Obnoxious Facility Location Problems: Strategyproof Mechanisms Optimizing L_p-Aggregated Utilities and Costs

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ABSTRACT

We study the problem of locating a single obnoxious facility on the normalized line segment $[0, 1]$ with strategic agents from a mechanism design perspective. Each agent has a preference for the undesirable location of the facility and would prefer the facility to be far away from their location. We consider the utility of the agent, defined as the distance between the agent's location and the facility location, and the cost of each agent, equal to one minus the utility. Given this standard setting of obnoxious facility location problems, our goal is to design (group) strategyproof mechanisms to elicit agent locations truthfully and determine facility location approximately optimizing the L_p-aggregated utility and cost objectives, which generalizes the L_p-norm ($p \geq 1$) of the agents' utilities and agents' costs to any $p \in [-\infty, \infty]$, respectively. We establish upper and lower bounds on the approximation ratios of deterministic and randomized (group) strategyproof mechanisms for maximizing the L_p-aggregated utilities or minimizing the L_p-aggregated costs across the range of p-values. While there are gaps between upper and lower bounds for randomized mechanisms, our bounds for deterministic mechanisms are tight.

KEYWORDS

Mechanism Design; Obnoxious Facility Location; Strategyproofness

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1 INTRODUCTION

Over the past several decades, facility location problems have been extensively studied theoretically and practically in various fields, including operations research [13, 27], computer science [28, 29], and economics [3, 9]. In a typical facility location problem, a social planner seeks to locate one or more facilities to *best* serve a set of agents within a region, where *best* is often understood as optimizing a specific utility or cost objective based on the distances between the agents' preferred facility locations and the locations of the facilities. Classic practical applications of facility location problems include selecting a location for a public school, library, or hospital to serve the surrounding communities, while taking into consideration the facility location preferences of the agents in those communities.

Obnoxious Facility Location Problems. In the most-studied facility location problems, such as locating a library, park, or transit station, agents often view these facilities as desirable and prefer them to be located closer to their preferred locations. In contrast, a social planner might sometimes locate obnoxious facilities (e.g., nuclear power plants, landfill sites, chemical factories, and detention centers) that are essential for society's functioning but are viewed as undesirable or unpleasant by the agents. For instance, landfill sites may expose residents to unpleasant odors and hazardous gases (e.g., lead, ammonia, and hydrogen sulfide) [2, 30]. Nuclear radiation from power plants has been associated with higher risks of cancer and leukemia [4, 19]. Naturally, the agents prefer these obnoxious facilities to be located far away from their undesirable facility locations. Therefore, another active area of research [10, 11] examines obnoxious facility location problems that deal with locating obnoxious facilities optimizing the objective based on their undesirable facility location preferences.

Mechanism Design for Obnoxious Facility Location Problems. Because agents' undesirable location preferences are often unknown, there have been significant efforts to design mechanisms to elicit agent preferences and use these preferences to determine optimal facility locations based on the objective. However, it is well known that each agent may have an incentive to misreport their true preference to manipulate the mechanisms' locations to benefit themselves [6, 25, 28]. Therefore, a major multidisciplinary research area in mechanism design for obnoxious facility location problems [6, 8, 12, 17, 18, 20, 21, 31] examines the design of strategyproof mechanisms that elicit agents' undesirable location preferences truthfully on a line segment and determine facility locations that (approximately) optimize a specific utility or cost objective. The most commonly studied utility objectives include the total utility [8, 12, 31], minimum utility [12], and sum of squared utilities [31], which are defined based on the agents' (individual) utilities, where each agent's utility is defined to be the distance between the facility location and the agent's undesirable location. For the cost objectives where each agent incurs a cost defined as the segment length minus their utility, existing studies examine the total cost and maximum cost [21].

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Because existing obnoxious facility location studies focus on designing specialized strategyproof mechanisms optimizing these utility or cost objectives separately, we investigate the extent to which we can design unified strategyproof mechanisms optimizing a family of L_p-*aggregated* utility or cost objectives for $p \in [-\infty, \infty]$. The L_p-aggregated objectives are generalization of L_p-norm ($p \geq 1$) on the agents' utilities or costs, and contain the above mentioned objectives as special cases (e.g., total utility when $p = 1$ and minimum utility when $p = -\infty$). That is, the L_p-aggregated objectives encapsulate a spectrum of social objectives, where varying p mediates the trade-off between utilitarian efficiency and egalitarian fairness [13]. Specifically, when p increases, higher costs have a disproportionately larger influence on the objective, and the social planner gives more weights to agents who experience larger costs. In practice, the social planner can choose different values of p to implement certain fairness and trade-offs.

While the standard L_p-norm cost objectives have been examined in the mechanism design for classic (desirable) facility location problems [5, 7, 13, 15, 16, 22, 24], we are the first to explore the L_p-norm objectives, and more generally, the L_p-aggregated utility and cost objectives with any value of p, in mechanism design for obnoxious facility location.

1.1 Our Contributions

We study the design of strategyproof (SP) and group strategyproof (GSP) mechanisms for the standard setting [6, 8, 12, 17, 18, 20, 21, 31] of obnoxious facility location problems with n agents under L_p-aggregated social utility (su) and social cost (sc) objectives. In this setting, a social planner aims to locate a single obnoxious facility on a normalized line segment $[0, 1]$, and each agent i has an undesirable facility location at $x_i \in [0, 1]$. We provide approximation guarantees or ratios for the designed SP/GSP mechanisms and any SP/GSP mechanisms in terms of upper bounds and lower bounds, respectively, across the range of p-values, including boundary cases such as $p = \pm\infty$ and $p \to 0^+$. Table 1 summarizes our upper and lower bounds using intervals or single numbers. All of our bounds for deterministic SP/GSP mechanisms are tight.

