Metric Hedonic Games on the Line

Merlin de la Haye
Hasso Plattner Institute
Potsdam, Germany
Merlin.delaHaye@hpi.de

Pascal Lenzner
University of Augsburg
Augsburg, Germany
pascal.lenzner@uni-a.de

Farehe Soheil
Hasso Plattner Institute
Potsdam, Germany
Farehe.Soheil@hpi.de

Marcus Wunderlich
University of Augsburg
Augsburg, Germany
marcus.wunderlich@uni-a.de

ABSTRACT

Hedonic games are fundamental models for investigating the formation of coalitions among a set of strategic agents, where every agent has a certain utility for every possible coalition of agents it can be part of. To avoid the intractability of defining exponentially many utilities for all possible coalitions, many variants with succinct representations of the agents' utility functions have been devised and analyzed, e.g., modified fractional hedonic games by Monaco et al. [36]. We extend this by studying a novel succinct variant that is related to modified fractional hedonic games. In our model, each agent has a fixed type-value and an agent's cost for some given coalition is based on the differences between its value and those of the other members of its coalition. This allows to model natural situations like athletes forming training groups with similar performance levels or voters that partition themselves along a political spectrum.

In particular, we investigate natural variants where an agent's cost is defined by distance thresholds, or by the maximum or average value difference to the other agents in its coalition. For these settings, we study the existence of stable coalition structures, their properties, and their quality in terms of the price of anarchy and the price of stability. Further, we investigate the impact of limiting the maximum number of coalitions. Despite the simple setting with metric distances on a line, we uncover a rich landscape of models, partially with counter-intuitive behavior. Also, our focus on both swap stability and jump stability allows us to study the influence of fixing the number and the size of the coalitions. Overall, we find that stable coalition structures always exist but that their properties and quality can vary widely.

KEYWORDS

Game Theory; Coalition Formation; Hedonic Games; Clustering

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1 INTRODUCTION

Consider athletes preparing for a marathon who want to form running groups that are supervised by coaches. Each athlete has an individual performance level (e.g. their pace) and wants to be in a group with other athletes who are on a similar level. The utility of an athlete then only depends on the comparison between their individual performance level and those of the other athletes in their running group. As there is only a limited number of coaches available, the maximum number of running groups is restricted.

This setting belongs to the research field of agent-based coalition formation that has been widely studied in the last decades, starting with the work of Miller [35] on coalition formation games and later the seminal concept of hedonic games studied by Bogomolnaia and Jackson [14], Drèze and Greenberg [23] and Banerjee et al. [10]; see also the surveys by Aziz and Savani [9] and Hajduková [28].

The key feature of hedonic games is that an agent's utility depends only on the members of its coalition and not on external factors such as the composition or size of other coalitions. In our athletics example, the training quality of an individual athlete only depends on the other athletes running in the same group with them.

In their full generality, hedonic games are hard to describe and analyze due to the existence of exponentially many possible coalitions and valuations. However, many natural coalition formation settings can be captured by hedonic games that have a succinct representation. Classical examples are additively separable hedonic games [14], and fractional hedonic games [6, 36], where agents value each other individually and their utility in a coalition is the sum or average over all the valuations of agents in their coalition.

Numerous succinct variants have been proposed; see the survey by Kerkmann et al. [30]. However, none of them closely captures our setting, in which agents have fixed real-valued types and derive their utility from the distances between these types.

In our model, instead of considering utilities, we take the equivalent cost-based approach and focus on three natural cost functions that depend only on the type differences within a coalition:

- Average: the cost of an agent is the average type difference to the other agents (without self-effect),
- Maximum: the cost of an agent is the maximum type difference to the other agents,
- Cutoff: the cost of an agent is the number of other agents who have a type difference larger than a given threshold λ.

See Example 1.1 for an illustration of the cost functions. These cost functions serve as a proof of concept for capturing natural agent

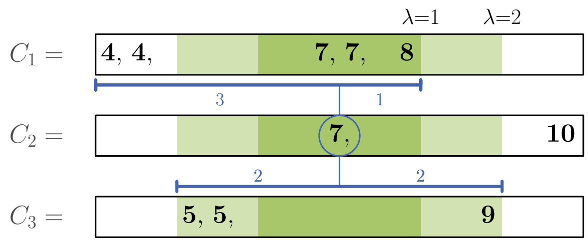

Figure 1: Coalition structure $\mathcal{C} = \{C_1, C_2, C_3\}$ with $n = 10$ agents and $k = 3$ coalitions. Bold black numbers represent agents with this value. Green areas are the intervals with radius $\lambda = 1$ (green) and $\lambda = 2$ (light green) around the value 7. The blue lines show the maximum distance of the circled agent to both sides for coalitions C_1 and C_3.

behavior. Moreover, they have been recently used in a closely related agent-based model investigating residential segregation [11].

Example 1.1. Consider the game instance with coalition structure, i.e., a partition of the agents into disjoint coalitions, $\mathcal{C} = \{C_1, C_2, C_3\}$, shown in Figure 1. In this example the circled agent with value 7, called agent x, is currently in its worst coalition C_2 (for all three cost functions) but would prefer to join either coalition C_1 or C_3, depending on the cost function (and on λ for Cutoff). With the Average cost function, the agents in coalitions C_1 and C_3 have an average distance of 1.4 and 2 to agent x, respectively. Thus, agent x prefers C_1 over C_3 in this setting. With the Maximum cost function, on the other hand, agent x prefers coalition C_3 over C_1, because the largest distance from x to another agent is smaller in C_3 (distance 2) than in C_1 (distance 3). For the Cutoff cost function, agent x's preference might depend on the threshold value λ we set. When $\lambda = 2$, all agents in coalition C_3 are within the threshold distance to agent x, while some agents in C_1 are outside. Thus, agent x prefers C_3 over C_1 when $\lambda = 2$. However, with a smaller threshold of $\lambda = 1$, agent x slightly prefers C_1 over C_3. See Section 1.1 for the exact definition of the cost functions and the full version of this paper [21] for more details. ◁

For our coalition formation problem with three natural cost functions and possibly a fixed number of coalitions k, we establish the existence of stable coalition structures. Furthermore, we study structural properties of both stable and socially optimal states and analyze the quality of the stable states.

1.1 Model

Agents and Coalition Structures. In our model there are n agents denoted by $[n] \coloneqq \{1, 2, \dots, n\}$, each having a fixed real value $v_i \in \mathbb{R}$ (for all agents $i \in [n]$). We assume, wlog that the values are sorted in non-decreasing order, i.e., $L = v_1 \leq v_2 \leq \dots \leq v_n = H$. Multiple agents may share the same value, which we call *repetition*.

Every subset of the agents is called a *coalition*. The coalition $[n]$ is called the *grand coalition*, while a coalition $\{i\} \subseteq [n]$ consisting of a single agent $i \in [n]$ is called a *singleton coalition*. A set of coalitions $\mathcal{C}$ is called a *coalition structure* if every agent is part of exactly one coalition in $\mathcal{C}$. For explicit coalition structures, we often represent an agent by its value. If a value $v \in \mathbb{R}$ or an entire coalition $\{a, b, c\}$ occurs $x \in \mathbb{N}$ times, we denote this by

$$\overbrace{v \dots v}^{x}, \text{ resp. } \overbrace{\{a, b, c\}}^{x}.$$

In this paper, we only focus on coalition structures consisting of at most $k \leq n$ many coalitions, and write $\mathcal{C} \coloneqq \{C_i\}_{i \in [k]}$, where $k \in \mathbb{N}$ is a model parameter. Given a coalition structure $\mathcal{C}$, we denote the coalition of agent $i \in [n]$ as $\mathcal{C}(i)$. We say that a coalition structure $\mathcal{C}$ (with possibly empty coalitions) is *sorted*, if for all agents i, j in some coalition $C \in \mathcal{C}$, it holds that all agents with a value between v_i and v_j are also part of the same coalition C. In a sorted coalition structure, the coalitions can be sorted by the value of their agents. Then, w.l.o.g., we name the coalitions to be ascending with respect to the value of any agents in each coalition, with C_1 being the coalition with the agents with lowest value, etc.

