On the Fairness of Additive Welfarist Rules

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ABSTRACT

Allocating indivisible goods is a ubiquitous task in fair division. We study *additive welfarist rules*, an important class of rules which choose an allocation that maximizes the sum of some function of the agents' utilities. Prior work has shown that the maximum Nash welfare (MNW) rule is the unique additive welfarist rule that guarantees envy-freeness up to one good (EF1). We strengthen this result by showing that MNW remains the only additive welfarist rule that ensures EF1 for identical-good instances, two-value instances, as well as normalized instances with three or more agents. On the other hand, if the agents' utilities are integers, we demonstrate that several other rules offer the EF1 guarantee, and provide characterizations of these rules for various classes of instances.

KEYWORDS

Welfarist rules; Fair division; Indivisible goods

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1 INTRODUCTION

The distribution of limited resources to interested agents is a fundamental problem in society. Fairness often plays a key role in the distribution process—whether it be dividing inheritance between family members, allocating the national budget across competing sectors, or sharing credit and responsibility amongst participants in collaborative efforts. The complexity of this problem has led to the research area of *fair division*, which develops methods and algorithms to ensure that all agents feel fairly treated [4, 17, 18].

What does it mean for an allocation of the resources to be "fair"? The answer depends on the context and the specific fairness benchmark applied. One of the most prominent fairness criteria is *envy-freeness*, which states that no agent should prefer another agent's share to her own. In the ubiquitous setting of allocating *indivisible goods*—such as houses, cars, jewelry, and artwork—it is sometimes infeasible to attain envy-freeness. For example, if a single valuable good is to be allocated between two or more agents, only one of the agents can receive the good, thereby incurring envy from the remaining agents. In light of this, envy-freeness is often relaxed to *envy-freeness up to one good (EF1).* The EF1 criterion allows an agent to envy another agent provided that the removal of some good in the latter agent's bundle would eliminate the former agent's envy. It is known that an EF1 allocation always exists, and that such an allocation can be computed in polynomial time [5, 15].

In addition to fairness, another important property of allocations is *efficiency.* A well-known efficiency notion is *Pareto optimality (PO)*, which stipulates that no other allocation makes some agent better off and no agent worse off. Caragiannis et al. [6] proved that a *maximum Nash welfare (MNW)* allocation, which maximizes the product of the agents' utilities across all possible allocations,[1] satisfies both EF1 and PO, thereby offering fairness and efficiency simultaneously. In spite of this, the MNW rule does have limitations. For example, it avoids giving an agent zero utility at all costs—an allocation that gives most agents a tiny positive utility is preferred to another allocation that gives one agent zero utility and every other agent a large utility, as illustrated in the following example.

Example 1.1. Consider an instance with $n \geq 4$ agents and n goods $g_1, \ldots, g_n$ such that the utilities of the goods are as follows.

- $u_1(g_1) = n$.
- For $i \in \{2, \ldots, n\}$, $u_i(g_{i-1}) = n - 1$ and $u_i(g_i) = 1$.
- $u_i(g) = 0$ for all other pairs (i, g).

The only allocation $\mathcal{A}$ that gives positive utility to every agent is the one that assigns good g_i to agent i for all $i \in \{1, \ldots, n\}$—this is also the MNW allocation. The sum of the agents' utilities in this allocation is $n + (n - 1) = 2n - 1$. On the other hand, consider the allocation $\mathcal{B}$ that gives good g_1 to agent 1, g_i to agent $i + 1$ for $i \in \{2, \ldots, n-1\}$, and g_n to agent n. The sum of the agents' utilities in this allocation is $n^2 - 2n + 3$, which is much larger than that in $\mathcal{A}$. Not only is $\mathcal{B}$ also EF1 and PO, but one agent receives the same utility, all remaining agents except one receive much higher utility, while the exceptional agent receives only marginally lower utility.

The MNW rule belongs to the class of *additive welfarist rules*—rules that choose an allocation that maximizes the sum of some function of the agents' utilities for their bundles [17, p. 67]. Additive welfarist rules have the advantage that they satisfy PO by definition. Suksompong [19] showed that MNW is the only additive welfarist rule[2] that guarantees EF1 for all instances. Nevertheless, there exist other additive welfarist rules that guarantee EF1 for restricted classes of instances. For example, Montanari et al. [16, Appendix C] proved that the *maximum harmonic welfare (MHW)* rule ensures EF1 for the class of *integer-valued* instances. Note that this is an important class of instances in practice—for example, the popular fair division website Spliddit [11] only allows each user to specify integer values for goods summing up to 1000. In Example 1.1, allocation $\mathcal{B}$ is the MHW allocation, and we saw earlier that $\mathcal{B}$ has certain advantages compared to the MNW allocation

[1]This is equivalent to maximizing the sum of the logarithm of the agents' utilities.
[2]Yuen and Suksompong [20] extended this result by showing that MNW is the only (not necessarily additive) welfarist rule that guarantees EF1 for all instances.

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Table 1: Conditions on the functions defining the additive welfarist rules that guarantee EF1 allocations for different classes of instances (see Definition 2.1). The logarithmic function corresponds to the MNW rule. For normalized instances with two agents (resp. integer-valued instances with no other restrictions), we give necessary and sufficient conditions in Propositions 3.8 and 3.9 (resp. Propositions 4.14 and 4.15).

	No other restrictions	Two-value	Identical-good	Binary	Normalized
Real-valued	Logarithmic [6, 19]	Logarithmic (Th. 3.5)	Logarithmic (Th. 3.1)	Condition 4 (Th. 4.7)	$n \geq 3$: Logarithmic (Th. 3.6)
Integer-valued	?	Condition 5 (Th. 4.11)	Condition 3 (Th. 4.1)	Condition 4 (Th. 4.7)	?

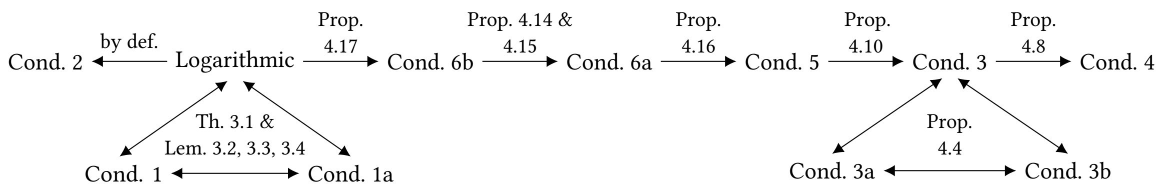

Figure 1: Relationships between the different conditions (see Definition 2.1).

$\mathcal{A}$. Beyond integer-valued instances, Eckart et al. [10] showed that certain additive welfarist rules based on *p-mean* functions, besides the MNW rule, also yield EF1 for the class of *normalized* instances with two agents—normalized instances have the property that the utility for the entire set of goods is the same for every agent.