Social Utility Objectives. Under the su objective, the utility of each agent at x_i is their distance to the facility location y. We consider maximizing the L_p-aggregated social utility objective $\sum_{i=1}^{n}(|x_i - y|^p)^{1/p}$. For deterministic SP/GSP mechanisms, we provide a tight bound of 2 for maximum utility (the case $p = +\infty$) and a tight bound of $(2^p + 1)^{1/p}$ for all $0 < p < \infty$. In the limit $p \to 0^+$ (which corresponds to Nash welfare up to a monotone transformation via the geometric mean), we show that the approximation ratio is unbounded. For randomized SP/GSP mechanisms, we show the following: for maximum utility, an upper bound of 4/3 and a lower bound of 6/5; for $1 \leq p < \infty$, an upper bound of $\left(\frac{2^p+1}{2^{p-1}+1}\right)^{1/p}$ and a lower bound of $\left(\frac{4(3^p+1)}{3(3^p+1)+2}\right)^{1/p}$; for $0 < p < 1$, an upper bound of $2^{1/p}$ with the same lower bound $\left(\frac{4(3^p+1)}{3(3^p+1)+2}\right)^{1/p}$; for $p \to 0^+$, an upper bound of $\sqrt{2} + 1$ and a lower bound of $\sqrt{6/5}$; and for minimum utility (the egalitarian objective, $p = -\infty$), an upper bound of $O(\sqrt{n})$, together with a lower bound of at least 3/2 in the limit $n \to \infty$ following [12] and a specialized lower bound of approximately 1.026 for $n = 2$.

Table 1: Main Upper and Lower Bound Results of SP/GSP Deterministic and Randomized Mechanisms.

Objectives	*Deterministic*	*Randomized*
su ($p = +\infty$)	2	$\left[\frac{6}{5}, \frac{4}{3}\right]$
su ($1 \leq p < \infty$)	$(2^p + 1)^{\frac{1}{p}}$	$\left[\left(\frac{4(3^p+1)}{3(3^p+1)+2}\right)^{\frac{1}{p}}, \left(\frac{2^p+1}{2^{p-1}+1}\right)^{\frac{1}{p}}\right]$
su ($0 < p < 1$)	$(2^p + 1)^{\frac{1}{p}}$	$\left[\left(\frac{4(3^p+1)}{3(3^p+1)+2}\right)^{\frac{1}{p}}, 2^{\frac{1}{p}}\right]$
su ($p \to 0^+$)	unbounded	$\left[\sqrt{\frac{6}{5}}, \sqrt{2} + 1\right]$
su ($p = -\infty$)	unbounded [12]	$[\frac{3}{2}$ [12], $O(\sqrt{n})]$
sc ($p = +\infty$)	2	≥ 1.008
sc ($1 \leq p < \infty$)	$(2^p + 1)^{\frac{1}{p}}$	$\geq (\frac{5}{4})^{\frac{1}{p}}$
sc ($p = 1$)	3	$\left[\frac{5}{4}, 2\right]$

Social Cost Objectives. Under the sc objective, the cost of each agent at x_i is one minus their distance to the facility location y [21]. We consider minimizing the L_p-aggregated social cost objective $\sum_{i=1}^{n}((1 - |x_i - y|)^p)^{1/p}$. For $p < 1$, the L_p-aggregated sc is a non-convex, non-norm aggregator that over-rewards concentrating harm on a few agents (risk-seeking), violating fairness and robustness. Practically, this undermines geometric intuition and lacks a clear welfare interpretation. Hence, we focus on $p \geq 1$. For deterministic SP/GSP mechanisms, we provide a tight bound of 2 for maximum cost (the egalitarian objective, $p = +\infty$) and a tight bound of $(2^p + 1)^{1/p}$ for all $1 \leq p < \infty$. For randomized SP/GSP mechanisms, we obtain the following: for maximum cost, a lower bound of approximately 1.008; for $1 \leq p < \infty$, an upper bound of 2 when $p = 1$ and a general lower bound of $(5/4)^{1/p}$. Moreover, we show that no *two-candidate* randomized mechanism can achieve a ratio better than $(2^{p-1} + 1)^{1/p}$ for any $pge1$, including the maximum-cost limit $p = +\infty$.

Outline. After discussing the related work below, the paper is structured as follows. Section 2 presents the preliminaries, including the basic setting, objective functions, and the SP/GSP mechanisms under consideration. Sections 3 and 4 analyze the approximation bounds of SP/GSP mechanisms for L_p-aggregated social utilities and social costs, respectively.

1.2 Related Work

We review mechanism design studies most relevant to obnoxious facility location problems and the consideration of L_p-norm social costs in (desirable) facility location problems. We refer readers to [6] for a comprehensive survey on mechanism design for general facility location problems, e.g., [1, 14, 15, 23, 24], which are first considered by [25, 28].