Cost Functions. The *distance* $\mathrm{d}(i, j)$ between two agents $i, j \in [n]$ is defined as the Euclidean distance between the respective values of the agents, i.e., $|v_i - v_j|$. To calculate the cost of an agent i, there are many ways to aggregate the distances between it and a set of agents. In this paper, we explore three natural cost functions (average distance, maximum distance and cutoff), where for agent i, we aggregate its distances to *all other agents in its current coalition* $\mathcal{C}(i)$ while calculating its cost. Such cost functions are called *without self-effect*. Because of this, singleton coalitions need to be treated separately: We differentiate between the *happy in isolation (HIS)* and the *unhappy in isolation (UIS)* case, where the cost of an agent $i \in [n]$ in a singleton coalition $\{i\}$ is zero or infinity, respectively.

For coalitions with at least two agents, we define the three cost functions as follows. The Average cost function

$$\mathrm{cost}_{\mathrm{AVG}}(i, C) \coloneqq \frac{1}{|C \setminus \{i\}|} \sum_{j \in C \setminus \{i\}} \mathrm{d}(v_i, v_j)$$

takes the average distance to all other agents. The Max cost function

$$\mathrm{cost}_{\mathrm{MAX}}(i, C) \coloneqq \max_{j \in C \setminus \{i\}} \mathrm{d}(v_i, v_j)$$

takes the maximum distance to all other agents in the coalition. To define the Cutoff cost function, we need an additional model parameter $\lambda > 0$. Based on that, we partition the coalition C into *friends* and *enemies* of agent i. The set of friends $N_i^+(C)$ of agent i in coalition C is the set of all other agents $j \in C \setminus \{i\}$ whose distance to agent i is at most λ. All other agents in coalition C are the *enemies* $N_i^-(C) \coloneqq C \setminus (N_i^+(C) \cup \{i\})$ of agent i. The Cutoff cost function with respect to $\lambda > 0$ is then

$$\mathrm{cost}_{\mathrm{CUT}-\lambda}(i, C) \coloneqq \frac{|N_i^-(C)|}{|C \setminus \{i\}|},$$

i.e., the fraction of enemies of agent i in coalition C among all other agents in the coalition C. When it is clear from the context, we omit the subscript of the cost functions.

Stability Concepts. Given a coalition structure $\mathcal{C}$, an agent i has an *improving move* if the cost of agent i in its current coalition $C \in \mathcal{C}$ is strictly higher than its cost in some other coalition $C' \in \mathcal{C} \setminus \{C\}$. A coalition structure is called *jump stable* or a *jump equilibrium* if no agent has an improving move. We refer to this as *jump stability*[1].

[1] In some related works this is called *Nash stability*.

Similarly, given a coalition structure $C \coloneqq \{C_i\}_{i\in[k]}$, there is an *improving swap* between agents i and j if both can strictly decrease their cost by swapping their coalitions, i.e., both leaving their current coalition and joining the other coalition. A coalition structure is called *swap stable* if there is no improving swap.

Terminology. We use the scheme COST-ST-XIS for naming combinations of cost functions and stability concepts, where

- COST indicates the cost function, which is either Avg (for Average), Max (for Maximum) or Cutoff_λ (for the Cutoff cost function with parameter λ),
- ST indicates the stability concept, which is either Jump for jump stability or Swap for swap stability,
- XIS indicates whether we are in the happy in isolation (HIS) or unhappy in isolation (UIS) setting.

Note that, when considering swap stability, the sizes of the coalition cannot change. Therefore, singleton coalitions stay as they are. Moreover, every agent in a singleton coalition in the initial coalition structure cannot leave, since either it does not want to (HIS) or no other agent wants to swap into a singleton coalition (UIS). As singletons thus do not contribute to any game dynamic in the swap setting, we will only consider games without singletons and omit the UIS and HIS flag in the terminology for Swap games.

All game instances are defined by the number of agents n, their values $(v_i)_{i\in[n]}$ and the maximum number of allowed coalitions, k. Indeed, every instance of a Jump game is defined by the tuple $(n, (v_i)_{i\in[n]}, k)$. For Swap games, we additionally need to fix the sizes of the coalitions, i.e., every instance of a Swap game is defined by the tuple $(n, (v_i)_{i\in[n]}, k, (k_i)_{i\in[k]})$.

Quality of Equilibria. The quality of a coalition structure C is measured by the *social cost*, which we define as the sum of costs over all agent's cost in the game, i.e., $\text{SC}(C) \coloneqq \sum_{i\in[n]} \text{cost}(i, C(i))$. A coalition structure with the smallest social cost is called an *optimum* $\text{OPT}(\mathcal{I})$ of a game instance $\mathcal{I}$. The quality of equilibria is measured by the Price of Anarchy (PoA) and the Price of Stability (PoS) [5, 31]. Let $\mathcal{G}$ be a set of game instances and EQ the set of equilibria (given some stability concept). Then the PoA (PoS) is defined to be the worst ratio between the cost of the worst (best) equilibrium and the optimum cost over all instances of $\mathcal{G}$, i.e.,

$$\text{PoA}(\mathcal{G}) \coloneqq \sup_{\mathcal{I}\in\mathcal{G}} \sup_{C\in\text{EQ}(\mathcal{I})} \frac{\text{SC}(C)}{\text{OPT}(\mathcal{I})},$$

$$\text{PoS}(\mathcal{G}) \coloneqq \sup_{\mathcal{I}\in\mathcal{G}} \inf_{C\in\text{EQ}(\mathcal{I})} \frac{\text{SC}(C)}{\text{OPT}(\mathcal{I})}.$$

Regarding game dynamics, a game has the *finite improvement property* (FIP), if any sequence of improving moves is finite, i.e., eventually must end in an equilibrium. We will use *ordinal potential functions* [37] to prove this property in a setting, or show the existence of an *improving response cycle* (IRC) to disprove it.

To discuss the PoS for Cutoff_λ-Jump games, we use the technical concept of a λ-block. Given some $\lambda > 0$, a λ*-block* is an interval $I \subseteq \mathbb{R}$ of size $|I| = \lambda$. A (multi-)set of values V is said to be *covered* by a set $\mathcal{B}$ of λ-blocks, if for every value $v \in V$ there is at least one λ-block $B \in \mathcal{B}$ that includes the values v, i.e., $v \in B$. We call a Cutoff_λ-Jump game instance $(n, (v_i)_{i\in[n]}, k)$ *nice* if the set of values $\{v_i\}_{i\in[n]}$ can be covered by a set of at most k λ-blocks.

1.2 Related Work

Given the breadth of variants of hedonic games, we focus our discussion on models with *cardinal*-based comparisons, where each agent has a valuation function scoring every coalition and prefers the higher (lower) valued coalition in the utility (cost) variant. However, some models based on ordinal comparisons employ ideas similar to the cost functions we study but their results cannot be applied to our model. Examples include: hedonic games with $\mathcal{W}$-preferences [8, 16, 17], where coalitions are compared based on the worst-ranked agent (similar to our Max cost function), and hedonic games with friends-and-enemies preferences [22, 32, 39], where each agent partitions others into friends and enemies (similar to our Cutoff cost function) and compares coalitions based on the number of friends and enemies they include.

