**
Mahyar Afshinmehr, Alireza Danaei, Mehrafarin Kazemi, Kurt Mehlhorn, and Nidhi Rathi. 2025. EFX Allocations and Orientations on Bipartite Multi-graphs: A Complete Picture. In *Proc. of the 24th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2025), Detroit, Michigan, USA, May 19 – 23, 2025*, IFAAMAS, 9 pages.

This work is licensed under a Creative Commons Attribution International 4.0 License.

*Proc. of the 24th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2025), Y. Vorobeychik, S. Das, A. Nowé (eds.), May 19 – 23, 2025, Detroit, Michigan, USA.* © 2025 International Foundation for Autonomous Agents and Multiagent Systems (www.ifaamas.org).

## 1 INTRODUCTION

The theory of *Fair Division* formalizes the classic problem of dividing a collection of resources to a set of participating players (referred to as *agents*) in a *fair* manner. This problem forms a key concern in the design of many social institutions and arises naturally in many real-world scenarios such as dividing business assets, assigning computational resources in a cloud computing environment, air traffic management, course assignments, divorce settlements, and so on [12, 22, 29, 31, 37]. The fundamental problem of fair division lies at the crossroads of economics, social science, mathematics, and computer science, with its formal exploration beginning in 1948 [34]. In recent years, this field has experienced a flourishing flow of research; see [5, 9, 10, 33] for excellent expositions.

In this work, we consider the setting where we wish to divide a set of $m$ indivisible items to a set $\{1, 2, \ldots, n\}$ of $n$ agents with each item being allocated wholly to some agent. The standard notion of fairness is *envy-freeness*, which entails a division (allocation) as fair if every agent values her bundle as much as any other bundle in the allocation [24]. Since envy-free allocations do not always exist for the case of indivisible items,[1] several variants of envy-freeness have been explored in the literature. Among all, *envy-freeness up to any item* (EFX) is considered to be the flag-bearer of fairness for the setting of indivisible items (introduced by [15]). We say an allocation $X = (X_1, X_2, \ldots, X_n)$ where bundle $X_i$ is for agent $i \in [n]$ is EFX if for every pair $i, j \in [n]$ of agents, agent $i$ prefers her bundle $X_i$ over $X_j \setminus \{g\}$ for *every* item $g \in X_j$. EFX is considered to be the "closest analogue of envy-freeness" for the discrete setting [14].

It remains a major open problem to understand whether there always exists an EFX allocation [32]. Despite significant efforts, it is not known whether EFX allocations exist for four or more agents, even for additive valuations [17]. This naturally points towards the complexity of this problem and motivates the study of various kinds of relaxation. That is, one may begin to understand the concept of EFX either by providing approximate EFX allocations, or by studying special valuation classes, or by relaxing the notion of EFX via *charity* or its *epistemic* form. We refer the readers to Section 1.2 for further discussion.

[1] Consider an instance with two agents valuing a single item positively. Here, the agent who does not receive the item will envy the other. In contrast, envy-free allocations always exist when the resource to be allocated is divisible (see [35, 36]).

A question of interest is understanding for what scenarios or valuation classes EFX allocations are guaranteed to exist. To this effect, in this work, we consider a recently introduced model by Christodoulou et al. [19] where agent valuations are represented via a graph (or a multi-graph). Here, every item can have a positive value for at most two agents and that item appears as an edge between these two agents, i.e., an item (edge) is valued at zero for all agents that are not its endpoints. In other words, the vertices of a given multi-graph represent agents while the multi-edges between two agents $i$ and $j$ correspond to the items that are valued only by agents $i$ and $j$. We assume *additive* valuations $v_i$ for agent $i \in [n]$ such that an edge $g$ has a positive value ($v_i(g) > 0$) if and only $g$ is incident to $i$. We call this a *fair division problem on multi-graphs* with the goal of finding complete EFX allocations.

This model addresses situations where the number of agents interested in a particular item is restricted, in particular when an item is *relevant* to at most two agents. The graph-based approach is inspired by geographic contexts, where agents only care about resources nearby and show no interest in those located further away. For example, constructing a trade corridor or establishing a natural gas line between neighboring countries where the other (distant) countries may extract no value from these projects. Another example is when football matches in the New Champions League Format must be played on the home grounds of either of the participating teams, and other teams do not care for it.

Christodoulou et al. [19] proved that EFX allocations exist for all simple graphs, i.e., where any two agent vertices share at most *one* edge item. A natural question, therefore, arises as to what happens when we consider classes of multi-graphs, i.e., where two agents can have multiple relevant items in common?

## 1.1 Our Contribution and Techniques

In this work, we answer the above question raised by [19] and study fair division instances where agent valuations are represented via *bipartite multi-graphs* and provide a complete picture of EFX allocations for *additive* valuations. Note that bipartite multi-graphs are a huge class consisting of multi-trees, multi-cycles with even length, and planer multi-graphs with faces of even length, to name a few.

We also study special types of (*non-wasteful*) allocations—EFX *orientations*—where every item is allocated to one of the agent endpoints. Note that EFX orientations are desirable since every item is allocated to an agent who values it, and hence, is *non-wasteful*. Our main results are as follows.

| Parameters | EFX Orientation |
|---|---|
| *acyclic*, $q = 2$, $d(G) \leq 4$ | exists (Theorem 3.3) |
| *acyclic*, $q = 2$, $d(G) > 4$ | may not exist (Theorem 3.5) |
| *acyclic*, $q > 2$, $d(G) \leq 2$ | exists (Proposition 4.3) |
| *acyclic*, $q > 2$, $d(G) > 2$ | may not exist (Theorem 3.4) |
| *cyclic*, $q \geq 2$, $d(G) \geq 2$ | may not exist (Theorem 3.2) |

**Table 1: A complete picture for EFX orientations on bipartite multi-graphs based on $q$ and $d(G)$.**

- EFX allocations are guaranteed to always exist for fair division instances on *bipartite multi-graphs* with *additive* valuations. Moreover, we can compute an EFX allocation in polynomial time for these instances (Theorem 4.9).
  We answer the open problem listed in [19] and push the frontiers of the scenarios where EFX allocations are known to exist for an arbitrary number of agents, thereby enhancing our understanding of the notion of EFX.
- While extending our techniques beyond bipartite multi-graphs, we also prove that EFX allocations exist and can be computed in polynomial time on multi-cycle graphs with additive valuations (Theorem 5.1).
- EFX orientations may not always exist for fair division instances on bipartite multi-graphs; in particular, they may not exist even in very simple settings of multi-trees. Nonetheless, we provide an exhaustive list of scenarios where EFX orientations exist depending on two parameters: $q$ (the maximum number of edges shared between any two agents) and $d(G)$ (diameter of the graph $G$); see Table 1.
  The fact that orientations do not always exist can be seen as a proof that such inefficiency is inherent or that approximations are necessary. We show that there exist orientations on bipartite multi-graphs where at least $\lceil \frac{n}{2} \rceil$ agents are EFX and the remaining agents are 1/2-EFX (Theorem 4.10).
- We also show that we can compute EFX orientations in polynomial time when the diameter of the acyclic bipartite multi-graph is at most four and any two adjacent vertices share at most two edge items (Theorem 3.3).
- It is NP-complete to decide whether a given fair division instance on bipartite multi-graphs admits an EFX orientation, even with a constant number of agents (Theorem 3.6).