In this paper, we study the necessary and sufficient conditions for additive welfarist rules to guarantee EF1 for different classes of instances. These classes include integer-valued instances, identical-good instances, two-value instances, normalized instances, as well as some of their combinations. We also establish relationships between the conditions for different classes of instances.

1.1 Our Results

In our model, we assume that each agent has an additive utility function over a set of indivisible goods. We consider additive welfarist rules, each of which is defined by a function f; the rule chooses an allocation that maximizes the sum of the function f applied to the agents' utilities for their own bundles. In order for the additive welfarist rules to ensure that agents receive as much utility as possible, we assume that f is strictly increasing.[3] For our characterizations, we only consider instances that are *positive-admitting*, i.e., instances in which there exists an allocation that gives positive utility to every agent.[4] Our model is described formally in Section 2.

We then examine conditions for an additive welfarist rule to always choose EF1 allocations, starting with general (real-valued) instances. As mentioned earlier, the MNW rule is the unique additive welfarist rule that guarantees EF1 for all (positive-admitting) instances [19]. In Section 3, we strengthen this result by showing that, for identical-good instances as well as for two-value instances, MNW remains the unique additive welfarist rule that ensures EF1. Furthermore, we prove that the same holds for normalized instances with three or more agents—this extends a result of Eckart et al. [10, Theorem 6], which only handles additive welfarist rules defined by p-mean functions. For normalized instances with two agents, however, this characterization ceases to hold, and we provide a necessary condition and a sufficient condition for additive welfarist rules to guarantee EF1.

In Section 4, we consider integer-valued instances. For such instances, not only the MNW rule but also the MHW rule guarantees EF1 [16]. We shall attempt to characterize rules with this property. We show that for identical-good, binary, and two-value instances (which are also integer-valued), the respective functions defining the additive welfarist rules are characterized by Conditions 3, 4, and 5 respectively (see Definition 2.1). This allows much larger classes of functions beyond the MNW rule or the MHW rule. We provide examples and non-examples of modified logarithmic functions and modified harmonic functions defining the additive welfarist rules that guarantee EF1 in these classes of instances. For the larger class of all integer-valued instances, we give a necessary condition and a sufficient condition for such additive welfarist rules.

For all of our characterization results (Theorems 3.1, 3.5, 3.6, 4.1, 4.7, and 4.11), we show that if a function defining an additive welfarist rule guarantees EF1 for that particular class of instance for *some* number of agents n, then it also guarantees EF1 for that class for *every* number of agents n.

Our results are summarized in Table 1. The relationships between the different conditions are summarized in Figure 1, and the relationships between additive welfarist rules that guarantee EF1 for different classes of instances are illustrated in Figure 2. All omitted proofs can be found in the full version of our paper [8].

2 PRELIMINARIES

Let $N = \{1, \ldots, n\}$ be a set of $n \geq 2$ agents and $M = \{g_1, \ldots, g_m\}$ be a set of goods. Each agent $i \in N$ has a *utility function* $u_i : 2^M \to \mathbb{R}_{\geq 0}$ such that $u_i(S)$ is i's utility for a subset S of goods; we write $u_i(g)$ instead of $u_i(\{g\})$ for a single good $g \in M$. The utility function is *additive*, i.e., for each $S \subseteq M$, $u_i(S) = \sum_{g \in S} u_i(g)$. An *instance* consists of N, M, and $(u_i)_{i \in N}$.

[3] If f is not strictly increasing, then the allocation chosen by the additive welfarist rule may not even be PO. In fact, we show in our full paper [8] that even if f is *non-decreasing* but not *strictly increasing*, then there exists an identical-good normalized instance such that the additive welfarist rule with function f does *not* always choose an EF1 allocation. This means that it is desirable for f to be strictly increasing.

[4] Even the MNW rule does not always return EF1 for instances that are not positive-admitting, unless there are additional tie-breaking mechanisms [6].

An allocation $\mathcal{A} = (A_1, \dots, A_n)$ is an ordered partition of M into n bundles $A_1, \dots, A_n$ such that A_i is allocated to agent $i \in N$. An allocation is *envy-free up to one good (EF1)* if for all $i, j \in N$ with $A_j \neq \emptyset$, there exists a good $g \in A_j$ such that $u_i(A_i) \geq u_i(A_j \setminus \{g\})$.

We say that an instance is

- *integer-valued* if $u_i(g) \in \mathbb{Z}_{\geq 0}$ for all $i \in N$ and $g \in M$;
- *identical-good* if for each $i \in N$, there exists $a_i \in \mathbb{R}_{>0}$ such that $u_i(g) = a_i$ for all $g \in M$;
- *binary* if $u_i(g) \in \{0, 1\}$ for all $i \in N$ and $g \in M$;
- *two-value* if there exist distinct $a_1, a_2 \in \mathbb{R}_{\geq 0}$ such that $u_i(g) \in \{a_1, a_2\}$ for all $i \in N$ and $g \in M$;[5]
- *normalized* if $u_i(M) = u_j(M)$ for all $i, j \in N$;
- *positive-admitting* if there exists an allocation $(A_1, \dots, A_n)$ such that $u_i(A_i) > 0$ for all $i \in N$.

Let an instance be given. An *additive welfarist rule* is defined by a strictly increasing function $f : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$ such that the rule chooses an allocation $\mathcal{A} = (A_1, \dots, A_n)$ that maximizes $\sum_{i \in N} f(u_i(A_i))$; if there are multiple such allocations, the rule chooses one arbitrarily. Since f is strictly increasing, $f(x) = -\infty$ implies that $x = 0$, which means that $f(x) \in \mathbb{R}$ for all $x \in \mathbb{R}_{>0}$.

A function f is *strictly concave* if for any distinct $x, y \in \mathbb{R}_{\geq 0}$ and $\alpha \in (0, 1)$, we have $f(\alpha x + (1 - \alpha)y) > \alpha f(x) + (1 - \alpha) f(y)$. We use log to denote the natural logarithm, and define $\log 0$ as $-\infty$.