The most prominent cardinal-based hedonic games with succinct representations are *additively separable hedonic games (ASHGs)* introduced by Bogomolnaia and Jackson [14] and *fractional hedonic games (FHGs)* introduced by Aziz et al. [6]. In both games, every agent i has a valuation $v_i(j) \in \mathbb{R}$ for every agent j in the game. In ASHGs, the utility of an agent is the sum of these valuations over all agents in its coalition, while in FHGs, the agents take the average of the values in its coalition. Recently, Monaco et al. [36] studied a variant of FHGs called *modified FHGs* where every agent takes the average only over all *other* agents' valuations, excluding its own (without self-effect). Indeed, the Avg-Jump-UIS and Cutoff_λ-Jump-UIS games can be seen as cost-based variants of these modified FHGs provided that the number of coalitions is not restricted. For the Cutoff_λ games, the valuations are additionally in $\{0, 1\}$, which is why they are a subclass of so called *unweighted* modified FHGs. Also, Monaco et al. [36] showed that in modified FHGs with non-negative and symmetric valuations, there is always a jump stable outcome (the grand coalition) and the PoA and PoS is linear in n. However, these results do not hold for our cost-based variants; especially when restricting the number of coalitions.

Several hedonic game variants use some notion of distance in their models. Rey and Rey [38] use the distance between an agent's ordinal preference list and its coalition to determine their valuation of that coalition. Further, some variants of FHGs position the agents on an unweighted host graph G and use the distance in G to get the valuations between agents, i.e., *distance hedonic games* (DHGs) by Flammini et al. [26] and *social distance games* (SDGs) by Brânzei and Larson [15]. In social distance games, the valuation of an agent i for an agent $j \neq i$ in a coalition C is $\frac{1}{d_{G[C]}(i,j)}$, where $d_{G[C]}(i, j)$ is the distance from node i to node j in the subgraph of G induced by the node-set C. Distance hedonic games (DHGs) generalize SDGs by adding a scoring vector α mapping each possible distance $d_{G[C]}(i, j)$ to a real number. Some instances of our Avg-Jump game can be modeled as DHGs. Here, agents have consecutive natural numbers as fixed values and the matching DHG instance is played on a path graph with a decreasing scoring vector α. To the best of our knowledge, there no results are known for this case.

Only a few models consider distances between points embedded in the Euclidean space. In *hedonic clustering games* introduced by Feldman et al. [25], coalitions are interpreted as clusters, and each agent's cost is defined by its distance to the cluster center or median. Also, they examine the "Fixed Clustering" setting, in which the

	Average			Cutoff$_\lambda$			Max		
	Swap	J-HIS	J-UIS	Swap	J-HIS	J-UIS	Swap	J-HIS	J-UIS
EQ Existence	✓✓C2.2	✓T2.4 [34]	✓GC	✓✓C2.2	✓T3.6	✓GC	✓✓T1 [11]	✓✓L3.2	
EQ sorted	∀∃ L2.3	∀∃ T2.4 [34]	∀∃ GC	∀∃ L2.3	∀∃ T3.6	∀∃ GC	∀∃ T2.3	∀∃ T3.6	∀∃ GC
	∃∀̸ L2.4	∃∀̸ E2.3 [34]		∃∀̸ L2.4	∃∀̸ L3.9		∃∀̸ L2.4	∃∀̸ L4.3	
OPT sorted	∃∄ L2.4	(∀∃) *Conjecture*		∃∄ L2.4			∃∄ L2.4	∃∄ L3.11	
PoA	∞ L4.3			∞ L4.2			∞ L4.3		
PoS	=1 C4.7	>1 L4.9		=1 C4.7	=1 (nice) / >1 (else) L4.10		=1 L4.8	>1 L4.9	

Table 1: Result Overview. The symbol "✓✓" means that this setting has the FIP, while "✓" only indicates simple equilibrium existence for all instances (e.g. GC stands for grand coalition). As for the logic quantifier, the first is over the set of instances of this setting while the second one is over the set of equilibria of the respective instance, i.e., "∃∀̸" can be read as "There is an instance such that not all equilibria are sorted"; "∀∃" means "For all instances there is at least one sorted equilibrium / optimum"; "∃∄" means "There are instances where no optimum is sorted". The "∞" in the PoA row indicates an unbounded PoA.

number of coalitions (or clusters) is fixed. Both variants are related to our model, but incomparable due to the different cost functions.

Regarding limiting the number of coalitions, Sless et al. [40] study ASHGs where exactly k coalitions must be formed. Li [33] extends this by considering FHGs with an upper bound on the number of coalitions. Milchtaich and Winter [34] analyze a model incorporating all three key aspects of our models: limited number of coalitions, geometric embedding, and ignoring the self-effect within coalitions. In particular, they study exactly the Avg-Jump-HIS games and show that sorted PNE always exist, although unsorted PNE can also occur, and an IRC exists even if there are only two coalitions. In order to show the existence of sorted PNE, they propose an algorithm that we will extend and adapt to our other cost functions. Interestingly, they also consider the Distance-to-Average cost function in higher dimensions, where the existence of PNE is no longer guaranteed.

A model similar to our Avg-Jump-HIS game is also analyzed by Ahmadi et al. [3]. For the general setting in multiple dimensions, they show NP-hardness and for the one-dimensional average setting with Euclidean distances, they rediscover the $O(kn)$ algorithm by Milchtaich and Winter [34]. Their model was later generalized by Aamand et al. [1], which allows for arbitrary distance aggregation functions for the Jump-HIS setting. With this, they investigate a minimum distance and maximum distance function, similar to our Max-Jump-HIS setting, where they show approximation results on the optimum in a multidimensional setting.

Another way to limit the number of coalitions is by fixing the size of each coalition. As jumps would change coalition sizes, swap stability is more applicable in this setting. However, we note that some studies on hedonic games consider upper bounds on coalition sizes, which allows jumps [13, 19]. Swap stability has been studied in various models including matching markets [4, 7, 20], Schelling Games [2, 12, 18, 24, 29], and also recently in hedonic games with $\mathcal{W}$-preferences [19] and ASHGs [13]. Schelling Games [2, 18], in particular, can be seen as hedonic games with overlapping coalitions. In these games, agents of different types occupy nodes on a host graph and aim to optimize their neighborhood. Formally, hedonic games are Schelling games on a host graph consisting of disconnected cliques. We highlight the model by Bilò et al. [11], as they study the same set of cost functions and stability concepts as those used in our hedonic games. Using the reduction mentioned above, we obtain that our Max-Swap games are potential games. Furthermore, they show that the PoA is unbounded in almost all cases except for Max-Swap and Avg-Swap, where the PoA is linear in n. They also show that the PoS is 1 for Avg-Swap and Cutoff$_\lambda$-Swap games on regular graphs. Note that the latter result does not directly apply to our models, as the reduction to Schelling games does not create regular connected graphs as used by Bilò et al. [11].

1.3 Our Contribution

We propose and analyze a novel succinct variant of hedonic games where each agent has a fixed real-valued type and costs are induced by type-distances. To capture natural settings where agents cannot opt out to form a singleton coalition, the maximum number of coalitions might be fixed. For an overview of our results with the respective references, see Table 1.

For all combinations of cost functions (average distance, distance cutoff, maximum distance; in columns) and stability concepts (swap and jump; in sub-columns), we show that: every game instance admits at least one equilibrium (table row 1), either because they fulfill the FIP, or as we show a way to construct one for any instance. Further, every game instance admits a sorted equilibrium (row 2) and there exists at least one game instance in each setting admitting an unsorted equilibrium (row 3). Furthermore, we identify game instances in which all socially optimal structures are unsorted in every setting (row 4), except for Avg-Jump, where we conjecture that all optimal structures are sorted. To substantiate this, we provide partial results in this direction in the full version [21]. Regarding the quality of equilibria, we provide examples showing that the price of anarchy (PoA) is unbounded for most settings (row 5), and we characterize the price of stability (PoS) for all Swap games. For Jump games, the PoS is strictly greater than 1, except for a specific subclass of Cutoff$_\lambda$-Jump games (row 6).