**Technical Overview:** We will give a description of the main techniques we develop in this work in order to prove the existence (and polynomial-time computation) of EFX allocations on bipartite multi-graphs with additive valuations. For a given bipartite multi-graph $G = (S \sqcup T, E)$, let us consider two adjacent vertices $i \in S$ and $j \in T$. The starting point of our technique is inspired by the *cut-and-choose* protocol that is used to prove the existence of EFX allocations for two agents.[2] Based on this protocol, we will define one specific *configuration* for the set of items, $E(i, j)$, between $i$ and $j$. In our configuration, agent $j$ in set $T$ will cut the set $E(i, j)$ into two bundles $C_1$ and $C_2$ such that she is EFX-happy with both bundles. We call this as *$j$-cut configuration*. Note that, this can be computed in polynomial time for additive valuations.

The idea is to find a partial EFX orientation $X$ that satisfies a set of five useful properties. These properties pave a simple way for us to extend $X$ to a complete allocation while maintaining EFX guarantees. One of them ensures that for any $i \in S$, $j \in T$, either (i) no item from $E(i, j)$ is allocated, (ii) exactly one of $C_1$ or $C_2$ is allocated to either $i$ or $j$, or (iii) both $C_1$ and $C_2$ are allocated to $i$ and $j$, such that one receives $C_1$ and other $C_2$ in $X$.

To create such a partial orientation, we start with a greedy algorithm that allocates a set of items to each agent such that every agent in $T$ is non-envied. This initial step ensures that the set of

[2]For two agents, the first agent divides the set of items into two EFX-feasible bundles (for her), and the second agent chooses her favorite bundle of the two.

items, $X_i$, allocated to an agent $i \in [n]$ is such that $X_i \subseteq E(i, j)$ for some $j \in [n]$. Moreover, this partial orientation is EFX where if a vertex is envied, the vertex that envies her is certainly non-envied. We then try to orient unallocated items incident to a non-envied vertex to her while maintaining partial EFX until the only unallocated edges are between a non-envied and an envied vertex.

Once we find a partial EFX orientation with certain useful properties, we can go two ways (of our choice) to have a complete allocation. We can either compute, in polynomial time, (i) an EFX orientation where at least $n/2$ agents are EFX, and the remaining agents are 1/2-EFX, or (ii) an exact EFX allocation. For the latter, we know that since EFX orientations do not necessarily exist for bipartite multi-graphs, we have to allocate the remaining edges to a vertex other than their endpoints, which will create a wasteful (albeit an EFX) allocation.

Before finding a complete allocation, we have ensured, by one of our properties, that both non-envied and envied vertices are satisfied enough with what is allocated to them that they will not envy if we give all the unallocated edges adjacent to them to a specific third vertex. We, therefore, safely allocate the remaining items from the set $E(i, j)$ to a specific agent $k \neq i, j$, and finally compute an EFX allocation.

## 1.2 Further Related Work

For the notion of EFX, [30] proved its existence for two agents with monotone valuations. For three agents, a series of works proved the existence of EFX allocations when agents have additive valuations [17], *nice-cancelable* valuations [8], and finally when two agents have monotone valuations and one has an *MMS-feasible* valuation [2]. EFX allocations exist when agents have identical [30], binary [25], or bi-valued [4] valuations. The study of several approximations [6, 16, 18, 23] and relaxations [3, 4, 7, 8, 13, 14, 26, 28] of EFX have become an important line of research in discrete fair division.

Another relaxation of envy-freeness proposed in discrete fair division literature is that of *envy-freeness up to some item* (EF1), introduced by [11]. It requires that each agent prefers her own bundle to the bundle of any other agent after removing some item from the latter. EF1 allocations always exist and can be computed efficiently [27]. Epistemic EFX is another relaxation of EFX that was recently introduced by [13], where they showed its existence and polynomial-time tractability for additive agents.

Following the work of [19], recent works have started to focus on EFX and EF1 orientations and allocations on graph setting. [39] studies the mixed manna setting with both goods and chores and proves that determining the existence of EFX orientations on simple graphs for agents with additive valuations is NP-complete and provides certain special cases like trees, stars, and paths where it is tractable. [38] relates the existence of EFX orientations and the chromatic number of the graph. Recently, [20] showed that EF1 orientations always exist for monotone valuations and can be computed in pseudo-polynomial time.

*Proportionality* [21, 34] and *maximin fair share* [11] are two other important fairness notions; we refer the readers to an excellent recent survey by [5] (and references within) for further details.

# 2 NOTATION AND DEFINITIONS

For any positive integer $k$, we use $[k]$ to denote the set $\{1, 2, \ldots, k\}$. We consider a set $[m]$ of $m$ goods (items) that needs to be allocated among a set $[n] = \{1, 2, \ldots, n\}$ of $n$ agents in a fair manner. For ease of notation, we will use $g$ instead of $\{g\}$ for an item $g \in [m]$. We begin by defining allocations and various fairness notions.

**Definition 2.1.** *(Allocations). A partial allocation $X = (X_1, X_2, \ldots, X_n)$ is an ordered tuple of disjoint subsets of $[m]$, i.e, for every pair of distinct agents $i$ and $j$ we have $X_i, X_j \subseteq [m]$ and $X_i \cap X_j = \emptyset$. Here, $X_i$ denotes the bundle allocated to agent $i \in [n]$ in $X$. We say an allocation $X$ is complete if $\bigcup_{i \in [n]} X_i = [m]$.*

**Valuation Functions and Fairness Notions:** Each agent $i \in [n]$ specifies her preferences using a valuation function $v_i : 2^{[m]} \rightarrow R^+$, that assigns a non-negative value to every subset of items. This work focuses on additive valuations that can be represented via *multi-graphs*, defined below.

**Definition 2.2.** *(Additive Valuations). A valuation function $v : 2^{[m]} \rightarrow \mathbb{R}^+$ is said to be* additive*, if for every subset $S \subseteq M$ of items, we have $v(S) = \sum_{g \in S} v(g)$, where $v(g)$ is the value for the item $g$ according to $v$.*

We denote a fair division instance as $\mathcal{I} = ([n], [m], \{v_i\}_{i \in [n]})$ where $v_i$'s are additive. Recently, [19] proposed a valuation class that can be represented by graphs. Here, vertices correspond to agents, and edges correspond to items such that an item (edge) is valued positively only by the two agents at its endpoints. [19] studied simple graphs, i.e., there is at most one edge between every pair of adjacent vertices. In this work, we focus on a natural extension of these instances to *multi-graphs* where we allow multiple edges between agents.