Mechanism Design for Obnoxious Facility Location Problems. Most studies focus on single obnoxious facility on a normalized interval. Ibara and Nagamochi [17] characterize deterministic SP mechanisms as two-candidate mechanisms. Cheng et al. [8] initiate the study of approximate mechanism design for this setting, aiming to maximize total utility (i.e., $p = 1$). They give a 3-approximation deterministic mechanism and a 1.5-approximation randomized mechanism. Feigenbaum and Sethuraman [12] show the tightness of the above bounds. For the minimum-utility objective (i.e., $p = -\infty$), they further show that no deterministic strategyproof mechanism achieves a bounded approximation ratio, while any randomized mechanism has a ratio of at least 1.5. Ye et al. [31] study both objectives of maximizing total utility and the sum of squared utilities; their results for randomized mechanisms imply an upper bound of $\sqrt{5/3}$ and a lower bound of 1.021 for the L_2-norm social utility—where our work improves the latter. The tree network is studied in [26]. Some recent studies consider fairness [20, 21] and locating a single obnoxious facility with "predictions" [18].

L_p-norm Social Costs. While the L_p-norm utility or cost objective has not been considered in obnoxious facility location problems, it has been considered in the classic (single desirable) facility location problems. On the real line, Feigenbaum et al. [13] show that the median mechanism achieves a tight $2^{1-1/p}$-approximation for minimizing L_p-norm social cost, and they also provide some results for randomized mechanisms In the 2-d Euclidean space $\mathbb{R}^2$, Goel and Hann-Caruthers [15] analyze the L_p-norm social cost objective and show that the coordinate-wise median (CM) mechanism has the best approximation ratio among deterministic, anonymous, strategyproof mechanisms. For $p = 1$ with an odd number of agents n, CM attains an approximation ratio of $\sqrt{2}\,\frac{\sqrt{n^2+1}}{n+1}$. For $p \geq 2$, its ratio is bounded above by $2^{\frac{3}{2}-\frac{2}{p}}$.

Another line of work studies d-dimensional L_p spaces, where agent costs (rather than the social objective) are defined via L_p-norm distances. Lin [22] characterizes the two-agent SP mechanisms in L_p spaces. Chan et al. [7] show that CM mechanism is 2-approximation for the maximum cost in any two-dimensional L_p space. Gravin and Jia [16] show that, in any L_p space, CM achieves at most a 3-approximation for minimizing total cost, improving upon the previously best $\sqrt{d}$ approximation [5, 24].

2 PRELIMINARIES

There are n agents $N = \{1, 2, \dots, n\}$ and an interval $[0, 1]$, where the distance of points $a, b \in [0, 1]$ is $d(a, b) = |a - b|$. Each agent $i \in N$ reports a location $x_i \in [0, 1]$ which is their preferred (undesirable) location. We want to determine the location $y \in [0, 1]$ of a single obnoxious facility. A *deterministic* mechanism is a function $f : [0, 1]^n \to [0, 1]$ that maps each location profile $\mathbf{x} = (x_1, \dots, x_n)$ to a facility location. A *randomized* mechanism is a function $f : [0, 1]^n \to \Delta([0, 1])$ that maps each location profile $\mathbf{x}$ to a probability distribution over $[0, 1]$, where the facility is located by sampling from the distribution $f(\mathbf{x})$.

In the setting of strategic agents, each agent's preferred undesirable location is private information, and mechanisms are required to be *strategyproof*—for every agent i, reporting their *true* preferred location x_i a (weakly) dominant strategy. Given facility location y, we assume that each agent i either has a utility $u(x_i, y) = d(x_i, y) = |x_i - y|$ that equals to their distance to y (see [8, 12]), or incurs a cost $c(x_i, y) = 1 - |x_i - y|$ that equals to the interval length minus the distance (see [21]). If the facility location is random and follows a distribution π, then the utility and cost of agent i are simply the expectations $u(x_i, \pi) = \mathbb{E}_{y\sim\pi} u(x_i, y)$ and $c(x_i, \pi) = \mathbb{E}_{y\sim\pi} c(x_i, y)$.

A mechanism f is *strategyproof* (SP) if for all $\mathbf{x} \in [0, 1]^n, i \in N, x'_i \in [0, 1]$, we have $d(x_i, f(\mathbf{x})) \geq d(x_i, f(x'_i, \mathbf{x}_{-i}))$, where $\mathbf{x}_{-i}$ is the profile of all agents but i. Further, a mechanism is *group strategyproof* (GSP) if no group of agents can collude to misreport in a way that makes every member better off. Formally, f is GSP if for all $\mathbf{x} \in [0, 1]^n, S \subseteq N, \mathbf{x}'_S \in [0, 1]^{|S|}$, there exists $i \in S$ so that $d(x_i, f(\mathbf{x})) \geq d(x_i, f(\mathbf{x}'_S, \mathbf{x}_{-S}))$.

2.1 L_p-aggregated Objectives

The mechanisms are evaluated by the worst-case performance on some social objectives. In this paper we consider L_p-aggregated objectives in terms of both social utility (su) and social cost (sc).

Social utility. Given a facility location y when the profile is $\mathbf{x} = (x_1, \dots, x_n)$, for any $p \in \mathbb{R}$, we define the L_p *social utility* as

$$\mathrm{su}_p(y, \mathbf{x}) = \left(\sum_{i\in N} u(x_i, y)^p\right)^{\frac{1}{p}}.$$

If y follows a probability distribution $\mathcal{P}$, then the L_p social utility is the expectation $\mathbb{E}_{y\sim\mathcal{P}} su_p(y, \mathbf{x})$.