In Sections 2 and 3, we address swap stability and jump stability, separately. Each section first establishes the existence of equilibria and then investigates their structural properties, along with those of the optimal structures. Finally, in Section 4, we analyze the quality of both swap and jump equilibria in terms of the PoA and PoS.

All omitted details can be found in the full version [21].

2 SWAP STABILITY

In this section, we study the existence and properties of swap equilibria as well as the properties of the respective optima.

2.1 Equilibrium Existence

In this subsection, we show that all Swap games have at least one equilibrium. Bilò et al. [11] give potential functions for all three cost functions for Schelling games with continuous types on regular graphs (Theorem 2 and Proposition 3) and for Max-Swap even on general graphs (Theorem 1). As our hedonic games can be transformed to their Schelling games on a set of (different-sized) cliques, we already have equilibrium existence for Max-Swap. For Avg-Swap and Cutoff$_\lambda$-Swap, they showed that the social cost is a potential function. Bilò et al. [13] showed that the social welfare (the utility equivalent of social cost) is also a potential function for ASHG and FHG restricted for swap stability. In the following Lemma 2.1, we show that this also applies to a subclass of modified FHGs [36] that includes utility variants for our Avg and Cutoff$_\lambda$ cost functions. The proof can be found in the full version [21].

Lemma 2.1. *Every modified FHG with symmetric and non-negative valuation has a swap equilibrium, and they have the FIP.*

Using Lemma 2.1 for Avg- and Cutoff$_\lambda$-Swap games, as well as the potential function for Max-Swap Schelling games by Bilò et al. [11], we prove Corollary 2.2 in the full version [21].

Corollary 2.2. *Every Max-, Avg- and Cutoff$_\lambda$-Swap game has an equilibrium, and all three fulfill the FIP.*

2.2 Properties of Equilibria and Optima

In this section, we study whether or when swap equilibria and optima of Swap games are sorted. First, we show the intuitive statement, that sorted coalition structures are swap stable for all three cost functions (see the full version [21] for the proof).

Theorem 2.3. *Every sorted coalition is swap stable for Max, Avg and Cutoff$_\lambda$ games.*

Although Theorem 2.3 shows that all sorted coalition structures are swap stable, this does not necessarily hold in the other direction. In Lemma 2.4, we give examples indicating that neither swap equilibria nor (swap) optima are always sorted.

Lemma 2.4. *For Max-, Avg- and Cutoff$_\lambda$-Swap games, there is an instance with only unsorted optima and an unsorted equilibrium.*

Proof. Consider the following game instance with $n \coloneqq 9$ agents, $k \coloneqq 2$ coalitions of size $k_1 \coloneqq 4$ and $k_2 \coloneqq 5$ and values $(v_i)_{i\in[9]} \coloneqq (1, 1, 2, 2, 2, 2, 2, 3, 3)$. For all cost functions, we show in the full version [21] that the unsorted coalition structure

$$C^* \coloneqq \big\{\{1, 1, 3, 3\}, \{2, 2, 2, 2, 2\}\big\}$$

is the social optimum and a swap equilibrium. For the Cutoff$_\lambda$ cost function, we choose $\lambda < 1$. However, one can scale the values of this example to get the same results for an arbitrary λ. □

3 JUMP STABILITY

Section 2 shows that swap equilibria are easy to find due to several potential functions. We show that equilibria are harder to find for Jump games, since the FIP only holds for Max-Jump games. Also, sorted coalition structures and social optima are not always stable.

3.1 Equilibrium Existence

First, note that in the unhappy in isolation (UIS) setting, the grand coalition is a stable coalition structure, as no agent would jump out into a singleton coalition. Thus, equilibrium existence is given for all cost functions in this case.

Observation 3.1. *For all three Jump-UIS game variants, the grand coalition is always stable.*

Moreover, Max-Jump-UIS games and also Max-Jump-HIS games are potential games, which we show in the following Lemma 3.2. Bilò et al. [11] show that the potential function for Max-Swap in Schelling games with continuous types fails on regular graphs where the number of empty nodes is higher than the maximal degree in the host graph. However, as Lemma 3.2 implies, this potential function works on any set of independent cliques in the HIS setting, and with minor adjustments also for the UIS setting.

Lemma 3.2. *Every Max-Jump-UIS, and -HIS game has an equilibrium, and they fulfill the FIP.*

Proof sketch. We show that both kinds of games are potential games. Both potential functions are based on the idea that the non-increasingly sorted cost vector of all agents decreases lexicographically with every improving jump. Formally, let $\Phi_{\text{cost}}(C)$ be the vector of the costs of all agents in the HIS setting that is sorted non-increasingly. Then, we show that $\Phi_{\text{cost}}(C)$ is a potential function for Max-Jump-HIS games and that $\Phi(C) \coloneqq (|C|_{\neq\emptyset}, \Phi_{\text{cost}}(C))$ is a potential function for Max-Jump-HIS games, where $|C|_{\neq\emptyset}$ is the number of non-empty coalitions in the given coalition structure C. Details can be found in the full version [21]. □

Next, we provide an IRC for Avg- and Cutoff$_\lambda$-Jump games in Theorem 3.3 showing that both are not potential games in comparison to the Max-Jump games. Note that, Milchtaich and Winter [34] provided an IRC for Avg-Jump games. However, we want to highlight that the IRC given in Theorem 3.3 works for both Avg- and Cutoff$_\lambda$-Jump games.

Theorem 3.3. *Avg- and Cutoff$_\lambda$-Jump games admit an IRC.*

Proof. Consider the IRC given in Table 2, which works for both cost functions. Please note that this IRC also holds for both the HIS and the UIS setting, as it does not include singleton coalitions. □

Although we know by Theorem 3.3 that Avg-Jump and Cutoff$_\lambda$-Jump are not potential games, there is at least one jump equilibrium for each such game instance. Indeed, Milchtaich and Winter [34] gave an algorithm that calculates a jump stable coalition structure for all Avg-Jump-HIS game instances that is also sorted [34, Theorem 2.4]. Now, we show that this algorithm also works for a large class of games, including Avg-, Cutoff$_\lambda$- and Max-Jump-HIS games.

In order to explain and analyze this algorithm, we introduce some additional notation and terminology regarding a sorted coalition structure $C \coloneqq \{C_i\}_{i\in[k]}$. For every coalition $C_i \in C$, we call the