Definition 2.1. Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$ be a strictly increasing function. For each $k \in \mathbb{Z}_{\geq 0}$, define $\Delta_{f,k} : \mathbb{R}_{>0} \to \mathbb{R} \cup \{\infty\}$ such that $\Delta_{f,k}(x) = f((k+1)x) - f(kx)$. The function f is said to satisfy

- *Condition 1* if $\Delta_{f,k}(b) > \Delta_{f,k+1}(a)$ for all $k \in \mathbb{Z}_{\geq 0}$ and $a, b \in \mathbb{R}_{>0}$;
- *Condition 1a* if $\Delta_{f,k}$ is a constant function (on domain $\mathbb{R}_{>0}$) for every $k \in \mathbb{Z}_{>0}$;
- *Condition 2* if $f(a) + f(b) < f(c) + f(d)$ for all $a, b, c, d \in \mathbb{R}_{\geq 0}$ such that $\min\{a, b\} \leq \min\{c, d\}$ and $ab < cd$;
- *Condition 3* if $\Delta_{f,k}(b) > \Delta_{f,k+1}(a)$ for all $k \in \mathbb{Z}_{\geq 0}$ and $a, b \in \mathbb{Z}_{>0}$;
- *Condition 3a* if $\Delta_{f,\ell}(b) > \Delta_{f,k}(a)$ for all $k, \ell \in \mathbb{Z}_{\geq 0}$ and $a, b \in \mathbb{Z}_{>0}$ such that $\ell < k$;
- *Condition 3b* if $\Delta_{f,k}(1) > \Delta_{f,k+1}(a) > \Delta_{f,k+2}(1)$ for all $k \in \mathbb{Z}_{\geq 0}$ and $a \in \mathbb{Z}_{>0}$;
- *Condition 4* if $\Delta_{f,k}(1) > \Delta_{f,k+1}(1)$ for all $k \in \mathbb{Z}_{\geq 0}$;
- *Condition 5* if $f((\ell + 1)b + ra) - f(\ell b + ra) > \Delta_{f,k+1}(a) > \Delta_{f,k+2}(1)$ for all $a, b \in \mathbb{Z}_{>0}$ and $k, \ell, r \in \mathbb{Z}_{\geq 0}$ such that $a \geq b$ and $(k + 1)b > \ell b + ra$;
- *Condition 6a* if $f((k + 1)b - 1) - f(kb - 1) > \Delta_{f,k}(a)$ for all $k, a, b \in \mathbb{Z}_{>0}$;
- *Condition 6b* if $f(y + b) - f(y) > f(x + a) - f(x)$ for all $a, b \in \mathbb{Z}_{>0}$ and $x, y \in \mathbb{Z}_{\geq 0}$ such that $x/a \geq (y + 1)/b$.

Note that for $k \in \mathbb{Z}_{\geq 0}$ and $x \in \mathbb{R}_{>0}$, we have $f((k+1)x) > f(kx)$ since f is strictly increasing, so $\Delta_{f,k}$ is well-defined and positive. Moreover, we have $\Delta_{f,k}(x) = \infty$ if and only if $k = 0$ and $f(0) = -\infty$.

2.1 Examples of Additive Welfarist Rules

The *maximum Nash welfare (MNW) rule* is the additive welfarist rule defined by the function log. We consider variations of this rule by defining, for each $c \in \mathbb{R}_{\geq 0}$, the *modified logarithmic function* $\lambda_c : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$ such that $\lambda_c(x) = \log(x + c)$.

[5] Two-value instances have received attention in fair division, particularly in relation to the MNW rule [1, 2].

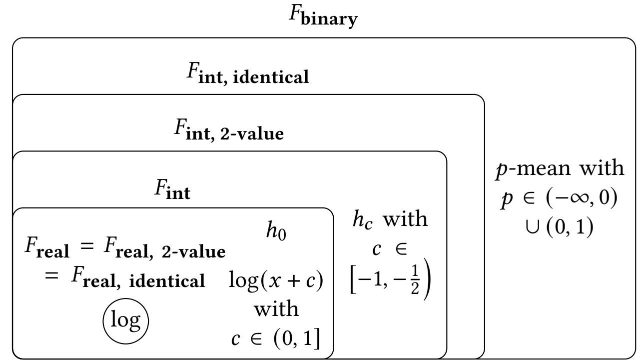

Figure 2: Venn diagram showing the relationships between additive welfarist rules that guarantee EF1 for different classes of instances. Each region, labeled by F with a subscript, represents the set of functions defining the additive welfarist rules that guarantee EF1 for the class of instances corresponding to the subscript. Some examples of these functions are given in the respective regions.

The *maximum harmonic welfare (MHW) rule* is the additive welfarist rule defined by the function $h_0(x) = \sum_{t=1}^{x} 1/t$. Note that this definition only makes sense when x is a non-negative integer; hence, we extend its definition to the non-negative real domain using the function $h_0(x) = \int_0^1 \frac{1-t^x}{1-t}\, \mathrm{d}t$ [12]. We also consider variations of this rule. We define, for each $c \geq -1$, the *modified harmonic number* $h_c : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$ such that[6]

$$h_{-1}(x) = \int_0^1 \frac{1 - t^{x-1}}{1 - t}\, \mathrm{d}t \quad \text{and} \quad h_c(x) = \int_0^1 \frac{t^c - t^{x+c}}{1 - t}\, \mathrm{d}t$$

for $c > -1$. When the domain of h_c is restricted to the set of non-negative integers, we can rewrite[7] the values as

$$h_{-1}(x) = \begin{cases} \sum_{t=1}^{x-1} \frac{1}{t} & \text{if } x \geq 1; \\ -\infty & \text{if } x = 0 \end{cases} \quad \text{and} \quad h_c(x) = \sum_{t=1}^{x} \frac{1}{t + c}$$

for $c > -1$ and all $x \in \mathbb{Z}_{\geq 0}$.

An *MNW allocation* (resp. *MHW allocation*) is an allocation chosen by the MNW rule (resp. MHW rule).

For $p \in \mathbb{R}$, the *(generalized) p-mean rule* is an additive welfarist rule defined by the function φ_p, where

$$\varphi_p(x) = \begin{cases} x^p & \text{if } p > 0; \\ \log x & \text{if } p = 0; \\ -x^p & \text{if } p < 0. \end{cases}$$

Hence, the MNW rule is also the 0-mean rule.

3 REAL-VALUED INSTANCES

We start with the general case where the utility of a good can be any (non-negative) real number. As mentioned earlier, the only

[6] This family of functions is defined for $c \in [-1, 0]$ in the domain of non-negative integers in previous work [16].

[7] We prove this equivalence in the full version of our paper [8].

additive welfarist rule that guarantees EF1 for all such instances is the MNW rule [19]. In this section, we consider the subclasses of *identical-good* instances (Section 3.1), *two-value* instances (Section 3.2), and *normalized* instances (Section 3.3). We demonstrate that the characterization of MNW continues to hold for the first two classes, as well as for the third class when there are three or more agents. To this end, we show that the only additive welfarist rule that ensures EF1 for these classes of instances is defined by the logarithmic function, which is equivalent to Conditions 1 and 1a.

3.1 Identical-Good Instances

Recall that identical-good instances have the property that each agent assigns the same utility to every good, although the utilities may be different for different agents. We state the characterization for such instances.