Our goal now is to find SP/GSP mechanisms that (approximately) maximize the social utility. Let $OPT_p(\mathbf{x}) = \max_{y\in[0,1]} \mathrm{su}_p(y, \mathbf{x})$ be the optimal L_p social utility. A mechanism is said to be α-*approximation* (or have an approximation ratio α) for the L_p social utility, if $\frac{OPT_p(\mathbf{x})}{\mathrm{su}_p(f(\mathbf{x}),\mathbf{x})} \leq \alpha$ for all profile $\mathbf{x} \in [0, 1]^n$.

One closely related term in mathematics is called the *power mean*: the p-power mean of utilities is $\left(\sum_{i\in N} \frac{1}{n} \cdot u(x_i, y)^p\right)^{1/p}$, which differs with $\mathrm{su}_p(y, \mathbf{x})$ by a normalization constant $\frac{1}{n^{1/p}}$. Hence, a mechanism has the same approximation ratio for maximizing the L_p social utility and the p-power mean.

We interpret the L_p social utilities su_p as follows. For $p \geq 1$, su_p is the L_p-norm of the utility vector, and for $0 < p < 1$ it is the L_p quasi-norm. At $p = 1$, su_p coincides with the utilitarian objective (the sum of utilities). For finite negative p, su_p is generally ill-defined unless all utilities are strictly positive, since negative exponents at zero are undefined. As $p = +\infty$, it is the max-utility objective, whereas as $p = -\infty$, it is the egalitarian (min-utility) objective. A notable boundary case occurs as $p \to 0^+$: the unnormalized su_p becomes unbounded, so we instead consider the p-power mean $\lim_{p\to 0^+} \left(\sum_{i\in N} \frac{1}{n} \cdot u(x_i, y)^p\right)^{1/p} = \sqrt[n]{\prod_{i\in N} u(x_i, y)}$, which is the geometric mean of utilities. Maximizing this geometric mean is equivalent to maximizing the Nash welfare $\prod_{i\in N} u(x_i, y)$.

Social cost. For any $p \in \mathbb{R}$, given facility location y and profile $\mathbf{x}$, define the L_p *social cost* as

$$\mathrm{sc}_p(y, \mathbf{x}) = \left(\sum_{i\in N} c(x_i, y)^p\right)^{\frac{1}{p}}.$$

If y follows a distribution $\mathcal{P}$, then the L_p social cost is $\mathbb{E}_{y\sim\mathcal{P}} sc_p(y, \mathbf{x})$.

We want to minimize the social cost. Let the optimal L_p social cost be $OPT_p(\mathbf{x}) = \min_{y\in[0,1]} \mathrm{sc}_p(y, \mathbf{x})$. A mechanism is said to be α-*approximation* for the L_p social cost, if $\mathrm{sc}_p(f(\mathbf{x}), \mathbf{x}) \leq \alpha \cdot OPT_p(\mathbf{x})$

for all profile $\mathbf{x} \in [0,1]^n$. The interpretations of the L_p social costs parallel those of L_p social utilities, with the roles of min and max reversed: in particular, as $p = +\infty$, it is the egalitarian objective.

2.2 GSP Mechanisms

For a deterministic mechanism f, a point y is called a *candidate* if there is a profile $\mathbf{x}$ such that $f(\mathbf{x}) = y$. Ibara and Nagamochi [17] characterize all deterministic SP/GSP mechanism as 1-candidate mechanisms (that return a fixed point) and 2-candidate *valid threshold* mechanisms. Roughly, a 2-candidate valid threshold mechanism chooses between the two candidates by comparing the number of agents closer to one candidate against a tie-aware cutoff—one that stays within feasible bounds and shifts by at most one when an additional agent is exactly tied; crossing the cutoff switches the outcome (see the detailed definition in [17]).

Lemma 1. *[[17]] Every 2-candidate valid threshold mechanism is GSP, and every deterministic SP mechanism is either a 2-candidate valid threshold mechanism or a 1-candidate mechanism.*

For convenience, we suppose $x_1 \le x_2 \le \ldots \le x_n$. Given profile $\mathbf{x}$, let n_1 be the number of agents with $x_i \in [0, \frac{1}{2}]$ and n_2 be the number of agents with $x_i \in (\frac{1}{2}, 1]$. We consider the following mechanisms: the first is deterministic, and all others are randomized.

Mechanism 1. *[*Majority Vote *[8]] Given profile* $\mathbf{x}$, *if* $n_1 \le n_2$, *return* $y = 0$*; otherwise return* $y = 1$.

Mechanism 2. *Return the uniform distribution over interval* $[0, 1]$.

Mechanism 3. *Given profile* $\mathbf{x}$, *return* $y = 0$ *with probability* $\frac{n_2^2}{n_1^2+n_2^2}$ *and* $y = 1$ *with probability* $\frac{n_1^2}{n_1^2+n_2^2}$.

Mechanism 4. *Given profile* $\mathbf{x}$, *return* $y = 0$ *with probability* $P_0 = \frac{n_2^2+2^p n_1 n_2}{n_1^2+n_2^2+2^{p+1}n_1 n_2}$ *and* $y = 1$ *with probability* $1 - P_0 = \frac{n_1^2+2^p n_1 n_2}{n_1^2+n_2^2+2^{p+1}n_1 n_2}$.