Coalition Structure	Cutoff$_4$	Average
$\{14, 11, 5, 6, 7, 9\}$, $\{1, \overleftarrow{\mathbf{5}}, 5, 8, 10, 14\}$	$\frac{2}{5} \rightsquigarrow \frac{2}{6}$	$\frac{21}{5} \rightsquigarrow \frac{22}{6}$
$\{14, 11, 5, 5, 6, 7, 9\}$, $\{\overleftarrow{\mathbf{1}}, 5, 8, 10, 14\}$	$\frac{3}{4} \rightsquigarrow \frac{5}{7}$	$\frac{33}{4} \rightsquigarrow \frac{50}{7}$
$\{14, 11, 1, 5, 5, 6, 7, \overrightarrow{\mathbf{9}}\}$, $\{5, 8, 10, 14\}$	$\frac{2}{7} \rightsquigarrow \frac{1}{4}$	$\frac{28}{7} \rightsquigarrow \frac{13}{5}$
$\{14, 11, 1, 5, 5, 6, \overrightarrow{\mathbf{7}}\}$, $\{5, 8, 9, 10, 14\}$	$\frac{2}{6} \rightsquigarrow \frac{1}{5}$	$\frac{22}{6} \rightsquigarrow \frac{15}{5}$
$\{14, 11, 1, 5, \overrightarrow{\mathbf{5}}, 6\}$, $\{5, 7, 8, 9, 10, 14\}$	$\frac{2}{5} \rightsquigarrow \frac{2}{6}$	$\frac{20}{5} \rightsquigarrow \frac{23}{6}$
$\{14, 11, \overrightarrow{\mathbf{1}}, 5, 6\}$, $\{5, 5, 7, 8, 9, 10, 14\}$	$\frac{3}{4} \rightsquigarrow \frac{5}{7}$	$\frac{33}{4} \rightsquigarrow \frac{40}{7}$
$\{14, 11, 5, 6\}$, $\{1, 5, 5, 7, 8, \overleftarrow{\mathbf{9}}, 10, 14\}$	$\frac{2}{7} \rightsquigarrow \frac{1}{4}$	$\frac{25}{7} \rightsquigarrow \frac{14}{4}$
$\{14, 11, 5, 6, 9\}$, $\{1, 5, 5, \overleftarrow{\mathbf{7}}, 8, 10, 14\}$	$\frac{2}{6} \rightsquigarrow \frac{1}{5}$	$\frac{21}{6} \rightsquigarrow \frac{16}{5}$
$\{14, 11, 5, 6, 7, 9\}$, $\{1, \overleftarrow{\mathbf{5}}, 5, 8, 10, 14\}$	…	

Table 2: IRC for the given instance under the Cutoff$_4$-Jump setting, and Avg-Jump setting. The first column shows the coalition structures of the IRC (jumping agent in bold). The other columns indicate the cost change of the jumping agent.

agent with the smallest and highest value in C_i the *left-most* agent $L(C_i)$ and the *right-most* agent $R(C_i)$ of C_i, respectively. Having the same idea in mind, we also refer to coalition C_{i-1} and C_{i+1} as the coalitions *left* and *right* of coalition C_i, respectively[2]. If some left-most agent $x \coloneqq L(C_i)$ wants to move to the next coalition to its left, i.e., C_{i-1}, to reduce its cost, we call this move a *left-improving move* of agent x. Similarly, a *right-improving move* is a move of some right-most agent $x \coloneqq R(C_i)$ to coalition C_{i+1} such that agent x has higher cost in C_i than in C_{i+1}.

The idea of Milchtaich and Winter's algorithm is to start with a sorted "right-heavy" coalition structure, having the $k-1$ smallest values of a game instance in singleton coalitions and the rest in the remaining coalition. Then the algorithm allows the agents to perform left-improving moves and stops if no agent has a left-improving move. Milchtaich and Winter [34] show that, with the Average cost function, that the algorithm terminates after at most nk many left-improving moves in jump stable coalition structure. In Definition 3.4, we bundle sufficient properties of the Average cost function into a broad class of cost function inducing the same behavior to the algorithm.

Definition 3.4. A cost function cost_m is called *monotone*, if the following conditions hold for every game instance $(n, (v_i)_{i\in[n]}, k)$:

(i) For a coalition C and agents $x, y \notin C$:
if $v_x \leq v_y \leq v_{L(C)}$ or $v_{R(C)} \leq v_y \leq v_x$
then $\text{cost}_m(y, C) \leq \text{cost}_m(x, C)$.

(ii) For all coalitions C, D and agents $x \notin C \cup D$:
if $v_x \leq v_{L(C)} \leq v_{R(C)} \leq v_{L(D)}$ or $v_{R(D)} \leq v_{L(C)} \leq v_{R(C)} \leq v_x$
then $\text{cost}_m(x, C) \leq \text{cost}_m(x, C \cup D) \leq \text{cost}_m(x, D)$.

(iii) For all coalitions C, D over $\mathbb{R}$ and $c \in C$:
if $\text{cost}_m(c, C) > \text{cost}_m(c, D)$, then
- $\text{cost}_m\big(R(C), C\big) > \text{cost}_m\big(R(C), D\big)$ $\Big(\text{for } v_{R(C)} \leq v_{L(D)}\Big)$,
- $\text{cost}_m\big(L(C), C\big) > \text{cost}_m\big(L(C), D\big)$ $\Big(\text{for } v_{R(D)} \leq v_{L(C)}\Big)$.

In order to prove that the algorithm by Milchtaich and Winter [34] calculates a jump stable coalition structure for all monotone cost functions (see Theorem 3.6), we first show that we only need to rule out left- and right-improving moves to prove stability for a given sorted coalition structure. This property is captured in the following Lemma 3.5 and can be proven as an implication of property (iii) of Definition 3.4 (see the full version [21] for details).

Lemma 3.5. *Let $\mathcal{I} \coloneqq (n, (v_i)_{i\in[n]}, k)$ be a game instance with monotone cost function cost_m. Further, let $C \coloneqq \{C_i\}_{i\in[k]}$ be a sorted coalition structure of $\mathcal{I}$. Then, either C is a jump equilibrium, or there is a left-improving move or a right-improving move.*

Now, we are ready to prove that the properties of monotone cost functions are sufficient for the algorithm of Milchtaich and Winter [34] to work properly.

Theorem 3.6. *Every game instance $\mathcal{I} \coloneqq (n, (v_1, \ldots, v_n), k)$ with a monotone cost function cost_m in the HIS-setting has a sorted jump equilibrium, which can be found within kn many improving moves.*

Proof sketch. The algorithm starts with the "right-heavy" coalition structure $C_{\text{start}} \coloneqq \big\{\{v_1\}, \ldots, \{v_{k-1}\}, \{v_k, \ldots, v_n\}\big\}$. Clearly, C_{start} is sorted and stays sorted under left-improving moves. It is therefore valid to also number coalitions during the algorithm from C_1 to C_k according to the ordering of their values.

To prove that this algorithm works under the monotone cost function cost_m, we show by induction that there is no coalition structure during the algorithm admitting a right-improving move. By Lemma 3.5, this proves that the final coalition structure of this algorithm, for which by definition no left-improving move is possible, is indeed a jump equilibrium.

In the base case, C_{start} is sorted due to the assumed HIS setting. Now consider a sorted coalition structure $C \coloneqq \{C_i\}_{i\in[k]}$ which does not admit any right-improving move, but at least one left-improving move. Let $L(C_i)$ be the agent with a left-improving move in C, and $C' \coloneqq \{C'_i\}_{i\in[k]}$ the coalition structure C after the left-improving move of agent $L(C_i)$. See Figure 2 for an illustration.

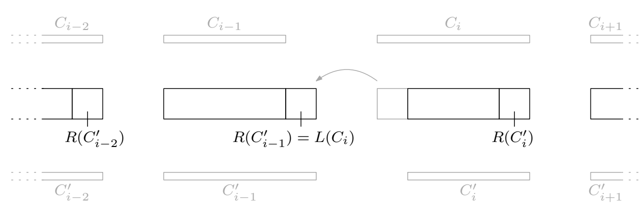

Figure 2: Sketch of a left-improving move from a coalition structure $\{C_i\}_{i\in[k]}$ to $\{C'_i\}_{i\in[k]}$.

The only coalitions that changed are C_i and C_{i-1}. Therefore, we only need to show that none of the agents $R(C'_{i-2})$, $R(C'_{i-1})$ and $R(C'_i)$ have a right-improving move. (See the full version [21].) □

Next, we show that all three cost functions are monotone, implying that Jump-HIS instances have at least one sorted equilibrium.