**Definition 2.3.** *(Multi-graph Instances). A fair division instance $\mathcal{I} = ([n], [m], \{v_i\}_{i \in [n]})$ on a multi-graph[3] is represented via a multi-graph $G = (V, E)$ where the $n$ agents appear as vertices in $V$ and the $m$ items form the edges in $E$ with the following structure: for every agent $i \in [n]$ and every item $g \in [m]$, $v_i(g) > 0$ if and only if $g$ is incident to $i$.*

For every multi-graph instance, we define $q$ as the maximum number of edges between any two adjacent vertices. Also, we denote the set of edges between agents $i$ and $j$ by $E(i, j)$. Note that $E(i, j) = E(j, i)$. In this paper, we use the words "agent" and "vertex" interchangeably, similarly for "item" and "edge".

**Definition 2.4.** *(Symmetric Instances) We say a multi-graph instance is symmetric if, for any edge $e \in E(i, j)$, it is identically valued by both $i$ and $j$, i.e., $v_i(g) = v_j(g)$.*

Let us now define the concept of *orientations* that are a kind of non-wasteful allocations in the context of graph-settings, where an item must be allocated to an agent who values it. More formally,

**Definition 2.5.** *((Partial) Orientation). A partial orientation is a partial allocation where an item $g$ (if allocated) is given to an agent $i$ such that $g$ is incident to $i$ in the given multi-graph. This can be*

[3]A multi-graph can have multiple edges between two vertices.

*represented by directing the edges of the graph towards the vertex receiving the edge.*

We use the popular notion of *envy-freeness up to any good (EFX)* as a standard fairness notion in our work. Let us first define the concept of *strong envy* and some useful constructs related to envy.

**Definition 2.6.** *(Envy and Strong Envy). Given an allocation $X = (X_1, X_2, \dots, X_n)$, we say $i$ envies $j$ if $v_i(X_j) > v_i(X_i)$, and we say $i$* strongly envies *$j$ if there exists an item $g \in X_j$ such that $v_i(X_i) < v_i(X_j \setminus g)$.*

**Definition 2.7.** *(Envy-Freeness Up to Any Good (EFX)). We say an allocation is EFX if there is no strong envy between any pair of agents. Moreover, we say an allocation $X$ is $\alpha$-EFX for an $\alpha \in (0, 1]$ if $v_i(X_i) \geq \alpha \cdot v_i(X_j \setminus g)$ for every $i, j \in [n]$ and $g \in X_j$.*

**Definition 2.8.** *(EFX-Feasibility). Given a partition $A = (A_1, A_2, \dots, A_n)$ of items into $n$ bundles, we say bundle $A_k$ is EFX-feasible for agent $i$ if we have $v_i(A_k) \geq \max_{j \in [n]} \max_{g \in A_j} v_i(A_j \setminus g)$.*

We say a bundle containing one item is a “singleton". Note that no agent strongly envies an agent owning a singleton.

Also, note that in an orientation on a given multi-graph $G$, a vertex $i \in [n]$ receives edges that are incident to her. Thus, it is only the set of $i$'s neighbors in $G$ that can possibly strongly envy her. This leads to the following observation, which we will use frequently in future sections.

**Observation 2.9.** *A partial orientation is EFX on a multi-graph if and only if no agent strongly envies her neighbor.*

Next, we demonstrate a useful property of EFX orientations on multi-graphs in Lemma 2.10. Its proof is omitted and can be found in the full version of the paper [1].

**Lemma 2.10.** *For a multi-graph instance, consider a partial EFX orientation $X$ where a vertex $i$ is envied by one of her neighbors $j$. Then, we must have $X_i \subseteq E(i, j)$. In particular, any vertex is envied by at most one neighbor in any EFX orientation.*

## 2.1 Graph Theory Definitions

Our work characterizes the existence of EFX orientations based on the parameter $q$ (the number of edges between any of agents in $G$) and the diameter of the multi-graph. We begin by defining some useful notions related to a multi-graph.

**Definition 2.11.** *(Skeleton of a Multi-graph). For a multi-graph $G = (V, E)$, we define its skeleton as a graph $G' = (V, E')$ where $G'$ has the same set of vertices, and there is a single edge between two vertices if they share at least one edge in $G$, i.e., $i$ is connected to $j$ in $G'$ if $E(i, j) \neq \emptyset$ in $G$.*

**Definition 2.12.** *($d(G)$ and $q$ of a Multi-graph $G$). We define $d(G)$ as the diameter of $G$, which is the length of the longest shortest path in the skeleton of $G$. And, we denote $q$ to be the maximum number of edges between any two agents in $G$.*

**Definition 2.13.** *(Center of a Multi-graph). For a multi-graph $G$, center $c \in V$ is as member of $\operatorname{argmin}_{x \in V} \max_{v \in V} d(x, v)$, where $d(x, v)$ is the distance between $x$ and $v$ in $G$. In this paper, we choose one arbitrarily if we have multiple centers.*

In this work, we focus on bipartite multi-graphs, that we define next.

**Definition 2.14.** *(Bipartite Multi-graph). A bipartite multi-graph $G = (V, E)$ has a skeleton that is a bipartite graph. We denote $V = S \sqcup T$ with two partitions $S$ and $T$ having no edge between them.*

**Definition 2.15.** *(Multi-star, Multi-$P_n$, Multi-cycle, and Multi-tree). A multi-H has a skeleton that is an $H$ graph, where $H$ can be a star, a path $P_n$ of length $n - 1$, a cycle, or a tree.*

Note that bipartite multi-graphs are a huge class consisting of multi-trees, multi-cycles of even length, and planer multi-graphs with faces of even length, to name a few.

# 3 EFX ORIENTATIONS ON BIPARTITE MULTI-GRAPHS

Christodoulou et al. [19] shows that EFX orientations may not always exist, even on simple graphs. Therefore, it is not surprising when we show the same on bipartite multi-graphs. In particular, we examine multi-cycles (a special kind of bipartite multi-graphs) and show that even for $q = 2$, EFX orientations may not always exist for four agents; see Theorem 3.2. Nonetheless, we identify the correct parameters to characterize the scenarios where EFX orientations are guaranteed to exist; see Table 1.

To do so, we proceed step by step, carefully considering all possible cases. Initially, we focus on bipartite multi-graphs with diameters, $d(G)$, of small numbers. For $d(G) = 1$, bipartite multi-graphs become multi-$P_2$, for which an EFX orientation can easily be achieved using the cut-and-choose protocol. As a warm-up, we show the existence of EFX orientations for multi-trees with $d(G) = 2$, i.e., for multi-stars. Here, we do so for $q = 2$, but the same approach can be generalized to any $q$, which we discuss in the next section (in Proposition 4.3).

In this section, we present counter-examples for various cases via figures, where we have symmetric instances, and the number on each edge depicts the value of that edge for both endpoints. A few proofs are omitted; these may be found in the full version of the paper [1].

**Proposition 3.1.** *EFX orientations exist on multi-stars for $q = 2$.*

However, an EFX orientation might not always exist on bipartite multi-graphs with $d(G) \geq 2$. We prove it by providing an example in the following theorem.