Theorem 3.1. *Let $n \geq 2$ be given, and let $f : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$ be a strictly increasing function continuous on $\mathbb{R}_{>0}$. Then, the following statements are equivalent:*

(a) For every positive-admitting identical-good instance with n agents, every allocation chosen by the additive welfarist rule with f is EF1.
(b) There exist constants $\alpha \in \mathbb{R}_{>0}$ and $\beta \in \mathbb{R}$ such that $f(x) = \alpha \log x + \beta$ for all $x \in \mathbb{R}_{\geq 0}$.

Note that the MNW rule is equivalent to the additive welfarist rule with a function that satisfies Theorem 3.1(b); therefore, the theorem essentially says that the only additive welfarist rule that guarantees EF1 for identical-good instances is the MNW rule.

We shall prove the theorem via a series of lemmas. We begin by showing that *Condition 1* is a necessary condition for a function f that satisfies the statement in Theorem 3.1(a).

Lemma 3.2. *Let $n \geq 2$ be given, and let $f : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$ be a strictly increasing function such that the statement in Theorem 3.1(a) holds. Then, f satisfies Condition 1.*

Proof. Let $k \in \mathbb{Z}_{\geq 0}$ and $a, b \in \mathbb{R}_{>0}$ be given. Consider an instance with n agents and $(k+1)n$ goods such that the utility of each good is a for agent 1 and b for the remaining agents. This instance is an identical-good instance; moreover, it is positive-admitting since the allocation where every agent receives $k+1$ goods gives positive utility to every agent.

Let $\mathcal{A} = (A_1, \ldots, A_n)$ be the allocation such that every agent receives $k+1$ goods. We have $\sum_{i \in N} f(u_i(A_i)) = f((k+1)a) + (n-1)f((k+1)b)$. Note that $\mathcal{A}$ is the only EF1 allocation; indeed, if some agent does not receive exactly $k+1$ goods, then some agent i receives fewer than $k+1$ goods while another agent j receives more than $k+1$ goods, so agent i envies agent j by at least two goods, thereby violating EF1. Let $\mathcal{B} = (B_1, \ldots, B_n)$ be the allocation such that agent 1 receives $k+2$ goods, agent 2 receives k goods, and every other agent receives $k+1$ goods. We have $\sum_{i \in N} f(u_i(B_i)) = f((k+2)a) + f(kb) + (n-2)f((k+1)b)$. Since $\mathcal{A}$ is the only EF1 allocation, the additive welfarist rule with f chooses (only) $\mathcal{A}$, and it holds that $\sum_{i \in N} f(u_i(A_i)) > \sum_{i \in N} f(u_i(B_i))$. Rearranging the terms, we get

$$f((k+1)b) - f(kb) > f((k+2)a) - f((k+1)a),$$

or equivalently, $\Delta_{f,k}(b) > \Delta_{f,k+1}(a)$. Since k, a, b were arbitrarily chosen, $\Delta_{f,k}(b) > \Delta_{f,k+1}(a)$ holds for all $k \in \mathbb{Z}_{\geq 0}$ and $a, b \in \mathbb{R}_{>0}$. Therefore, f satisfies Condition 1. □

Next, we show that Condition 1 is sufficient for the function f to satisfy Theorem 3.1(b). To this end, we prove in Lemmas 3.3 and 3.4 that Condition 1 implies Condition 1a, which in turn implies that f satisfies the statement in Theorem 3.1(b). The proofs are rather complex and involve careful analyses of functions; we outline their main steps here and leave the details to our full version [8].

Lemma 3.3. *Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$ be a strictly increasing function that satisfies Condition 1. Then, f satisfies Condition 1a.*

Proof sketch. Let $k \in \mathbb{Z}_{>0}$ be given. We show that $\sup \Delta_{f,k}$ and $\inf \Delta_{f,k}$ are both finite, and hence $d_{f,k} := \sup \Delta_{f,k} - \inf \Delta_{f,k}$ is well-defined and non-negative. We then prove that $d_{f,k} = 0$, so $\Delta_{f,k}$ is a constant function, and f satisfies Condition 1a. □

Lemma 3.4. *Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$ be a strictly increasing function continuous on $\mathbb{R}_{>0}$ that satisfies Condition 1a. Then, there exist constants $\alpha \in \mathbb{R}_{>0}$ and $\beta \in \mathbb{R}$ such that $f(x) = \alpha \log x + \beta$ for all $x \in \mathbb{R}_{\geq 0}$.*

Proof sketch. Define f_0 such that $f_0(x) = \alpha \log x + \beta$ where $\alpha = (f(2) - f(1))/\log 2 \in \mathbb{R}_{>0}$ and $\beta = f(1) \in \mathbb{R}$. We show that f and f_0 coincide on the set $\{2^t : t \text{ is rational}\}$. Since this set is dense in $\mathbb{R}_{>0}$ and f (and f_0) is continuous, f and f_0 coincide on $\mathbb{R}_{\geq 0}$. □

We remark that Suksompong [19, Lemma 2] proved a result similar to Lemma 3.4 but made the stronger assumption that the function f is *differentiable*. Our result only assumes *continuity*.

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. The implication (a) ⇒ (b) follows from Lemmas 3.2 to 3.4, while the implication (b) ⇒ (a) follows from the result of Caragiannis et al. [6]. □

Theorem 3.1 and Lemmas 3.2 to 3.4 imply that a continuous function having the logarithmic property is equivalent to the function satisfying Conditions 1 and 1a (separately). We view this as an interesting mathematical property in itself.

3.2 Two-Value Instances

We now consider two-value instances, which are instances with only two possible utilities for the goods. Note that the classes of identical-good instances and two-value instances do not contain each other. Indeed, an identical-good instance with $n \geq 3$ agents may be *n-value* rather than *two-value* since each of the n agents could assign a unique utility to the goods,[8] whereas a two-value instance allows an agent to have varying utilities for different goods.

Despite this incomparability, we show that the same characterization for identical-good instances also holds for two-value instances.

Theorem 3.5. *Let $n \geq 2$ be given, and let $f : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$ be a strictly increasing function continuous on $\mathbb{R}_{>0}$. Then, the following statements are equivalent:*

[8]However, for two agents, an identical-good instance is necessarily two-value.

(a) *For every positive-admitting two-value instance with n agents, every allocation chosen by the additive welfarist rule with f is EF1.*
(b) *There exist constants $\alpha \in \mathbb{R}_{>0}$ and $\beta \in \mathbb{R}$ such that $f(x) = \alpha \log x + \beta$ for all $x \in \mathbb{R}_{\geq 0}$.*

3.3 Normalized Instances

Next, we consider normalized instances, where agents assign the same utility to the entire set of goods M. We first consider the case of three or more agents. We show that just like for the cases of identical-good instances and two-value instances, the only additive welfarist rule that yields EF1 for this case is the MNW rule. This extends Theorem 6 of Eckart et al. [10], which only handles p-mean rules (as opposed to all additive welfarist rules).