Lemma 3.7. *The Max-, Avg- and Cutoff$_\lambda$ costs are monotone.*

3.2 Properties of Optima and Equilibria

First, let us observe that if we fix n, k, and the size of each coalition, the optimum structure for both Jump and Swap games is the same, as social costs are computed in the same way. But as the game dynamics of Jump games allow the coalition number and sizes to

[2]Note that there is no coalition left of coalition C_1 nor right of coalition C_k.

change dynamically, we consider optima only depending on n and maximum k. Especially for JUMP-HIS games, the stable coalition structures will always have k occupied coalitions, as singleton resp. single-value coalitions will always incur the minimum cost of 0 for each contained agent, and thus instances with a higher maximum k may have a lower optimum social cost. In contrast to this, in the UIS setting, agents in singleton coalitions will always leave them and never enter empty coalitions, and thus coalitions may stay empty, and the grand coalition will always be an equilibrium.

The following result has already been mentioned but not proven by Milchtaich and Winter [34, Example 2.3]. The proof and further observations can be found in the full version [21].

LEMMA 3.8. *For AVG-JUMP games, there is an instance with at least one unsorted equilibrium.*

LEMMA 3.9. *For CUTOFF$_\lambda$-JUMP games, there is an instance with only unsorted optima and at least one unsorted equilibrium.*

PROOF. We use a similar construction as in Lemma 2.4, but this time with only three agents with value 2: Consider the game instance with $n \coloneqq 7$ agents, $k \coloneqq 2$, some $\lambda < 1$ and values $(v_i)_{i\in[7]} \coloneqq (1, 1, 2, 2, 2, 3, 3)$. The unsorted coalition structure

$$C^* \coloneqq \big\{\{1, 1, 3, 3\}, \{2, 2, 2\}\big\}$$

is jump stable, as all but one coalition have only pairwise friends in it. In the full version [21], we show that it is also socially optimal. □

For MAX-JUMP games, there is a simple construction to show the existence of unsorted equilibria.

LEMMA 3.10. *For MAX-JUMP-{UIS,HIS} games and any $k \geq 2$ and $n \geq 4k$, there is an instance with at least one unsorted equilibrium.*

PROOF SKETCH. Here, we only give the worst-case example for $k = 2$ and $n = 4$ and refer to the full version [21] for the general construction. For this, consider the following instance

$$\big\{\{L, L, H, H\}, \{L, L, H, H\}\big\},$$

where no agent has an improving jump, as its valuation is $H - L$ in both coalitions. □

By adjusting the instance used in Lemma 2.4, we get that MAX games also admit unsorted optima.

LEMMA 3.11. *For MAX-JUMP-{UIS,HIS} games, there is an instance with only unsorted optima.*

Computing the optimal coalition structure in AVERAGE-JUMP games is challenging even for $k = 2$, due to the interdependence between the agents' values and their positions within the coalition structure. To gain a better understanding of these optima, we study their structural properties and prove several lemmata and observations (see the full version [21]), which lead us to the following conjecture.

CONJECTURE 3.12. *In AVERAGE-JUMP games with $k = 2$, the optimal coalition structure is sorted.*

This conjecture points to an important structural property of AVERAGE-JUMP games. If it holds, the optimal coalition structure is also sorted for any k, since otherwise two overlapping coalitions C' and C'' in C can be rearranged to reduce the total cost. In particular, the existence of sorted optimal structures hints at the possibility of a dynamic programming approach, similar to those used for one-dimensional clustering problems such as k-means and k-medians [27] and related clustering models [25]. However, it remains unclear whether such an approach can be directly applied or adapted to our setting, even if the conjecture holds, due to the dependencies between agents' costs and coalition boundaries.

4 QUALITY OF EQUILIBRIA

In this section, we study the impact of selfish behavior on the social cost, by looking at the PoA and PoS for both SWAP- and JUMP games.

4.1 Price of Anarchy (PoA)

Theorem 4.1 summarizes our results on the PoA, which is unbounded in most settings. We show results by providing worst-case instances for different game variants, split into Lemmas 4.2 to 4.5.

THEOREM 4.1. *The PoA is unbounded for all games with all $k \geq 2$, except for AVG-JUMP-HIS games with exactly two coalitions, where it is either 1 or at least $\Omega(n)$ depending on the agent values.*

LEMMA 4.2. *The PoA is unbounded for CUTOFF$_\lambda$-{SWAP, JUMP} games with all $k \geq 2$.*

PROOF SKETCH. We give two different game instances for $k = 2$ and $k > 2$ respectively, and show that they have an optimum C^* with social cost 0 and admit a coalition structure C with positive social cost that is both swap and jump stable.

For $k = 2$, we assume $\lambda = 1$ and consider a game with small $0.5 \geq \varepsilon > 0$ and the following two coalition structures

$$C^* \coloneqq \big\{\{0, 1-\varepsilon, 1\}, \{1+\varepsilon, 2\}\big\},$$
$$C \coloneqq \big\{\{0, 1, 2\}, \{1-\varepsilon, 1+\varepsilon\}\big\}.$$

For all $k > 2$, consider the game with $n \coloneqq 4(k-1)$ agents and coalitions of size $k_1 \coloneqq 2(k-1)$ and $k_i = 2$ for all $i \in [k] \setminus \{1\}$. Then

$$C^* \coloneqq \big\{\{\overbrace{0 \ldots 0}^{2(k-1)}\}, \{2, 2\}, \ldots, \{k, k\}\big\}, \text{ and}$$
$$C \coloneqq \big\{\overbrace{\{0, 0\}, \ldots, \{0, 0\}}^{k-1}, \{2, 2, 3, 3, \ldots, k, k\}\big\}$$

are the social optimum with cost 0 and an equilibrium with positive cost, respectively. □

LEMMA 4.3. *The PoA of {MAX, AVERAGE}-{SWAP, JUMP} games is unbounded for all $k \geq 3$ and $n \geq 2k + 2$, independently of whether UIS or HIS is considered. For MAX-{SWAP, JUMP} games, the number of coalitions can be reduced to $k = 2$.*

PROOF SKETCH. We state the worst-case examples for both functions. Refer to the full version [21] for the general constructions.

For AVERAGE, let $L < M < H$ and $M \coloneqq \frac{H+L}{2} + 1$, and consider a game with $k = 3$ and $n = 8$ admitting a coalition structure

$$\big\{\{L, L\}, \{L, L\}, \{M, M, H, H\}\big\}.$$

This is swap and jump stable for both the Max and the Average cost function. However, there is an optimum

$$\big\{\{L,L,L,L\},\{M,M\},\{H,H\}\big\}.$$

with a social cost of 0. A similar construction works for the Maximum cost function even with $k = 2$ and $n = 8$. Here,

$$\big\{\{L,L,L,L\},\{H,H,H,H\}\big\}$$

is the optimum with a social cost of 0 while

$$\big\{\{L,L,H,H\},\{L,L,H,H\}\big\}$$

is a swap and jump stable coalition structure with asymptotically maximal cost of $\Theta(n(H-L))$. ◻

Lemma 4.3 only shows the PoA for Avg games with $k \geq 3$ coalitions. With Lemma 4.4 and Lemma 4.5, we take a closer look at $k = 2$.

Lemma 4.4. *For Avg-Jump-HIS games, if* $\mathrm{SC}(\mathrm{OPT}) = 0$ *for* $k = 2$ *coalitions, then the social cost of any equilibrium is also 0. If* $\mathrm{SC}(\mathrm{OPT}) > 0$*, then the PoA is in* $\Omega(n)$ *and cannot be unbounded, even for instances with two coalitions and only 3 different agent values.*

Compared to Average-Jump-HIS games, where an unbounded PoA is impossible for $k = 2$, we show that Average-Jump-UIS games have an unbounded PoA, since the grand coalition is stable.