Theorem 3.2. *EFX orientations on cyclic bipartite multi-graphs with $d(G) \geq 2$ and any $q \geq 2$ may not exist (even on symmetric instances).*

Since EFX orientations do not exist even on bipartite multi-cyclic graphs with small diameters, we examine the existence of EFX orientation on multi-trees as an important subset of bipartite multi-graphs, and show the following.

Theorem 3.3. *EFX orientations exist and can be computed in polynomial time on multi-trees with $d(G) \leq 4$ and $q = 2$.*

Unfortunately, EFX orientations may not exist on multi-trees with a greater diameter or higher $q$, as shown in the following theorems.

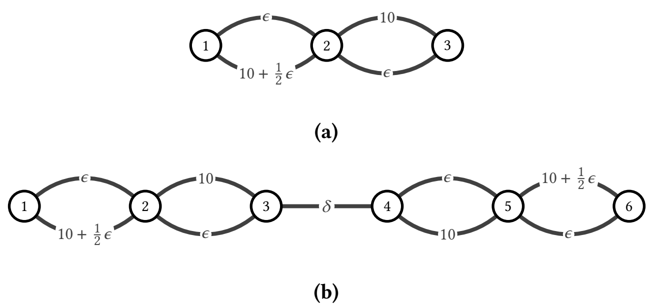

Figure 1: Instance (a) represents a multi-$P_3$ instance where every EFX orientation leaves agent 3 envied. We use this as a building block to give a multi-$P_6$ instance that does not admit any EFX allocation; see instance (b).

**Theorem 3.4.** *For multi-trees with $d(G) \geq 3$, EFX orientations may not exist for $q \geq 3$ (even on symmetric instances).*

**Theorem 3.5.** *For multi-trees with $d(G) \geq 5$, EFX orientations may not exist even for $q = 2$ (even on symmetric instances).*

*Proof.* We present a multi-$P_6$ instance with $q = 2$ that does not admit any EFX orientation. We use Figure 1 to represent our counter-example.

We first construct a special multi-$P_3$ instance where a specific node in any of its EFX orientations is envied. Consider the multi-$P_3$ instance in Figure 1(a). One can easily show that it admits exactly two EFX orientations, and agent 3 is envied in both.

Now we build the multi-$P_6$ instance in Figure 1(b) that is made by two copies of the above-mentioned special multi-$P_3$ instance. We connect them with an edge of value $\delta \ll \epsilon$ using the two vertices (vertex 3 and vertex 4) that are always envied in EFX orientations of the multi-$P_3$ parts. Without loss of generality, let agent 3 receive the edge $(3, 4)$ with value $\delta$. Now, since in any EFX orientation of the special multi-$P_3$ instance, agent 3 was always envied, the addition of edge with value $\delta$ to agent 3's bundle will therefore create strong envy against her.

One can extend the above counter-example by adding nodes to the graph with edges having value $\delta' \approx 0$ to achieve a counter-example for any $n \geq 6$ and any $d(G) \geq 5$. □

## 3.1 Hardness of Deciding the Existence of EFX Orientations on Bipartite Multi-Graphs

In this section, we consider the computational problem of deciding whether a given instance on a bipartite multi-graph (even a multi-tree with a constant number of agents) admits an EFX orientation. We reduce from the NP-complete problem of *Partition* to our problem.

*Partition Problem.* Consider a multi-set[4] $P = \{p_1, p_2, \ldots, p_k\}$ of $k$ non-negative integers. The problem is to decide whether $P$ can be partitioned into two multi-sets $P_1$ and $P_2$ such that $\sum_{p \in P_1} p = \sum_{p \in P_2} p$.

[4] A multi-set allows multiple instances for each of its elements.

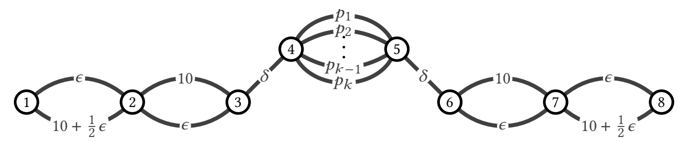

**Figure 2: The construction used in proof of Theorem 3.6. Here, $\delta \ll \epsilon \ll 1$.**

We prove a stronger claim and prove hardness for multi-tree instances.

**Theorem 3.6.** *The problem of deciding whether a fair division instance on a multi-tree (with additive valuations) admits an EFX orientation is NP-complete. It holds true even for symmetric instances with a constant number of agents.*

*Proof.* Given an orientation, it is easy to verify if it is EFX. Hence, the problem belongs in NP.

Let us now consider an instance $P = \{p_1, p_2, \ldots, p_k\}$ of the Partition problem. We will construct a fair division instance on a multi-tree with eight vertices, as depicted in Figure 2. Note that we have used the multi-$P_3$ instance used in the proof of Theorem 3.5 here as well. Following the similar lines, we can argue that in any EFX orientation in this instance, agent 2 envies agent 3, and agent 7 envies agent 6. Thus, the two edges $(3, 4)$ and $(5, 6)$ are allocated to agents 4 and 5 respectively. Now, one can observe that this instance admits an EFX orientation if and only if the set $P$ can be partitioned into two sets of equal sum. This completes our proof. □

Theorem 3.6 immediately implies the following, which is independently proved by [20]:

**Corollary 3.7.** *Deciding whether an EFX orientation exists on a multi-graph with additive valuations is NP-complete, even for symmetric instances with a constant number of agents.*

# 4 EXISTENCE OF EFX ALLOCATIONS ON BIPARTITE MULTI-GRAPHS

In this section, we prove our main positive result, showing that any fair division instance on a bipartite multi-graph with additive valuations always admits an EFX allocation. Furthermore, such allocations can be computed in polynomial time Note that, as we have already discussed in Section 3 that EFX orientations may not exist for these instances, we know that an EFX allocation may allocate some items to a third party, i.e., not the endpoints of its corresponding edge.

We will denote a fair division instance on a bipartite multi-graph by $G = (S \sqcup T, E)$, where $S$ and $T$ represent the two bi-partitions parts of its skeleton. We begin by discussing the main idea of our techniques (in Section 4.1) and then define some concepts and properties (in Section 4.2) useful for our proof (in Sections 4.3, 4.4, 4.5, and 4.6).

All the omitted proofs of this section can be found in the full version of the paper [1].

### 4.1 Main Idea

We introduce the concept of *configuration* to decide how to allocate the edges between any two adjacent vertices (in $G$) to their endpoints. It has a flavor that is similar to the cut-and-choose protocol (used for finding EFX allocations between two agents). In our proof, we will use this configuration to partially orient the edges between two agents. We define it as follows:

**Definition 4.1.** ***$T$*-cut *Configurations:*** *For any pair of agents $i \in S$ and $j \in T$, we let agent $j$ to partition the set $E(i,j)$ into two bundles $C_1$ and $C_2$ such that both are EFX-feasible for $j$ (for the items $E(i,j)$). We call the partition $(C_1, C_2)$ as the $j$-*cut *configuration between agents $i$ and $j$.*

We show that *$j$-cut* configurations can be computed in polynomial time (in Lemma 4.2).