Theorem 3.6. *Let $n \geq 3$ be given, and let $f : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$ be a strictly increasing function continuous on $\mathbb{R}_{>0}$. Then, the following statements are equivalent:*

(a) *For every positive-admitting normalized instance with n agents, every allocation chosen by the additive welfarist rule with f is EF1.*
(b) *There exist constants $\alpha \in \mathbb{R}_{>0}$ and $\beta \in \mathbb{R}$ such that $f(x) = \alpha \log x + \beta$ for all $x \in \mathbb{R}_{\geq 0}$.*

To prove Theorem 3.6, we cannot use the construction in the proof of Lemma 3.2 since the instance in that construction is not necessarily normalized. Instead, we show that the function defining the additive welfarist rule must satisfy *Condition 1a*. To this end, we augment the construction of Suksompong [19] by adding a highly valuable good and an extra agent who only values that good, so that the instance becomes normalized.

Lemma 3.7. *Let $n \geq 3$ be given, and let $f : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$ be a strictly increasing function continuous on $\mathbb{R}_{>0}$ such that the statement in Theorem 3.6(a) holds. Then, f satisfies Condition 1a.*

Proof. Assume, for the sake of contradiction, that f does not satisfy Condition 1a, i.e., there exist $k \in \mathbb{Z}_{>0}$ and $a, b \in \mathbb{R}_{>0}$ such that $f((k+1)a) - f(ka) > f((k+1)b) - f(kb)$. By the continuity of f, there exists $\epsilon \in (0, b)$ such that

$$f((k+1)a) - f(ka) > f((k+1)b - \epsilon) - f(kb - \epsilon). \qquad (1)$$

Consider an instance with n agents and $m = k(n-1) + 2$ goods. For ease of notation, let $c = \max\{kna, knb\}$, $N' = N \setminus \{1, n\}$, $g' = g_{m-1}$, $g'' = g_m$, and $M' = M \setminus \{g', g''\}$. The utilities of the goods are as follows.

- $u_1(g') = b - \epsilon$, $u_1(g'') = c - k(n-1)b - (b-\epsilon)$, and $u_1(g) = b$ for $g \in M'$.
- For $i \in N'$, $u_i(g') = 0$, $u_i(g'') = c - k(n-1)a$, and $u_i(g) = a$ for $g \in M'$.
- $u_n(g'') = c$, and $u_n(g) = u_n(g') = 0$ for $g \in M'$.

Note that $|N'| = n-2 \geq 1$, $|M'| = k(n-1) \geq |N'|$, the utility of each good is non-negative, and $u_i(M) = c$ for all $i \in N$. Therefore, this is a valid normalized instance; moreover, it is positive-admitting since the allocation where agent 1 receives g', agent n receives g'', and agent $i \in N'$ each receives at least one good from M' gives positive utility to every agent.

Let $\mathcal{A} = (A_1, \ldots, A_n)$ be an allocation chosen by the additive welfarist rule with f. We first show that $g' \in A_1$ and $A_n \subseteq \{g''\}$. If $g' \in A_j$ for some $j \in N \setminus \{1\}$, then giving g' to agent 1 increases $u_1(A_1)$ and does not change $u_j(A_j)$; the value of $\sum_{i \in N} f(u_i(A_i))$ increases since f is strictly increasing, which contradicts the assumption that $\mathcal{A}$ is chosen by the additive welfarist rule with f. Likewise, if $A_n \setminus \{g''\} \neq \emptyset$, then giving $A_n \setminus \{g''\}$ to agent 1 results in the same contradiction.

Next, we show that $A_n = \{g''\}$. Suppose on the contrary that $g'' \in A_j$ for some agent $j \in N \setminus \{n\}$. Let $\mathcal{A}' = (A'_1, \ldots, A'_n)$ be the allocation such that g'' is given instead to agent n, i.e., $A'_j = A_j \setminus \{g''\}$, $A'_n = \{g''\}$, and $A'_i = A_i$ for all $i \in N \setminus \{j, n\}$. Then,

$$\begin{aligned}
\sum_{i \in N} f(u_i(A'_i)) &= f(u_j(A'_j)) + f(u_n(A'_n)) + \sum_{i \in N \setminus \{j,n\}} f(u_i(A'_i)) \\
&\geq f(0) + f(c) + \sum_{i \in N \setminus \{j,n\}} f(u_i(A'_i)) \\
&= f(c) + f(0) + \sum_{i \in N \setminus \{j,n\}} f(u_i(A_i)) \\
&\geq f(u_j(A_j)) + f(u_n(A_n)) + \sum_{i \in N \setminus \{j,n\}} f(u_i(A_i)) \\
&= \sum_{i \in N} f(u_i(A_i)),
\end{aligned}$$

where the inequalities hold because $u_n(A'_n) = c$ and $u_n(A_n) = 0$. Now, at least one of the two inequalities must be strict because f is strictly increasing and we have $u_j(A'_j) > 0$ or $u_j(A_j) < c$. This shows that $\sum_{i \in N} f(u_i(A'_i)) > \sum_{i \in N} f(u_i(A_i))$. It follows that $\mathcal{A}$ is not chosen by the welfarist rule, a contradiction. Hence, $A_n = \{g''\}$.

Recall that $\mathcal{A}$ is EF1 since it is chosen by the additive welfarist rule with f. We show that every agent $i \in N \setminus \{n\}$ receives exactly k goods from M'. If agent 1 receives at most $k-1$ goods from M', then some agent $j \in N'$ receives at least $k+1$ goods from M'. Then, $u_1(A_1) \leq kb - \epsilon$ and $u_1(A_j \setminus \{g\}) \geq kb > u_1(A_1)$ for all $g \in A_j$, which shows that $\mathcal{A}$ is not EF1 for agent 1. Therefore, agent 1 receives at least k goods from M'. Likewise, if some agent $i \in N'$ receives at most $k-1$ goods from M', then some agent $j \in N \setminus \{i, n\}$ receives at least $k+1$ goods from M'. Then, $u_i(A_i) \leq (k-1)a$ and $u_i(A_j \setminus \{g\}) \geq ka > u_i(A_i)$ for all $g \in A_j$, which shows that $\mathcal{A}$ is not EF1 for agent i. Therefore, every agent $i \in N'$ receives at least k goods from M' as well. The only way for every agent in $N \setminus \{n\}$ to receive at least k goods from M' is when every agent in $N \setminus \{n\}$ receives *exactly* k goods from M'. Then, we have

$$\begin{aligned}
\sum_{i \in N} f(u_i(A_i)) &= f(u_1(A_1)) + \sum_{i \in N'} f(u_i(A_i)) + f(u_n(A_n)) \\
&= f((k+1)b - \epsilon) + (n-2)f(ka) + f(c).
\end{aligned}$$