Lemma 4.5. *For all three Jump-UIS games, even games with* $k = 2$ *coalitions and only two different agent values have unbounded PoA.*

4.2 Price of Stability (PoS)

While the results on the PoA are similar for Swap and Jump games, the PoS differs between Swap and Jump. Although the optima of all three Swap games are always stable, this is not necessarily true for the respective Jump games.

An overview of the results is given by Theorem 4.6. For that, we remind the reader of the definition of nice Cutoff games given at the end of Section 1.1.

Theorem 4.6. *The PoS for all Swap games and nice* Cutoff$_\lambda$ *games is 1. For all general Jump games, it is strictly larger than 1.*

For Cutoff$_\lambda$-Swap and Avg-Swap games, this follows directly from the social cost being potential functions, while the proof for Max-Swap games is more technical.

Corollary 4.7. *The PoS is exactly* 1 *for all* Cutoff$_\lambda$*-Swap and Avg-Swap games.*

Lemma 4.8. *The PoS is* 1 *for Max-Swap games.*

In contrast to Swap games, the PoS in Jump games is greater than 1 in general, as optima can be unstable. However, there is an interesting subclass of Cutoff$_\lambda$-Jump games with a PoS of 1.

Lemma 4.9. *For Avg- and Max-Jump games, it holds that PoS* > 1.

Proof sketch. Consider the following game instance with $n = 7$ agents, $k = 2$ coalitions and values allowing the coalition structure

$$C^* \coloneqq \big\{\{1,1,1\},\{4,6,8,8\}\big\}.$$

In the full version [21], we prove for both cost functions that this coalition structure C^* indeed (1) is the social optimum and (2) has an improving jump for the agent with value 4. ◻

Lemma 4.10. *The PoS of nice* Cutoff$_\lambda$*-Jump games is exactly* 1*, while the PoS over all other* Cutoff$_\lambda$*-Jump games is greater than* 1.

Proof sketch. For nice Cutoff$_\lambda$-Jump games, the optimum has a social cost of 0, since the agents can be partitioned into k coalitions based on the unique λ-blocks by which they are covered. As coalition structures with social cost of 0 are stable, the PoS is 1.

For the second part, consider a Cutoff$_\lambda$-Jump game instance with $n = 8$ agents, $k = 2$ coalitions and coalition structure

$$C^* \coloneqq \Big\{\{0,\tfrac{1}{4}\varepsilon,\tfrac{2}{4}\varepsilon,\tfrac{2}{4}\varepsilon\},\ \{\underline{\lambda+\tfrac{1}{4}\varepsilon},\lambda+\tfrac{3}{4}\varepsilon,2\lambda+\tfrac{1}{4}\varepsilon,2\lambda+\varepsilon\}\Big\}$$

for some $\varepsilon > 0$. In the full version [21], we show that this coalition structure C^* is an optimum and not jump stable, as the underlined agent has an improving jump. ◻

CONCLUSION

We consider a hedonic games model, where each agent has a fixed value, and evaluates the cost of each coalition by accumulating the values of all other agents in its coalition, using three natural cost functions. It is clear that in this setting, many more variants could be implemented, e.g. it would also be interesting to explore alternative cost functions, such as vectors of distances, squared distances, or x-nearest-neighbor formulations. Within this setting, we study two stability concepts and show the existence of equilibria and their quality compared to the utilitarian social optimum. While unsorted optima exist for most variants, it is yet unclear whether they also exist for Avg-Jump games.

The structural property of being sorted, proposed in our conjecture, is key to understanding how optimal coalition structures behave in Avg-Jump games. Confirming or disproving this conjecture would therefore be an important step toward obtaining a clearer picture of both equilibrium quality and social optimality, and could guide future algorithmic work. Still, proving this property is far from easy, even for the simple case of $k = 2$.

Since there is a substantial gap between unbounded PoA and the constant lower bound on PoS, and because the worst-case instances appear somewhat artificial, future work could focus on random or real world instances, or on stability notions allowing more cooperation. Regarding strong stability, a result by Milchtaich and Winter [34] suggests instances with no equilibria, so models permitting limited but not arbitrary cooperation could be of interest, especially as some of the reported high PoA instances (e.g. Lemma 4.3) are clearly not stable if multiple agents could coordinate.

Another promising direction would be to limit the size of coalitions, e.g. no coalition can be larger than $x \in \mathbb{N}$, to prevent the formation of the grand coalition in the UIS setting. Moreover, an unbounded PoA provides limited insight. In particular, for the utility variant of nice Cutoff$_\lambda$ games, the PoA becomes bounded. Therefore, analyzing the utility variant may offer additional understanding of the game. Other avenues for research include analyzing the core and the computational complexity of finding equilibria.

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REFERENCES

[1] Anders Aamand, Justin Y. Chen, Allen Li, Sandeep Silwal, Pattara Sukprasert, Ali Vakilian, and Fred Zhang. 2023. Constant approximation for individual preference stable clustering. In *NeuIPS 2023*. Article 1892, 16 pages. http://papers.nips.cc/paper_files/paper/2023/hash/881259965dacb9f42967aae84a157283-Abstract-Conference.html

[2] Aishwarya Agarwal, Edith Elkind, Jiarui Gan, Ayumi Igarashi, Warut Suksompong, and Alexandros A. Voudouris. 2021. Schelling Games on Graphs. *Artif. Intell.* 301 (2021), 103576. https://doi.org/10.1016/J.ARTINT.2021.103576

[3] Saba Ahmadi, Pranjal Awasthi, Samir Khuller, Matthäus Kleindessner, Jamie Morgenstern, Pattara Sukprasert, and Ali Vakilian. 2022. Individual Preference Stability for Clustering. In *ICML 2022*. 197–246. https://proceedings.mlr.press/v162/ahmadi22a.html

[4] José Alcalde. 1994. Exchange-Proofness or Divorce-Proofness? Stability in One-Sided Matching Markets. *Economic Design* 1, 1 (1994). https://doi.org/10.1007/BF02716626

[5] Elliot Anshelevich, Anirban Dasgupta, Jon M. Kleinberg, Éva Tardos, Tom Wexler, and Tim Roughgarden. 2008. The Price of Stability for Network Design with Fair Cost Allocation. *SIAM J. Comput.* 38, 4 (2008), 1602–1623. https://doi.org/10.1137/070680096

[6] Haris Aziz, Florian Brandl, Felix Brandt, Paul Harrenstein, Martin Olsen, and Dominik Peters. 2019. Fractional Hedonic Games. *ACM Trans. Econ. Comput.* 7, 2 (May 2019), 1–29. https://doi.org/10.1145/3327970

[7] Haris Aziz and Adrian Goldwaser. 2017. Coalitional Exchange Stable Matchings in Marriage and Roommate Markets. In *AAMAS 2017*. 1475–1477. https://dl.acm.org/doi/10.5555/3091125.3091334

[8] Haris Aziz, Paul Harrenstein, and Evangelia Pyrga. 2012. Individual-Based Stability in Hedonic Games Depending on the Best or Worst Players. In *AAMAS 2012*. 1311–1312. https://dl.acm.org/doi/10.5555/2343896.2343979

[9] Haris Aziz and Rahul Savani. 2016. Hedonic Games. In *Handbook of Computational Social Choice*. 356–376. https://doi.org/10.1017/cbo9781107446984.016

[10] Suryapratim Banerjee, Hideo Konishi, and Tayfun Sönmez. 2001. Core in a Simple Coalition Formation Game. *Soc. Choice Welf.* 18, 1 (2001), 135–153. https://doi.org/10.1007/S003550000067