Lemma 2. *Mechanism 1-4 are all GSP.*

Proof. Mechanism 1 is GSP as it is a 2-candidate valid threshold mechanism [8, 17]. Mechanism 2 is GSP because the output is a fixed distribution.

For Mechanism 3, suppose group S misreports locations and assume w.l.o.g. the probability of $y = 0$ decreases. In this way, only those agents with $x_i < \frac{1}{2}$ are better off and can join the group. However, without the attendance of agents located in $(\frac{1}{2}, 1]$, n_1 cannot increase, and thus the probability of $y = 0$, i.e. $\frac{(n-n_1)^2}{n_1^2+(n-n_1)^2}$ would not decrease (which is a decreasing function of n_1). This gives a contradiction to the assumption. The proof for Mechanism 4 follows from the same arguments, as P_0 is decreasing with n_1. □

3 L_p SOCIAL UTILITIES

The utility objectives have been widely studied in the literature, including the total utility [8, 12, 31], minimum utility [12], and sum of squared utilities [31]. All of them are special cases of L_p social utilities. We present a complete picture for any value of p. In Section 3.1 we study L_p norms ($p \ge 1$), quasi norms ($0 < p < 1$), and the maximum utility ($p = +\infty$). In Section 3.2 we study the minimum utility ($p = -\infty$). In Section 3.3 we consider the geometric mean and Nash welfare ($p \to 0^+$).

3.1 L_p Norms and Quasi Norms

For the total utility, i.e., $p = 1$, Cheng et al. [8] prove that Mechanism 1 (Majority Vote) is 3-approximation, and Feigenbaum and Sethuraman [12] show the tightness of this bound for deterministic SP mechanisms. For $p = 2$, Ye et al. [31] explore the sum of squared utilities and their results imply a $\sqrt{5}$-approximation for the L_2 social utility. We extend these results to general L_p norms ($p \ge 1$) and quasi norms ($0 < p < 1$).

Theorem 1. *Mechanism 1 is a* $(2^p + 1)^{1/p}$*-approximation for the* L_p *social utility for every finite* $p > 0$*, and is a 2-approximation for the* L_∞ *social utility (max-utility).*

Proof. By symmetry it suffices to analyze the case $n_1 \le n_2$, in which Mechanism 1 outputs $y = 0$. If $n_1 = 0$, then $y = 0$ is optimal for every $p > 0$, so we assume $n_1 \ge 1$ in the sequel.

For profile $\mathbf{x}$, we write

$$\mathrm{ALG}^p(\mathbf{x}) \coloneqq \sum_{i \in N} |x_i - 0|^p.$$

Let $\mathrm{OPT}^p(\mathbf{x}) \coloneqq \max_{y \in [0,1]} \left(\sum_{i\in N} |x_i - y|^p\right)$ denote the optimal p-power of the L_p social utility. The approximation ratio is the p-th root of $\mathrm{OPT}^p(x)/\mathrm{ALG}^p(x)$.

We prove the bound by partitioning agents into disjoint pairs that cross the midpoint and (possibly) one unpaired agent on the side $x_i \ge \frac{1}{2}$: - For each pair (i, j) with $x_i \le \frac{1}{2} < x_j$, we compare their contribution under $y = 0$ against their optimal contribution under any y. - For any agent k with $x_k \ge \frac{1}{2}$ that is not paired (this occurs only if $n_2 > n_1$), we compare their singleton contribution under $y = 0$ against the optimal singleton contribution.

The proof proceeds via the following two claims.

Claim 1 (Two-agent crossing pairs). Fix $x_i \le \frac{1}{2} < x_j$. Let

$$A^p \coloneqq (x_i - 0)^p + (x_j - 0)^p \quad \text{and} \quad O^p \coloneqq \max_{y\in[0,1]} \left(|x_i - y|^p + |x_j - y|^p\right).$$

Then $O^p \le 1 + \left(\frac{1}{2}\right)^p \le (2^p + 1)A^p$.

Proof of Claim 1. First, since $\frac{1}{2} < x_j$, we have $A^p \ge \left(\frac{1}{2}\right)^p$. We now upper bound O^p by considering the location of an optimal y^*.

(i) If $y^* \le x_i$, then

$$O^p = (x_i - y^*)^p + (x_j - y^*)^p \le x_i^p + x_j^p \le \left(\frac{1}{2}\right)^p + 1.$$

(ii) If $y^* \ge x_j$, by symmetry of (i), we again get $O^p \le 1 + \left(\frac{1}{2}\right)^p$.

(iii) If $x_i < y^* < x_j$, then $O^p = (y^* - x_i)^p + (x_j - y^*)^p$ with $(y^* - x_i) + (x_j - y^*) = x_j - x_i \le 1$. For $0 < p \le 1$, the function $t \mapsto t^p$ is concave, and for fixed sum the maximum of $a^p + b^p$ is attained at balance $a = b = \frac{x_j - x_i}{2}$, so

$$O^p \le 2\left(\frac{x_j - x_i}{2}\right)^p \le 2\left(\frac{1}{2}\right)^p < 1 + \left(\frac{1}{2}\right)^p.$$

For $p > 1$, the function is convex and by the power-mean inequality, $(a + b)^p \ge a^p + b^p$ for $a, b \ge 0$, hence

$$O^p = (y^* - x_i)^p + (x_j - y^*)^p \le (x_j - x_i)^p \le 1 < 1 + \left(\frac{1}{2}\right)^p.$$

Combining the three subcases yields $O^p \le 1 + \left(\frac{1}{2}\right)^p$. Since $A^p \ge (\frac{1}{2})^p$, we obtain

$$\frac{O^p}{A^p} \le \frac{1 + (\frac{1}{2})^p}{(\frac{1}{2})^p} = 2^p + 1,$$

which proves the claim.