Let $\mathcal{B} = (B_1, \ldots, B_n)$ be the allocation such that agent 1 receives g' and $k-1$ goods from M', agent 2 receives $k+1$ goods from M', each agent $i \in N' \setminus \{2\}$ receives k goods from M', and agent n receives g''. We have

$$\begin{aligned}
\sum_{i \in N} f(u_i(B_i)) &= f(u_1(B_1)) + \sum_{i \in N'} f(u_i(B_i)) + f(u_n(B_n)) \\
&= f(kb - \epsilon) + f((k+1)a) + (n-3)f(ka) + f(c).
\end{aligned}$$

Since $\mathcal{A}$ is chosen, we have $\sum_{i \in N} f(u_i(B_i)) \leq \sum_{i \in N} f(u_i(A_i))$. Rearranging the terms, we get

$$f((k+1)a) - f(ka) \leq f((k+1)b - \epsilon) - f(kb - \epsilon),$$

contradicting (1). Therefore, f satisfies Condition 1a. □

With this lemma in hand, we are ready to prove Theorem 3.6.

Proof of Theorem 3.6. The implication (a) ⇒ (b) follows from Lemmas 3.7 and 3.4, while the implication (b) ⇒ (a) follows from the result of Caragiannis et al. [6]. □

We now address the case of two agents. Eckart et al. [10] showed that in this case, the p-mean rule guarantees EF1 for all $p \leq 0$, which implies that there is a larger class of additive welfarist rules that guarantee EF1. Accordingly, our characterization in Theorem 3.6 does not work for two agents. The problem of finding a characterization for this case turns out to be rather challenging. We instead provide a necessary condition and a sufficient condition for the functions defining the additive welfarist rules that guarantee EF1 for normalized instances with two agents. Note that the sufficient condition in Proposition 3.9 is a generalization of the condition used by Eckart et al. [10, Lemma 3].

Proposition 3.8. *Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$ be a strictly increasing function continuous on $\mathbb{R}_{>0}$ such that for every positive-admitting normalized instance with two agents, every allocation chosen by the additive welfarist rule with f is EF1. Then, f is strictly concave.*

Proposition 3.9. *Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$ be a strictly increasing function that satisfies Condition 2. Then, for every positive-admitting normalized instance with two agents, every allocation chosen by the additive welfarist rule with f is EF1.*

We show in our full version [8] that beyond the p-mean rules for $p \leq 0$, there are (infinitely many) other additive welfarist rules that guarantee EF1 for normalized instances with two agents, and provide some insights on the extent to which the additive welfarist rules with the modified logarithmic function λ_c and the modified harmonic number h_c can ensure EF1 for such instances.

4 INTEGER-VALUED INSTANCES

In this section, we turn our attention to *integer-valued* instances, where the utility of each agent for each good must be a (non-negative) integer. For these instances, the MNW rule is no longer the unique additive welfarist rule that guarantees EF1: the MHW rule exhibits the same property [16]. We shall explore the conditions for rules to satisfy this property, and provide several examples of such rules.

4.1 Identical-Good Instances

We begin with the class of (integer-valued) identical-good instances. We show that the functions defining the additive welfarist rules that guarantee EF1 for such instances are precisely those that satisfy *Condition 3.* Perhaps unsurprisingly, Condition 3 is similar to Condition 1 (which is the condition corresponding to *real-valued* identical-good instances), with the difference being that the values of a and b in the conditions are positive real numbers in Condition 1 and positive integers in Condition 3.

Theorem 4.1. *Let $n \geq 2$ be given, and let $f : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$ be a strictly increasing function. Then, the following statements are equivalent:*

(a) For every positive-admitting integer-valued identical-good instance with n agents, every allocation chosen by the additive welfarist rule with f is EF1.
(b) f satisfies Condition 3.

Due to the similarity between Conditions 1 and 3, the proof for (a) ⇒ (b) in Theorem 4.1 follows similarly to that in the real-valued case (Lemma 3.2). To prove (b) ⇒ (a), we show that Condition 3 implies Condition 3a, which in turn implies (a).

Lemma 4.2. *Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$ be a strictly increasing function that satisfies Condition 3. Then, f satisfies Condition 3a.*

Proof. Let $k, \ell \in \mathbb{Z}_{\geq 0}$ and $a, b \in \mathbb{Z}_{>0}$ be given such that $\ell < k$. By applying Condition 3 repeatedly, we have $\Delta_{f,\ell}(b) > \Delta_{f,\ell+1}(a) > \cdots > \Delta_{f,k}(a)$, so f indeed satisfies Condition 3a. □

Lemma 4.3. *Let $n \geq 2$ be given, and let $f : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$ be a strictly increasing function that satisfies Condition 3a. Then, the statement in Theorem 4.1(a) holds.*

Proof of Theorem 4.1. The implication (a) ⇒ (b) follows similarly as Lemma 3.2, while the implication (b) ⇒ (a) follows from Lemmas 4.2 and 4.3. □

Since Conditions 1 and 1a are equivalent (see Theorem 3.1 and Lemmas 3.2 to 3.4), one may be tempted to think that there is a condition analogous to Condition 1a that Condition 3 is equivalent to, perhaps "$\Delta_{f,k}$ is a constant function (on domain $\mathbb{Z}_{>0}$) for each $k \in \mathbb{Z}_{>0}$". However, this condition is too strong, and is not implied by Condition 3.[9] Instead, we show that Condition 3 is equivalent to Condition 3b.

Proposition 4.4. *Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$ be a strictly increasing function. Then, the following statements are equivalent:*

(a) f satisfies Condition 3.
(b) f satisfies Condition 3a.
(c) f satisfies Condition 3b.

We next give examples of additive welfarist rules that guarantee EF1 for all integer-valued identical-good instances. We study two families of functions: modified logarithmic functions λ_c and modified harmonic numbers h_c. By leveraging Proposition 4.4 and Theorem 4.1, we can determine which of these additive welfarist rules ensure EF1 for integer-valued identical-good instances.

Proposition 4.5. *The function λ_c satisfies Condition 3b if and only if $0 \leq c \leq 1$.*

Proposition 4.6. *The function h_c satisfies Condition 3b if and only if $-1 \leq c \leq 1/\log 2 - 1$ (≈ 0.443).*

Propositions 4.5 and 4.6 show that beyond the MNW rule and the MHW rule, many other additive welfarist rules guarantee EF1 for integer-valued identical-good instances. We provide more discussion in the full version of our paper [8].