**Lemma 4.2.** *For any additive valuation function $v$ over a set of items $S$, there always exist a partition $(C_1, C_2)$ of $S$ such that both $C_1$ and $C_2$ are EFX-feasible with respect $v$. Moreover, this partition can be computed in polynomial time.*

As a warm-up, we will use this configuration to prove the existence of EFX orientations for multi-stars with any $q$. Previously, (in Proposition 3.1), we proved the existence of EFX orientations for multi-stars with $q = 2$.

**Proposition 4.3.** *EFX orientations exist and can be computed in polynomial time for multi-stars with any $q$.*

### 4.2 Some Useful Notions

For a partial allocation $X = (X_1, X_2, \ldots, X_n)$, we define the followings:

- For any two adjacent agents $i \in S$ and $j \in T$, we define $A_{i,j}(X)$ as the set of *available* edges in $E(i,j)$ for $i$ and $A_{j,i}(X)$ as the set of *available* edges in $E(i,j)$ for $j$. Formally, we define these sets as follows[5]. Let us assume the $j$-cut configuration of $E(i,j)$ is $(C_1, C_2)$.

$$A_{i,j}(X) = \begin{cases} \arg\max\limits_{C_1,C_2}\{v_i(C_1), v_i(C_2)\}, & E(i,j)\cap X_k = \emptyset \text{ for all } k \in [n] \\ E(i,j) \setminus X_j, & X_i \cap E(i,j) = \emptyset, X_j \cap E(i,j) \neq \emptyset \\ \emptyset, & X_i \cap E(i,j) \neq \emptyset, X_j \cap E(i,j) = \emptyset \end{cases}$$

$$A_{j,i}(X) = \begin{cases} \arg\max\limits_{C_1,C_2}\{v_j(C_1), v_j(C_2)\}, & E(i,j)\cap X_k = \emptyset \text{ for all } k \in [n] \\ \emptyset, & X_i \cap E(i,j) = \emptyset, X_j \cap E(i,j) \neq \emptyset \\ E(i,j) \setminus X_i, & X_i \cap E(i,j) \neq \emptyset, X_j \cap E(i,j) = \emptyset \end{cases}$$

- For $i \in [n]$, we define $A_i(X)$ to be her *available set* of edges, i.e., $A_i(X) = \bigcup_{j \neq i, j \in [n]} A_{i,j}(X)$.
- For $i \in [n]$, $U_i(X)$ is the set of all unallocated edges incident to $i$. Note that $A_i(X) \subseteq U_i(X)$.
- For $i \in [n]$, $B_i(X)$ is the set of all *available bundles* for $i$, i.e. $B_i(X) = \{A_{i,j}(X) : j \neq i, j \in [n]\}$.
- For any envied agent $i \in [n]$, we define $S_i(X) \subseteq [n]$ to be her *safe* set, as follows,

$$S_i(X) = \{k \in [n] : k \text{ is non-envied in } X \text{ and } v_i(X_i) \geq v_i(X_k \cup A_i(X))\}.$$

That is, $i$ will not envy $k$ even if we allocate her whole available set $A_i(X)$ to $k$.

To achieve a complete EFX allocation, we will use a similar approach as [19]. We will first find a partial orientation with some nice properties and then allocate the remaining edges to some agent who is not incident to them. Identifying these key nice properties is a non-trivial challenge that we address next.

*Key Properties:* We search for a partial allocation $X = (X_1, X_2, \ldots, X_n)$ with the following properties:

(1) $X$ is an EFX orientation.
(2) For any two adjacent agents $i \in S$ and $j \in T$, items in $E(i,j)$ must be allocated based on the $j$-*cut* configuration $(C_1, C_2)$ to either one of their endpoints. By this property, we mean that one of the following cases must happen for $X$ (following the Definition 4.1):
  - Either $C_1 \subseteq X_i, C_2 \subseteq X_j$ or $C_2 \subseteq X_i, C_1 \subseteq X_j$.
  - One of the bundles[6] $C_1$ or $C_2$ is allocated to agent $i$, and the other bundle is unallocated in $X$.
  - One of the bundles $C_1$ or $C_2$ is allocated to agent $j$, and the other bundle is unallocated in $X$.
(3) For any agent $i \in [n]$ and a set $B \in B_i(X)$, we have $v_i(X_i) \geq v_i(B)$.
(4) For any non-envied agent $i \in [n]$, we have $A_i(X) = \emptyset$.
(5) For any envied agent $i \in [n]$, let $j$ envies $i$. Then, we have $j \in S_i(X)$.

We are now finally equipped to present our algorithm. We will give a step-by-step procedure to satisfy each key property in the above order. These key properties ensure that there is an easy way to then convert a partial EFX orientation to a complete EFX allocation (see Section 4.6).

For the sake of a better comprehension, we illustrate the three main steps of our algorithm via a running example and show the results after each step in the full version of the paper [1].

### 4.3 Satisfying Properties (1)-(3)

We present a greedy algorithm that assigns a set of items to each agent and satisfies the first three properties. It works in the following manner.

Let $S = \{i_1, i_2, \ldots, i_{|S|}\}$ and $T = \{j_1, j_2, \ldots, j_{|T|}\}$ be the two bipartitions. We fix a picking sequence $\sigma = [i_1, \ldots, i_{|S|}, j_1, \ldots, j_{|T|}]$ that decides the order in which an agent comes and selects her most valuable available bundle. Since the definition of $A_{i,j}(X)$ is dynamic, the set of available bundles for some agents may change after another agent picks her favorite bundle in the picking sequence. Algorithm 1 illustrates this procedure. The properties of this algorithm are further formalized in Lemmas 4.4 and 4.5.

**Lemma 4.4.** *For a fair division instance on bipartite multi-graph, the output allocation of Algorithm 1 satisfies properties (1)-(3). Moreover, the algorithm runs in polynomial time.*

[5] In other cases, $A_{i,j}(X)$ and $A_{j,i}(X)$ are not defined.

[6] $X_i$ can also have other items.

**Algorithm 1:** Greedy Orientation: Properties (1)-(3)

**Input:** The fair division instance on a bipartite multi-graph $G = (S \sqcup T, E)$
**Output:** A partial EFX orientation satisfying properties (1)-(3)

**for** $(l \leftarrow 1; l \leq |S|\ ; l \mathrel{+}= 1)$ **do**
  $k \leftarrow \arg\max_{k \in [n] \setminus \{i_l\}} v_{i_l}(A_{i_l,k}(X))$
  $X_{i_l} \leftarrow A_{i_l,k}(X)$
**for** $(l \leftarrow 1; l \leq |T|\ ; l \mathrel{+}= 1)$ **do**
  $k \leftarrow \arg\max_{k \in [n] \setminus \{j_l\}} v_{j_l}(A_{j_l,k}(X))$
  $X_{j_l} \leftarrow A_{j_l,k}(X)$

**Lemma 4.5.** *In the output allocation $X$ of Algorithm 1, every envied vertex belongs to the set $S$.*

Lemma 4.5 will continue to hold while we obtain our desired orientation. As demonstrated in the following sections, we will not produce new envied vertices when we modify our allocation $X$ to satisfy properties (1)-(5). And hence, Lemma 4.5 will continue to hold in the above process.