Claim 2 (Unpaired agent on the right). If $x_k \geq \frac{1}{2}$, then under $y = 0$ the agent's contribution is $(x_k - 0)^p$, and it is optimal.

The proof follows from $x_k \geq \frac{1}{2}$ and $\arg\max_{y\in[0,1]} |x_k - y|^p = 0$.

We now complete the proof by aggregating disjoint contributions. Form a disjoint collection $\mathcal{P}$ of n_1 crossing pairs (i, j) (each with $x_i \leq \frac{1}{2} < x_j$), and let $\mathcal{U}$ be the set of remaining unpaired agents located in $(\frac{1}{2}, 1]$ (there are $n_2 - n_1$ such agents if any). By Claims 1 and 2, for each pair $(i, j) \in \mathcal{P}$,

$$\max_y \big(|x_i - y|^p + |x_j - y|^p\big) \;\leq\; (2^p + 1)\big(x_i^p + x_j^p\big),$$

and for each $k \in \mathcal{U}$,

$$\max_y |x_k - y|^p \;\leq\; (2^p + 1)x_k^p.$$

Summing these inequalities over the disjoint partition $(\mathcal{P}, \mathcal{U})$ gives

$$\mathrm{OPT}^p(\mathbf{x}) \;\leq\; (2^p + 1)\sum_{i\in N} x_i^p \;=\; (2^p + 1)\,\mathrm{ALG}^p(\mathbf{x}).$$

Taking the p-th root yields the approximation ratio $(2^p + 1)^{1/p}$ for every finite $p > 0$.

Finally, consider the limit $p \to \infty$ (max-utility). Since at least one agent lies in $(\frac{1}{2}, 1]$, the maximum utility under $y = 0$ is at least $\frac{1}{2}$, while the optimal maximum utility cannot exceed 1. Hence the ratio is at most 2, which coincides with $\lim_{p\to\infty}(2^p + 1)^{1/p} = 2$. □

We now show that this guarantee is tight among all deterministic strategyproof mechanisms.

Theorem 2. *No deterministic SP mechanism can achieve an approximation ratio better than $(2^p + 1)^{1/p}$ for the L_p social utility for any finite $p > 0$, or an approximation ratio better than 2 for the L_∞ social utility.*

Proof. By Lemma 1, any deterministic SP mechanism on the line either returns a fixed point or is a 2-candidate valid threshold mechanism. The first case is clearly unbounded. We show that the latter has a ratio at least $(2^p + 1)^{1/p}$ for any finite positive p.

Let the two candidates be $a \leq b$. If $\frac{1}{2} \leq a \leq b$, consider the profile (a, b). The optimal solution is 0, and the p-power of the optimal social utility is $\mathrm{OPT}^p = a^p + b^p$. The mechanism selects either a or b, and the p-power of the social utility is $\mathrm{ALG}^p = 0^p + (b - a)^p = (b - a)^p$. The ratio of them is

$$\frac{\mathrm{OPT}^p}{\mathrm{ALG}^p} = \frac{a^p + b^p}{(b - a)^p}.$$

Let $d = b - a > 0$ and $t = \frac{a}{b-a} = \frac{a}{d}$. Then

$$\frac{a^p + b^p}{(b - a)^p} = \frac{(td)^p + (td + d)^p}{d^p} = t^p + (t + 1)^p.$$

From the constraints $1 \geq b \geq a \geq \frac{1}{2}$, we have $d \leq \frac{1}{2}$ and $t = \frac{a}{d} \geq 1$. Thus, the ratio is at least $t^p + (t + 1)^p \geq 1 + 2^p$. Symmetrically, if $a \leq b \leq \frac{1}{2}$, the same profile (a, b) forces a ratio at least $(2^p + 1)^{1/p}$.

Therefore, it suffices to consider the case with $a < \frac{1}{2} < b$. In our analysis we only consider the instances with $n = 2$ agents in which no one is located at $\frac{a+b}{2}$ (with no ties). Then, a valid threshold mechanism must have a cutoff $s \in \{0, 1, 2\}$ such that the mechanism returns a whenever the number of agents preferring a is at least s, and returns b otherwise. We treat the three possibilities separately.

Case $s = 0$. The mechanism always outputs $y = a$ for any profile under consideration. On the profile (a, a), the mechanism's social utility is zero, while the optimal utility is strictly positive; the approximation ratio is unbounded.

Case $s = 1$. Consider the profile $\big(a, \frac{a+b}{2} + \varepsilon\big)$ with $\varepsilon > 0$ arbitrarily small. The second agent prefers a, so the mechanism outputs $y = a$. Denote

$$\mathrm{ALG}^p \;=\; |a - a|^p + \left|\tfrac{a+b}{2} + \varepsilon - a\right|^p \;=\; \left(\tfrac{b-a}{2} + \varepsilon\right)^p.$$

For the optimal solution, placing the facility at $y = 1$ gives

$$\mathrm{OPT}^p \;\geq\; (1 - a)^p + \left(1 - \tfrac{a+b}{2} - \varepsilon\right)^p.$$

Letting $\varepsilon \to 0^+$ yields

$$\frac{\mathrm{OPT}^p}{\mathrm{ALG}^p} \geq \frac{(1 - a)^p + \left(1 - \frac{a+b}{2}\right)^p}{\left(\frac{b-a}{2}\right)^p} = 2^p\frac{(1 - a)^p}{(b - a)^p} + \frac{(2 - a - b)}{(b - a)^p} \geq 2^p + 1.$$

Thus the ratio is at least $(2^p + 1)^{1/p}$.