[11] Davide Bilò, Vittorio Bilò, Michelle Döring, Pascal Lenzner, Louise Molitor, and Jonas Schmidt. 2023. Schelling Games with Continuous Types. In *IJCAI 23*. 2520–2527. https://doi.org/10.24963/ijcai.2023/280

[12] Davide Bilò, Vittorio Bilò, Pascal Lenzner, and Louise Molitor. 2022. Topological Influence and Locality in Swap Schelling Games. *Auton. Agents Multi Agent Syst.* 36, 2 (2022), 47. https://doi.org/10.1007/S10458-022-09573-7

[13] Vittorio Bilò, Gianpiero Monaco, and Luca Moscardelli. 2022. Hedonic Games with Fixed-Size Coalitions. In *AAAI 2022*. 9287–9295. https://doi.org/10.1609/AAAI.V36I9.21156

[14] Anna Bogomolnaia and Matthew O. Jackson. 2002. The Stability of Hedonic Coalition Structures. *Games Econ. Behav.* 38, 2 (2002), 201–230. https://doi.org/10.1006/game.2001.0877

[15] Simina Brânzei and Kate Larson. 2011. Social Distance Games. In *IJCAI 2011*. 91–96. https://doi.org/10.5591/978-1-57735-516-8/IJCAI11-027

[16] Katarína Cechlárová and Jana Hajduková. 2004. Stable Partitions with W-preferences. *Discret. Appl. Math.* 138, 3 (2004), 333–347. https://doi.org/10.1016/S0166-218X(03)00464-5

[17] Katarína Cechlárová and Antonio Romero-Medina. 2001. Stability in Coalition Formation Games. *Int. J. Game Theory* 29, 4 (May 2001), 487–494. https://doi.org/10.1007/s001820000053

[18] Ankit Chauhan, Pascal Lenzner, and Louise Molitor. 2018. Schelling Segregation with Strategic Agents. In *SAGT 2018*, Vol. 11059. 137–149. https://doi.org/10.1007/978-3-319-99660-8_13

[19] Ágnes Cseh, Tamás Fleiner, and Petra Harján. 2019. Pareto Optimal Coalitions of Fixed Size. *jMID* 4, 1 (2019), 87–108. https://doi.org/10.22574/jmid.2019.11.003

[20] Anastasia Damamme, Aurélie Beynier, Yann Chevaleyre, and Nicolas Maudet. 2015. The Power of Swap Deals in Distributed Resource Allocation. In *AAMAS 2015*. 625–633. https://dl.acm.org/doi/10.5555/2772879.2773235

[21] Merlin de la Haye, Pascal Lenzner, Farehe Soheil, and Marcus Wunderlich. 2026. Metric Hedonic Games on the Line. arXiv:2602.05888 https://arxiv.org/abs/2602.05888 Full version of this paper.

[22] Dinko Dimitrov, Peter Borm, Ruud Hendrickx, and Shao Chin Sung. 2006. Simple Priorities and Core Stability in Hedonic Games. *Soc. Choice Welf.* 26, 2 (2006), 421–433. https://doi.org/10.1007/S00355-006-0104-4

[23] Jacques H Drèze and Joseph Greenberg. 1980. Hedonic Coalitions: Optimality and Stability. *Econometrica* 48, 4 (1980), 987–1003. https://doi.org/10.2307/1912943 arXiv:1912943

[24] Hagen Echzell, Tobias Friedrich, Pascal Lenzner, Louise Molitor, Marcus Pappik, Friedrich Schöne, Fabian Sommer, and David Stangl. 2019. Convergence and Hardness of Strategic Schelling Segregation. In *WINE 2019*, Vol. 11920. 156–170. https://doi.org/10.1007/978-3-030-35389-6_12

[25] Moran Feldman, Liane Lewin-Eytan, and Joseph (Seffi) Naor. 2015. Hedonic Clustering Games. *ACM Trans. Parallel Comput.* 2, 1 (May 2015), 1–48. https://doi.org/10.1145/2742345

[26] Michele Flammini, Bojana Kodric, Martin Olsen, and Giovanna Varricchio. 2021. Distance Hedonic Games. In *SOFSEM 2021: Theory and Practice of Computer Science*. Vol. 12607. 159–174. https://doi.org/10.1007/978-3-030-67731-2_12

[27] Allan Grønlund, Kasper Green Larsen, Alexander Mathiasen, and Jesper Sindahl Nielsen. 2017. Fast Exact K-Means, k-Medians and Bregman Divergence Clustering in 1D. *CoRR* (2017). https://doi.org/10.48550/arXiv.1701.07204

[28] Jana Hajduková. 2006. Coalition Formation Games: A Survey. *IGTR* 8, 4 (2006), 613–641. https://doi.org/10.1142/S0219198906001144

[29] Panagiotis Kanellopoulos, Maria Kyropoulou, and Alexandros A. Voudouris. 2021. Modified Schelling Games. *Theor. Comput. Sci.* 880 (2021), 1–19. https://doi.org/10.1016/J.TCS.2021.05.032

[30] Anna Maria Kerkmann, Jérôme Lang, Anja Rey, Jörg Rothe, Hilmar Schadrack, and Lena Schend. 2020. Hedonic Games with Ordinal Preferences and Thresholds. *J. Artif. Intell. Res.* 67 (April 2020), 705–756. https://doi.org/10.1613/jair.1.11531

[31] Elias Koutsoupias and Christos H. Papadimitriou. 2009. Worst-Case Equilibria. *Comput. Sci. Rev.* 3, 2 (2009), 65–69. https://doi.org/10.1016/J.COSREV.2009.04.003

[32] Jérôme Lang, Anja Rey, Jörg Rothe, Hilmar Schadrack, and Lena Schend. 2015. Representing and Solving Hedonic Games with Ordinal Preferences and Thresholds. In *AAMAS 2015*. 1229–1237. https://doi.org/10.5555/2772879.2773308

[33] Fu Li. 2021. Fractional Hedonic Games With a Limited Number of Coalitions. In *ICTCS 2021*, Vol. 3072. 205–218. https://ceur-ws.org/Vol-3072/paper18.pdf

[34] Igal Milchtaich and Eyal Winter. 2002. Stability and Segregation in Group Formation. *Games Econ. Behav.* 38, 2 (Feb. 2002), 318–346. https://doi.org/10.1006/game.2001.0878

[35] Charles E Miller. 1980. Coalition Formation in Characteristic Function Games: Competitive Tests of Three Theories. *J. Exp. Soc. Psychol.* 16, 1 (1980), 61–76. https://doi.org/10.1016/0022-1031(80)90036-0

[36] Gianpiero Monaco, Luca Moscardelli, and Yllka Velaj. 2020. Stable Outcomes in Modified Fractional Hedonic Games. *Auton. Agents Multi Agent Syst.* 34, 1 (April 2020), 4. https://doi.org/10.1007/s10458-019-09431-z

[37] Dov Monderer and Lloyd S. Shapley. 1996. Potential Games. *Games Econ. Behav.* 14, 1 (1996), 124–143. https://doi.org/10.1006/game.1996.0044

[38] Anja Rey and Lisa Rey. 2022. FEN-Hedonic Games with Distance-Based Preferences. *CoRR* (Jan. 2022). https://doi.org/10.48550/arXiv.2201.13158

[39] TJ Schlueter and Makoto Yokoo. 2024. Friend- and Enemy-Oriented Hedonic Games with Strangers. In *PRIMA 2024*, Vol. 15395. 421–427. https://doi.org/10.1007/978-3-031-77367-9_31

[40] Liat Sless, Noam Hazon, Sarit Kraus, and Michael Wooldridge. 2018. Forming k Coalitions and Facilitating Relationships in Social Networks. *Artif. Intell.* 259 (June 2018), 217–245. https://doi.org/10.1016/j.artint.2018.03.004