## 4.4 Satisfying Property (4)

Let us now focus on satisfying property (4) that requires $A_i(X) = \emptyset$ for any non-envied agent $i \in [n]$. Let us assume that the output allocation $X$ of Algorithm 1 violates property (4). Consider a non-envied agent $i \in [n]$ with $A_i(X) \neq \emptyset$. Therefore, an agent $j \in [n]$ exists such as $A_{i,j}(X) \neq \emptyset$. We will now allocate all of $A_{i,j}(X)$ either to $i$ or $j$, depending on the following three possible cases.

- **Case 1. A set of items in $E(i, j)$ is allocated to $j$:** Since $A_{i,j} \neq \emptyset$, by its definition and property (2), no edge in $E(i, j)$ is allocated to $i$. In this case, we can allocate $A_{i,j}(X)$, which is exactly the set $E(i, j) \setminus X_j$ to $i$. Since $X_j \cap E(i, j) \neq \emptyset$, $j$ chose the better bundle from the configuration of $E(i, j)$ during Algorithm 1. Also, the set $A_{i,j}(X)$ has value only to agents $i$ and $j$; therefore, since agent $i$ was non-envied before, the modified allocation remains EFX. One can observe that properties (1)-(3) remain satisfied. Observe that, in this case, $E(i, j)$ will be fully allocated.
- **Case 2. No item in $E(i, j)$ is allocated, and $j$ is non-envied:** Without loss of generality, we can assume $i \in S$ and $j \in T$. Let the partition $(C_1, C_2)$ be the *$j$-cut* configuration of the set $E(i, j)$. Let us assume $v_i(C_1) \geq v_i(C_2)$ (the other case is symmetric). Observe that by property (3) $v_i(X_i) \geq \max\{v_i(C_1), v_i(C_2)\}$ and $v_j(X_j) \geq \max\{v_j(C_1), v_j(C_2)\}$. Since the $X$ is an orientation, we have that $v_i(X_j) = v_j(X_i) = 0$. We now allocate $C_1$ to agent $i$ and $C_2$ to agent $j$ to obtain,
$$v_i(X_j \cup C_2) = v_i(C_2) \leq v_i(X_i), \text{ and } v_j(X_i \cup C_1) = v_j(C_1) \leq v_j(X_j)$$
Thus, the allocation remains EFX, and all the first three properties are still satisfied. Notice that $E(i, j)$ will be fully allocated in this case as well.
- **Case 3. No item in $E(i, j)$ is allocated and $j$ is envied:** In this case, Lemma 4.5 entails that $j \in S$ and $i \in T$. Let the partition $(C_1, C_2)$ be the *$i$-cut* configuration of items $E(i, j)$. By property (3) of $X$, we have $v_j(X_j) \geq \max\{v_j(C_1), v_j(C_2)\}$. Assuming $v_i(C_1) \geq v_i(C_2)$ (symmetric otherwise), we allocate $C_1$ to agent $i$. Agent $j$ will not envy $i$ and agent $i$ remains non-envied in the modified allocation. Hence, the allocation remains EFX, and the properties (1)-(3) are still satisfied.

**Algorithm 2:** Allocating to Non-Envied Vertices: Properties (1)-(3) + Property (4)

**Input:** A partial orientation $X \leftarrow$ Algorithm 1$(G)$ satisfying properties (1)-(3)
**Output:** An orientation $X$ satisfying properties (1)-(4)

**while** *there exists a non-envied agent $i \in [n]$ such that $A_i(X) \neq \emptyset$* **do**
  **while** *there exists an agent $j \in [n]$ such that $A_{i,j}(X) \neq \emptyset$* **do**
    **if** $X_j \cap E(i, j) \neq \emptyset$ **then**
      $X_i \leftarrow X_i \cup A_{i,j}(X)$
    **else if** $X_j \cap E(i, j) = \emptyset$ *and $j$ is non-envied* **then**
      W.l.o.g, let $i \in S, j \in T$.
      $(C_1, C_2) \leftarrow$ the *$j$-cut* configuration of $E(i, j)$.
      $C_\ell \leftarrow \arg\max_{C_1,C_2}\{v_i(C_1), v_i(C_2)\}$.
      $X_i \leftarrow X_i \cup C_\ell$
      $X_j \leftarrow X_j \cup C_{3-\ell}$
    **else if** $X_j \cap E(i, j) = \emptyset$ *and $j$ is envied* **then**
      $(C_1, C_2) \leftarrow$ the *$i$-cut* configuration of $E(i, j)$.
      $C_\ell \leftarrow \arg\max_{C_1,C_2}\{v_i(C_1), v_i(C_2)\}$.
      $X_i \leftarrow X_i \cup C_\ell$

*Formalized protocol (Algorithm 2) to satisfy property (4) along with properties (1)-(3):* We repeat the following process as long as there is a non-envied agent $i \in [n]$ who violates property (4). We pick such a violator agent $i$. Then, for every agent $j \neq i$ such that $A_{i,j}(X) \neq \emptyset$, we allocate $A_{i,j}(X)$ according to the cases above. Note that we allocate at least one edge incident to $i$ at each step. Therefore, for each agent $i$, this step takes at most $O(m)$ iterations. Then, we repeat. At the end, for any non-envied agent $i \in [n]$, we ensure that $A_i(X) = \emptyset$, thereby satisfying property (4). Moreover, as discussed above, properties (1)-(3) remain satisfied as well. We abuse the notation and call the partial orientation we have built so far (that satisfies properties (1)-(4)) by $X$.

**Claim 4.6.** *After satisfying properties (1)-(4), if there exists a pair of agents $k, i \in [n]$ such that $A_{k,i}(X) \neq \emptyset$, then $k$ is an envied vertex, but $i$ is non-envied. Furthermore, $E(k, i) \setminus A_{k,i}(X)$ is allocated to $i$.*

**Claim 4.7.** *After satisfying properties (1)-(4), we have $v_i(U_i(X)) \leq v_i(X_i)$ for every non-envied vertex $i \in [n]$.*

## 4.5 Satisfying Property (5)

We now finally focus on satisfying property (5) where for any envied agent $i \in [n]$ who is envied by $j$ must be such that $j \in S_i(X)$. We present Algorithm 3 to detail the required modifications for reaching our desired partial orientation.

Algorithm 3 begins by identifying a pair of agents $(i, j)$ in $X$ where $i$ is envied by $j$ and $j \notin S_i(X)$ and swaps the bundles they possess from the $j$-*cut* configuration of the set $E(i, j)$. And then, we allocate the set $A_i(X)$ to $i$ as well. We will show (in Lemma 4.8) that the above procedure will make agent $i$ non-envied. Algorithm 3 presents the pseudo-code.