Case $s = 2$. This is symmetric to the case $s = 1$. Taking the profile $\big(\frac{a+b}{2} - \varepsilon,\, b\big)$ with $\varepsilon \to 0^+$, only the first agent prefers a, and thus the mechanism outputs $y = b$. The same analysis as above yields the same lower bound $(2^p + 1)^{1/p}$.

The proof for the L_∞ utility is similar. When $\frac{1}{2} \leq a \leq b$ or $a \leq b \leq \frac{1}{2}$, the optimal maximum utility is b or $1-a$, which is at least twice the maximum utility induced by the mechanism $b - a$. When $a < \frac{1}{2} < b$, consider the cutoff $s \in \{0, 1, 2\}$. If $s = 0$, the mechanism is unbounded for the profile (a, a). If $s = 1$, for the profile $(a, \frac{a+b}{2} + \epsilon)$, we have $\mathrm{OPT} \geq 1 - a$ and $\mathrm{ALG} \to \frac{b-a}{2} \leq \frac{\mathrm{OPT}}{2}$. If $s = 2$, for the profile $(\frac{a+b}{2} - \epsilon, b)$, we have $\mathrm{OPT} \geq b$ and $\mathrm{ALG} \to \frac{b-a}{2} \leq \frac{\mathrm{OPT}}{2}$. □

The lower bound in Theorem 2 shows that deterministic mechanisms are fundamentally limited in their approximability. This motivates the study of randomization: can random choices, while preserving strategyproofness in expectation, yield strictly better worst-case guarantees? We next turn to randomized mechanisms and establish that they can outperform.

Theorem 3. *Mechanism 4 is a $2^{1/p}$-approximation for the L_p social utility for every $0 < p < 1$, and a $(\frac{2^p+1}{2^{p-1}+1})^{1/p}$-approximation for every $1 \leq p < \infty$. For the L_∞ social utility (max-utility), it is a 4/3-approximation.*

Proof. We first consider the L_∞ social utility (max-utility). For any instance $\mathbf{x} = (x_1, \ldots, x_n)$, obviously an optimal solution with the largest maximum utility must be $y^* \in \{0, 1\}$. When $n_1 = 0$ (resp. $n_2 = 0$), the mechanism returns the optimal solution $y = 0$ (resp. $y = 1$) with probability 1. When $n_1, n_2 > 0$, the probability of $y = 0$ and $y = 1$ is both $\frac{1}{2}$. Assume w.l.o.g. that $y^* = 0$ is the optimal solution. The approximation ratio follows from

$$\frac{\mathrm{OPT}}{\mathrm{ALG}} = \frac{x_n - 0}{\frac{1}{2}x_n + \frac{1}{2}(1 - x_1)} \leq \frac{1}{\frac{1}{2} + \frac{1}{2}(1 - x_1)} \leq \frac{1}{\frac{1}{2} + \frac{1}{4}} = \frac{4}{3}.$$

For the case with finite positive p, if $n_1 = 0$ or $n_2 = 0$, again the mechanism returns the optimal solution deterministically. Assume $n_1, n_2 > 0$ and w.l.o.g. that $0 < n_1 \leq n_2$. We partition the agents into n_1 groups where each group consists of one agent at $x_i \leq \frac{1}{2}$ and $\frac{n_2}{n_1}$

agent(s) at the same point $x_j > \frac{1}{2}$, and we only need to prove the approximation ratio for each group. Whenever $\frac{n_2}{n_1}$ is not integral we can use $\lfloor \frac{n_2}{n_1} \rfloor$ or $\lceil \frac{n_2}{n_1} \rceil$ instead without affecting the analysis.

We explain why we can assume all the $\frac{n_2}{n_1}$ agents at the same location. Suppose there is a group in which one agent is at $x_i \le \frac{1}{2}$ and the $\frac{n_2}{n_1}$ agents are located at m different points $z_1, z_2, \ldots, z_m > \frac{1}{2}$. For each $j \in [m]$, let λ_j be the number of agents at z_j, and $\lambda_1 + \lambda_2 + \ldots + \lambda_m = \frac{n_2}{n_1}$. If y and y^* are the solution of a mechanism and the optimal solution, respectively, denote by $a_j = |z_j - y|^p$ and $b_j = |z_j - y^*|^p$. We write both ALG^p and OPT^p (defined within this group) as linear combinations of a_j and b_j:

$$\frac{\mathrm{ALG}^p}{\mathrm{OPT}^p} = \frac{\lambda_1 a_1 + \ldots + \lambda_m a_m + |x_i - y|^p}{\lambda_1 b_1 + \ldots + \lambda_m b_m + |x_i - y^*|^p} \ge \min\left(\frac{a_1 n_2/n_1 + |x_i - y|^p}{b_1 n_2/n_1 + |x_i - y^*|^p}, \ldots, \frac{a_m n_2/n_1 + |x_i - y|^p}{b_m n_2/n_1 + |x_i - y^*|^p}\right).$$

Thus, there exists a location $z_k \in \{z_1, \ldots, z_m\}$ such that the ratio when all the $\frac{n_2}{n_1}$ agents are at z_k is no better than the ratio for the group under consideration.