[9] Indeed, the MHW rule—which corresponds to the function h_0—guarantees EF1 for integer-valued instances [16], but $\Delta_{h_0,1}$ is not a constant function as $\Delta_{h_0,1}(1) = h_0(2) - h_0(1) = 1/2 < 7/12 = h_0(4) - h_0(2) = \Delta_{h_0,1}(2)$.

4.2 Binary Instances

Next, we consider binary instances, where each good is worth either 0 or 1 to each agent. We show that the functions defining the additive welfarist rules that guarantee EF1 for all binary instances are those that satisfy *Condition 4*, which is equivalent to the function being strictly concave in the domain of non-negative integers.

Theorem 4.7. *Let $n \geq 2$ be given, and let $f : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$ be a strictly increasing function. Then, the following statements are equivalent:*

(a) For every positive-admitting binary instance with n agents, every allocation chosen by the additive welfarist rule with f is EF1.
(b) f satisfies Condition 4.

Proof. (a) $\Rightarrow$ (b): Let $k \in \mathbb{Z}_{\geq 0}$. Consider an instance with n agents and $2k + n$ goods, and let $M' = \{g_1, \ldots, g_{2k+2}\}$. The utilities of the goods are as follows.

- For $i \in \{1, 2\}$, $u_i(g) = 1$ for $g \in M'$, and $u_i(g) = 0$ otherwise.
- For $i \in \{3, \ldots, n\}$, $u_i(g_{2k+i}) = 1$, and $u_i(g) = 0$ for all other $g \in M$.

This instance is a binary instance; moreover, it is positive-admitting since any allocation where each agent $i \in N$ receives g_{2k+i} gives positive utility to every agent.

Let $\mathcal{A} = (A_1, \ldots, A_n)$ be an allocation chosen by the additive welfarist rule with f. Each good must be allocated to an agent who has utility 1 for the good; otherwise, transferring such a good to an agent who has utility 1 for the good increases the value of $\sum_{i \in N} f(u_i(A_i))$, contradicting the assumption that $\mathcal{A}$ is chosen by the additive welfarist rule with f. Accordingly, the goods in M' are allocated to agent 1 and 2, and for each $i \in \{3, \ldots, n\}$, g_{2k+i} is allocated to agent i.

Recall that $\mathcal{A}$ is EF1 since it is chosen by the additive welfarist rule with f. We claim that agent 1 and 2 each receives exactly $k + 1$ goods from M'. If not, then one of them receives at most k goods from M', and will envy the other agent (who receives at least $k + 2$ goods from M') by more than one good, making the allocation not EF1 and contradicting our assumption. Then, we have $\sum_{i \in N} f(u_i(A_i)) = 2f(k + 1) + (n - 2)f(1)$.

Let $\mathcal{B} = (B_1, \ldots, B_n)$ be the allocation such that agent 1 receives k goods from M', agent 2 receives $k+2$ goods from M', and for each $i \in \{3, \ldots, n\}$, agent i receives g_{2k+i}. We have $\sum_{i \in N} f(u_i(B_i)) = f(k) + f(k + 2) + (n - 2)f(1)$. Note that $\mathcal{B}$ is not EF1, and cannot be chosen by the additive welfarist rule with f. Therefore, we have $\sum_{i \in N} f(u_i(A_i)) > \sum_{i \in N} f(u_i(B_i))$. Rearranging the terms, we get

$$f(k + 1) - f(k) > f(k + 2) - f(k + 1),$$

or equivalently, $\Delta_{f,k}(1) > \Delta_{f,k+1}(1)$. Since $k \in \mathbb{Z}_{\geq 0}$ was arbitrarily chosen, $\Delta_{f,k}(1) > \Delta_{f,k+1}(1)$ holds for all $k \in \mathbb{Z}_{\geq 0}$. Therefore, f satisfies Condition 4.

(b) $\Rightarrow$ (a): Let a positive-admitting binary instance with n agents be given, and let $\mathcal{A} = (A_1, \ldots, A_n)$ be an allocation chosen by the additive welfarist rule with f. Assume, for the sake of contradiction, that $\mathcal{A}$ is not EF1. Then, there exist $i, j \in N$ such that agent i envies agent j by more than one good, i.e., $u_i(A_i) < u_i(A_j \setminus \{g\})$ for all $g \in A_j$. Note that we must have $u_i(g) \leq u_j(g)$ for all $g \in A_j$. Indeed, otherwise we have $u_i(g) = 1$ and $u_j(g) = 0$, and transferring g from agent j's bundle to agent i's bundle increases $f(u_i(A_i))$ and does not decrease $f(u_j(A_j))$, thereby increasing $\sum_{k \in N} f(u_k(A_k))$ and contradicting the assumption that $\mathcal{A}$ is chosen by the additive welfarist rule with f. Moreover, since $u_i(A_j) > 0$, there exists $g' \in A_j$ such that $u_i(g') = 1$ (and hence $u_j(g') = 1$).

Let $\mathcal{B} = (B_1, \ldots, B_n)$ be the same allocation as $\mathcal{A}$ except that g' is transferred from agent j's bundle to agent i's bundle, i.e., $B_i = A_i \cup \{g'\}$, $B_j = A_j \setminus \{g'\}$, and $B_k = A_k$ for all $k \in N \setminus \{i, j\}$. Note that $u_i(A_i) < u_i(A_j \setminus \{g'\}) = u_i(B_j)$, and that $u_i(B_j) \leq u_j(B_j)$ since $u_i(g) \leq u_j(g)$ for all $g \in B_j$. Therefore, $u_i(A_i) < u_j(B_j)$. By Condition 4, it holds that $\Delta_{f,u_i(A_i)}(1) > \cdots > \Delta_{f,u_j(B_j)}(1)$. On the other hand, we have $\sum_{k \in N} f(u_k(B_k)) = f(u_i(B_i)) + f(u_j(B_j)) + \sum_{k \in N \setminus \{i,j\}} f(u_k(A_k))$. Since $\mathcal{A}$ is chosen by the additive welfarist rule with f, we have $\sum_{k \in N} f(u_k(B_k)) \leq \sum_{k \in N} f(u_k(A_k))$. Rearranging the terms, we get

$$f(u_i(B_i)) - f(u_i(A_i)) \leq f(u_j(A_j)) - f(u_j(B_j)),$$

or equivalently, $\Delta_{f,u_i(A_i)}(1) \leq \Delta_{f,u_j(B_j)}(1)$, a contradiction. Therefore, $\mathcal{A}$ is EF1. □

Condition 4 is weaker than Condition 3. Indeed, if f satisfies Condition 3, then it satisfies Condition 4 by definition. However, the converse is not true: the function $f(x) = \sqrt{x}$ satisfies Condition 4, but it does not satisfy Condition 3 since $\Delta_{f,0}(1) = 1$ is not greater than $\Delta_{f,1}(6) \approx 1.01$. This means that each additive welfarist rule that guarantees EF1 for every positive-admitting integer-valued identical-good instance also guarantees EF1 for every positive-admitting binary instance, but the converse does not hold.