**Algorithm 3:** Safe Set: Properties (1)-(4)+Property (5)

**Input:** Allocation X satisfying properties (1)-(4)
**Output:** Allocation X satisfying properties (1)-(5)

**while** *there exists an* $i \in [n]$ *who is envied by* $j \notin S_i(X)$ **do**
  Let the partition $(C_1, C_2)$ be the $j$-*cut* configuration of the set $(i, j)$.
  Swap the bundles $C_1$ and $C_2$ between agents $i$ and $j$.
  $X_i \leftarrow X_i \cup A_i(X)$

**Lemma 4.8.** *Algorithm 3 terminates and outputs a partial allocation that satisfies properties (1)-(5). Moreover, the algorithm runs in polynomial time.*

Note that Claim 4.6 still holds since properties (1)-(4) are satisfied.

## 4.6 Allocating the Remaining Items

For a given bipartite multi-graph, we execute Algorithms 1, 2, and 3 (in that order) and reach a desired partial EFX orientation $X$ that satisfies properties (1)-(5). What is remaining is to now make $X$ complete by assigning the unallocated items while maintaining EFX guarantees. We will show that the five properties of $X$ make it easy to do the above.

By Claim 4.6, we know that the only unallocated edges in $X$ are between two vertices $i$ and $j$ where $i$ is envied, and $j$ is non-envied. We call the unallocated set of items between $i$ and $j$ as $C$. We allocate $C$ to an agent $k$ that envies $i$. Note the $k \neq j$, since otherwise, $j$ envies $i$, and hence, $i$ would have received something from $E(i, j)$, which is a contradiction.

Using property (5), we know that $k \in S_i(X)$, i.e., $v_i(X_i) \geq v_i(X_k \cup A_i(X))$. Moreover, using Claim 4.7, we know that $v_j(X_j) \geq v_j(U_j(X))$. Note that $v_j(X_k) = 0$, since $k$ and $j$ both belong to the set $T$. Hence, allocating $C$ (which is subset of $A_i(X)$ and $U_j(X)$) to $k$ does not make either $i$ or $j$ envious of $k$. Therefore, we reach a complete EFX allocation. Moreover, one can easily observe that every step in our algorithm can be performed in polynomial time. That is, we have our following main theorem.

Theorem 4.9. *For any fair division instance on bipartite multi-graph with additive valuations, EFX allocations always exist and can be computed in polynomial time.*

Now, as a corollary, we obtain the following theorem that says that we can compute EFX orientations that are 1/2-EFX in polynomial time.

Theorem 4.10. *There is an EFX orientation for any fair division instance on bipartite multi-graph where at least* $\lceil \frac{n}{2} \rceil$ *of agents are EFX and the remaining agents are* $\frac{1}{2}$*-EFX. Furthermore, such an orientation can be computed in polynomial time.*

# 5 FURTHER IMPROVEMENTS AND LIMITATIONS

Theorem 4.9 motivates the question of what happens if the graph skeleton contains cycles of odd length. In this section, we prove, using our technique, that any multi-cycle instance with additive valuations admits an EFX allocation (see Theorem 5.1, with its proof in the full version of the paper [1]). This demonstrates the power of our technique while providing an insight and strong hope for potentially proving the existence of EFX on general multi-graphs.

Theorem 5.1. *For any fair division instance on multi-cycles with additive valuations, EFX allocations always exist and can be computed in polynomial time.*

*Why do our techniques fail for general multi-graphs?* Theorems 4.9 and 5.1 make it hopeful that one can adapt/modify our techniques to prove the existence of EFX allocations in general multi-graphs. In our proof, we never had to deal with the case (after Algorithm 1) where there were two envied vertices $i$ and $j$ in the graph such that no edge from $E(i, j)$ was allocated. This turns out to be the most complicated case for the general multi-graph structure. As a solution concept, one can aim to achieve a partial orientation with $|S_i(X) \cap S_j(X)| \geq 2$ for any adjacent envied vertices $i, j$. If this happens, we can let $(C_1, C_2)$ be the $i$-*cut* configuration of items $E(i, j)$ and then allocate $C_1$ and $C_2$ to two different vertices in $S_i(X) \cap S_j(X)$. Also, we believe that we have to use both configurations $i$-*choose* and $j$-*choose* for every pair of adjacent vertices $i, j$ for extending our result to general multi-graphs.

*Conjecture:* Any fair division instance on a multi-graph admits an EFX allocation.

# 6 CONCLUSION

In this work, we study a model that captures the setting where every item is relevant to at most two agents and any two agents can have multiple relevant items in common (represented by multi-graphs). We prove that EFX allocations exist and can be computed in polynomial time for fair division instances on bipartite multi-graphs and multi-cycles. An immediate question for future research work is to understand EFX allocations on general multi-graphs, as discussed in Section 5.

The fact that EFX orientations may not exist on bipartite multi-graphs implies the fact that *wastefulness* is inherent in EFX allocations in these instances. In this work, we prove the existence of orientations that are EFX for half the agents and 1/2-EFX for the remaining agents. Another immediate question is, therefore, to improve this factor. For understanding the trade-offs with efficiencies, it would be interesting to see what one can say about approximating social welfare or Nash social welfare of EFX allocations.

Ultimately, we hope that insights gained from EFX allocations in the multi-graph setting will contribute to advancements in the broader challenge of EFX allocations in a general setting.


# ACKNOWLEDGMENTS

Part of this work was done when M. Afshinmehr and M. Kazemi were research interns at MPI-INF, Germany. N. Rathi is supported by the Lise Meitner Postdoc Fellowship awarded by MPI-INF, Germany.

# REFERENCES


[1] Mahyar Afshinmehr, Alireza Danaei, Mehrafarin Kazemi, Kurt Mehlhorn, and Nidhi Rathi. 2024. EFX Allocations and Orientations on Bipartite Multi-graphs: A Complete Picture. *arXiv preprint arXiv:2410.17002* (2024).

[2] Hannaneh Akrami, Noga Alon, Bhaskar Ray Chaudhury, Jugal Garg, Kurt Mehlhorn, and Ruta Mehta. 2024. EFX: a simpler approach and an (almost) optimal guarantee via rainbow cycle number. *Operations Research* (2024).

[3] Hannaneh Akrami, Rojin Rezvan, and Masoud Seddighin. 2022. An EF2X Allocation Protocol for Restricted Additive Valuations. In *Proceedings of the Thirty-First International Joint Conference on Artificial Intelligence (IJCAI)*. 17–23.

[4] Georgios Amanatidis, Georgios Birmpas, Aris Filos-Ratsikas, Alexandros Hollender, and Alexandros A. Voudouris. 2020. Maximum Nash welfare and other stories about EFX. In *Proceedings of the 29th International Joint Conference on Artificial Intelligence (IJCAI)*. 24–30.

[5] Georgios Amanatidis, Georgios Birmpas, Aris Filos-Ratsikas, and Alexandros A. Voudouris. 2022. Fair Division of Indivisible Goods: A Survey. In *Proceedings of the Thirty-First International Joint Conference on Artificial Intelligence, IJCAI-22*. 5385–5393.

[6] Georgios Amanatidis, Evangelos Markakis, and Apostolos Ntokos. 2020. Multiple birds with one stone: Beating 1/2 for EFX and GMMS via envy cycle elimination. *Theoretical Computer Science* 841 (2020), 94–109.