Proposition 4.8. *Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$ be a strictly increasing function. If f satisfies Condition 3, then it satisfies Condition 4.*

We mentioned earlier that Condition 4 is equivalent to the function being strictly concave in the domain of non-negative integers. On the other hand, the function φ_p defining the p-mean rule is strictly concave if and only if $p < 1$. We prove that, indeed, φ_p satisfies Condition 4 exactly when $p < 1$.

Proposition 4.9. *The function φ_p satisfies Condition 4 if and only if $p < 1$.*

4.3 Two-Value Instances

We now consider integer-valued instances which are two-value, a strict generalization of binary instances. Recall from Section 3.2 that the classes of identical-good instances and two-value instances are not subclasses of each other (except when $n = 2$). Therefore, the techniques used in Section 4.1 cannot be used in this section.[10]

We show that the characterization for integer-valued two-value instances is given by *Condition 5*. Before we state the characterization, we examine the relationship between Conditions 3 and 5.

Proposition 4.10. *Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$ be a strictly increasing function.*

(i) If f satisfies Condition 5, then f satisfies Condition 3.
(ii) If f satisfies Condition 3 and the property that for each $k \in \mathbb{Z}_{>0}$, the function $\Delta_{f,k}$ is either non-decreasing or non-increasing on the domain of positive integers, then f satisfies Condition 5.

[10]In particular, the proof that Condition 3a is a sufficient condition for the identical-good case (Lemma 4.3) does not apply to the two-value case.

We now state the characterization for integer-valued two-value instances.

Theorem 4.11. *Let $n \geq 2$ be given, and let $f : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$ be a strictly increasing function. Then, the following statements are equivalent:*

(a) *For every positive-admitting integer-valued two-value instance with n agents, every allocation chosen by the additive welfarist rule with f is EF1.*
(b) *f satisfies Condition 5.*

Proposition 4.10 says that Condition 5 is stronger than Condition 3. Therefore, Theorem 4.11 implies that every additive welfarist rule that guarantees EF1 for every positive-admitting integer-valued two-value instance also guarantees EF1 for every positive-admitting integer-valued identical-good instance, even though these two classes of instances are not subclasses of each other.

Next, we give examples of additive welfarist rules that ensure EF1 for all integer-valued two-value instances. We use Propositions 4.5, 4.6, and 4.10 to prove these results.

Proposition 4.12. *The function λ_c satisfies Condition 5 if and only if $0 \leq c \leq 1$.*

Proposition 4.13. *The function h_c satisfies Condition 5 if and only if $-1 \leq c \leq 1/\log 2 - 1$ (≈ 0.443).*

4.4 General Integer-Valued Instances

Finally, we consider the class of all integer-valued instances. It turns out to be challenging to find an exact characterization for this class of instances. Instead, we provide a necessary condition in the form of Condition 6a and a sufficient condition in the form of Condition 6b.

Proposition 4.14. *Let $n \geq 2$ be given, and let $f : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$ be a strictly increasing function such that for every positive-admitting integer-valued instance with n agents, every allocation chosen by the additive welfarist rule with f is EF1. Then, f satisfies Condition 6a.*

Proposition 4.15. *Let $n \geq 2$ be given, and let $f : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$ be a strictly increasing function that satisfies Condition 6b. Then, for every positive-admitting integer-valued instance with n agents, every allocation chosen by the additive welfarist rule with f is EF1.*

We demonstrate that the necessary condition given as Condition 6a is fairly tight, in the sense that it is stronger than Condition 5, which characterizes the functions defining the additive welfarist rules that guarantee EF1 for integer-valued two-value instances.

Proposition 4.16. *Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$ be a strictly increasing function. If f satisfies Condition 6a, then it satisfies Condition 5.*

We now present some examples. We show that just like for the subclasses of integer-valued identical-good instances (Proposition 4.5), binary instances (Proposition 4.8), and integer-valued two-value instances (Proposition 4.12), the additive welfarist rule with function λ_c for $0 \leq c \leq 1$ guarantees EF1 for all integer-valued instances.

Proposition 4.17. *The function λ_c satisfies Condition 6b if and only if $0 \leq c \leq 1$.*

On the other hand, leveraging Condition 6a, we show that the additive welfarist rule with function h_c for $-1 \leq c < -1/2$ does *not* guarantee EF1 for the class of all integer-valued instances. This stands in contrast to integer-valued identical-good instances (Proposition 4.6), binary instances (Proposition 4.8), and integer-valued two-value instances (Proposition 4.13).

Proposition 4.18. *The function h_c does not satisfy Condition 6a when $-1 \leq c < -1/2$.*

5 CONCLUSION

In this paper, we have characterized additive welfarist rules that guarantee EF1 allocations for various classes of instances. In the real-valued case, we strengthened the result of Suksompong [19] by showing that only the maximum Nash welfare (MNW) rule ensures EF1 even for the most restricted class of identical-good instances. This indicates that the unique fairness of MNW stems from its *scale-invariance*,[11] since for normalized identical-good instances, an additive welfarist rule with any strictly concave function f guarantees EF1. On the other hand, in the practically important case where all values are integers, we demonstrated that even for the most general setting with no additional restrictions, there is a wide range of additive welfarist rules that always return EF1 allocations.

Since we have established the existence of several alternatives that perform as well as MNW in terms of guaranteeing EF1 allocations for integer-valued instances, a natural follow-up would be to compare these rules with respect to other measures, both to MNW and to one another. As illustrated in Example 1.1, some additive welfarist rules may produce more preferable allocations for certain instances than others. It would be interesting to formalize this observation, for example, by comparing their prices in terms of social welfare, in the same vein as the "price of fairness" [3, 7, 14].

Other possible future directions include completely characterizing additive welfarist rules for normalized instances with two agents (Section 3.3) and for general integer-valued instances (Section 4.4) by tightening their respective necessary and sufficient conditions. Obtaining characterizations of welfarist rules that are not necessarily additive is also a possible extension of our work. Beyond EF1, one could consider other fairness notions from the literature, such as proportionality up to one good (PROP1). Finally, another intriguing avenue is to study the problem of computing (or approximating) an allocation produced by various additive welfarist rules, as has been done extensively for the MNW rule [1, 2, 9, 13].

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[11] That is, scaling the utility function of an agent by a constant factor does not change the outcome of the MNW rule.

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