[7] Benjamin Aram Berendsohn, Simona Boyadzhiyska, and László Kozma. 2022. Fixed-Point Cycles and Approximate EFX Allocations. In *Proceedings of the 47th International Symposium on Mathematical Foundations of Computer Science (MFCS)*. 17:1–17:13.

[8] Ben Berger, Avi Cohen, Michal Feldman, and Amos Fiat. 2022. Almost Full EFX Exists for Four Agents. In *Proceedings of the 36th AAAI Conference on Artificial Intelligence (AAAI)*. 4826–4833.

[9] Steven J Brams and Alan D Taylor. 1996. *Fair Division: From cake-cutting to dispute resolution.* Cambridge University Press.

[10] Felix Brandt and Ariel D Procaccia. 2016. *Handbook of computational social choice.* Cambridge University Press.

[11] Eric Budish. 2011. The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. *Journal of Political Economy* 119, 6 (2011), 1061–1103.

[12] Eric Budish and Estelle Cantillon. 2012. The multi-unit assignment problem: Theory and evidence from course allocation at Harvard. *American Economic Review* 102, 5 (2012), 2237–2271.

[13] Ioannis Caragiannis, Jugal Garg, Nidhi Rathi, Eklavya Sharma, and Giovanna Varricchio. 2023. New Fairness Concepts for Allocating Indivisible Items. In *Proceedings of the Thirty-Second International Joint Conference on Artificial Intelligence, IJCAI-23*. 2554–2562.

[14] Ioannis Caragiannis, Nick Gravin, and Xin Huang. 2019. Envy-freeness up to any item with high Nash welfare: The virtue of donating items. In *Proceedings of the 20th ACM Conference on Economics and Computation (EC)*. 527–545.

[15] Ioannis Caragiannis, David Kurokawa, Hervé Moulin, Ariel D. Procaccia, Nisarg Shah, and Junxing Wang. 2019. The Unreasonable Fairness of Maximum Nash Welfare. *ACM Transactions on Economics and Computation* 7, 3 (2019), 12:1–12:32.

[16] Hau Chan, Jing Chen, Bo Li, and Xiaowei Wu. 2019. Maximin-aware allocations of indivisible goods. In *Proceedings of the 28th International Joint Conference on Artificial Intelligence (IJCAI)*. 137–143.

[17] Bhaskar Ray Chaudhury, Jugal Garg, and Kurt Mehlhorn. 2020. EFX exists for three agents. In *Proceedings of the 21st ACM Conference on Economics and Computation (EC)*. 1–19.

[18] Bhaskar Ray Chaudhury, Telikepalli Kavitha, Kurt Mehlhorn, and Alkmini Sgouritsa. 2021. A little charity guarantees almost envy-freeness. *SIAM J. Comput.* 50, 4 (2021), 1336–1358.

[19] George Christodoulou, Amos Fiat, Elias Koutsoupias, and Alkmini Sgouritsa. 2023. Fair allocation in graphs. In *Proceedings of the 24th ACM Conference on Economics and Computation (EC)*. 473–488.

[20] Argyrios Deligkas, Eduard Eiben, Tiger-Lily Goldsmith, and Viktoriia Korchemna. 2024. EF1 and EFX Orientations. *arXiv preprint arXiv:2409.13616* (2024).

[21] L. E. Dubins and E. H. Spanier. 1961. How to Cut A Cake Fairly. *The American Mathematical Monthly* 68, 1 (1961), 1–17.

[22] Raul Etkin, Abhay Parekh, and David Tse. 2007. Spectrum sharing for unlicensed bands. *IEEE Journal on Selected Areas in Communications* 25, 3 (2007), 517–528.

[23] Alireza Farhadi, MohammadTaghi Hajiaghayi, Mohamad Latifian, Masoud Seddighin, and Hadi Yami. 2021. Almost envy-freeness, envy-rank, and Nash social welfare matchings. In *Proceedings of the AAAI Conference on Artificial Intelligence*, Vol. 35. 5355–5362.

[24] Duncan Karl Foley. 1966. *Resource Allocation and the Public Sector*. Vol. 7:45-98. Yale Economic Essays.

[25] Daniel Halpern, Ariel D Procaccia, Alexandros Psomas, and Nisarg Shah. 2020. Fair division with binary valuations: One rule to rule them all. In *Proceedings of the 16th International Conference on Web and Internet Economic (WINE)*. 370–383.

[26] Shayan Chashm Jahan, Masoud Seddighin, Seyed Mohammad Seyed Javadi, and Mohammad Sharifi. 2023. Rainbow Cycle Number and EFX Allocations: (Almost) Closing the Gap. In *Proceedings of the Thirty-Second International Joint Conference on Artificial Intelligence (IJCAI)*. 2572–2580.

[27] Richard J Lipton, Evangelos Markakis, Elchanan Mossel, and Amin Saberi. 2004. On Approximately Fair Allocations of Indivisible Goods. In *Proceedings of the 5th ACM Conference on Electronic Commerce (EC)*. 125–131.

[28] Ryoga Mahara. 2024. Extension of Additive Valuations to General Valuations on the Existence of EFX. *Mathematics of operations research* 49, 2, 1263–1277.

[29] Hervé Moulin. 2004. *Fair Division and Collective Welfare.* MIT press.

[30] Benjamin Plaut and Tim Roughgarden. 2020. Almost envy-freeness with general valuations. *SIAM Journal on Discrete Mathematics* 34, 2 (2020), 1039–1068.

[31] John Winsor Pratt and Richard Jay Zeckhauser. 1990. The fair and efficient division of the Winsor family silver. *Management Science* 36, 11 (1990), 1293–1301.

[32] Ariel D Procaccia. 2020. Technical perspective: An answer to fair division's most enigmatic question. *Commun. ACM* 63, 4 (2020), 118–118.

[33] Jack Robertson and William Webb. 1998. *Cake-cutting algorithms: Be fair if you can.* AK Peters/CRC Press.

[34] Hugo Steihaus. 1948. The Problem of Fair Division. *Econometrica* 16 (1948), 101–104.

[35] Walter Stromquist. 1980. How to Cut a Cake Fairly. *The American Mathematical Monthly* 87, 8 (1980), 640–644.

[36] Francis Edward Su. 1999. Rental Harmony: Sperner's Lemma in Fair Division. *The American Mathematical Monthly* 106, 10 (1999), 930–942.

[37] Thomas Vossen. 2002. *Fair Allocation Concepts in Air Traffic Management.* Ph.D. Dissertation. University of Maryland.

[38] Jinghan A Zeng and Ruta Mehta. 2024. On the structure of envy-free orientations on graphs. *arXiv preprint arXiv:2404.13527* (2024).

[39] Yu Zhou, Tianze Wei, Minming Li, and Bo Li. 2024. A Complete Landscape of EFX Allocations on Graphs: Goods, Chores and Mixed Manna. In *Proceedings of the Thirty-Third International Joint Conference on Artificial Intelligence (IJCAI)*. 3049–3056.