From the perspective of worst-case analysis, thus we can safely consider a group with one agent at $x_i \le \frac{1}{2}$ and $\frac{n_2}{n_1}$ agent(s) at the same location $x_j > \frac{1}{2}$. The p-power of the L_p social utility induced by the mechanism for this group is

$$\mathrm{ALG}^p = \frac{\left(n_2^2 + 2^p n_1 n_2\right)\left(x_i^p + \frac{n_2}{n_1} x_j^p\right)}{n_1^2 + n_2^2 + 2^{p+1} n_1 n_2} + \frac{\left(n_1^2 + 2^p n_1 n_2\right)\left((1 - x_i)^p + \frac{n_2}{n_1}(1 - x_j)^p\right)}{n_1^2 + n_2^2 + 2^{p+1} n_1 n_2}.$$

Then we discuss the optimal social utility OPT. There are three cases for the optimal solution y^*.

(i) $y^* \le x_i$, which means $y^* = 0$. The p-power of the optimal social utility is $\mathrm{OPT}^p = x_i^p + \frac{n_2}{n_1} x_j^p$, and

$$\begin{aligned}\frac{\mathrm{ALG}^p}{\mathrm{OPT}^p} &= \frac{n_2^2 + 2^p n_1 n_2}{n_1^2 + n_2^2 + 2^{p+1} n_1 n_2} + \frac{n_1^2 + 2^p n_1 n_2}{n_1^2 + n_2^2 + 2^{p+1} n_1 n_2} \cdot \frac{n_1(1 - x_i)^p + n_2(1 - x_j)^p}{n_1 x_i^p + n_2 x_j^p} \\ &\ge \frac{n_2^2 + 2^p n_1 n_2}{n_1^2 + n_2^2 + 2^{p+1} n_1 n_2} + \frac{n_1^2 + 2^p n_1 n_2}{n_1^2 + n_2^2 + 2^{p+1} n_1 n_2} \cdot \frac{n_1(1 - \frac{1}{2})^p + n_2(1 - 1)^p}{n_1(\frac{1}{2})^p + n_2 \cdot 1^p} \\ &= 1 - \frac{n_1^2 + 2^p n_1 n_2}{n_1^2 + n_2^2 + 2^{p+1} n_1 n_2} \cdot \frac{2^p n_2}{n_1 + 2^p n_2} \\ &= 1 - \frac{2^p n_1 n_2}{n_1^2 + n_2^2 + 2^{p+1} n_1 n_2} \ge 1 - \frac{2^p}{2^{p+1} + 2} = \frac{2^{p-1} + 1}{2^p + 1},\end{aligned}$$

which is larger than $\frac{1}{2}$. Hence, this gives a $2^{1/p}$-approximation when $0 < p < 1$ and a $\left(\frac{2^p+1}{2^{p-1}+1}\right)^{1/p}$-approximation when $p \ge 1$.

(ii) $y^* \ge x_j$, which means $y^* = 1$. The p-power of the optimal social utility is $\mathrm{OPT}^p = (1 - x_i)^p + \frac{n_2}{n_1}(1 - x_j)^p$. As the proof in (i) is independent of the condition $n_1 \le n_2$, the symmetric argument of (i) proves this case.

(iii) $x_i < y^* < x_j$. This case is similar. □

It is worth noting that the limit $\lim_{p \to \infty} (\frac{2^p+1}{2^{p-1}+1})^{1/p} = 1$ is not equal to the approximation ratio $\frac{4}{3}$ for the L_∞ utility. This is because for randomized mechanisms, the limit of expectations and the expectation of limits may not coincide. We give an example to show the approximation ratio $\frac{4}{3}$ is attainable: consider a profile $(0.5, 1)$, for which the optimal facility location is 0 and the optimal maximum utility is 1. Mechanism 4 returns 0 and 1 with probability $\frac{1}{2}$ each, and the maximum utility is $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$.

We complement the results by a randomized lower bound.

Theorem 4. *No randomized SP mechanism can achieve an approximation ratio better than* $(\frac{4(3^p+1)}{3(3^p+1)+2})^{\frac{1}{p}}$ *for the* L_p *social utility for any finite* $p > 0$. *No randomized SP mechanism can achieve an approximation ratio better than* $\frac{6}{5}$ *for the* L_∞ *social utility.*

We remark that when $p = 2$, our lower bound is $\frac{\sqrt{5}}{2} \approx 1.118$, which improves the previous result of $1.042^{1/2} \approx 1.021$ in [31].

3.2 Minimum Utility

When $p = -\infty$, the L_p social utility is the minium utility of agents, and the minimum-utility-maximization is the egalitarian objective. We first compute the optimal minimum utility, and then derive upper and lower bounds. For convenience, let $x_1 \le x_2 \le \ldots \le x_n$ in profile $\mathbf{x}$. The following proposition is clear.

Proposition 1. *The optimal minimum utility is*

$$OPT_{-\infty}(\mathbf{x}) = \max\left(x_1, 1 - x_n, \max_{1 \le i \le n-1} \frac{x_{i+1} - x_i}{2}\right).$$

If $OPT_{-\infty}(\mathbf{x}) = x_1$, *then* $y = 0$ *is an optimal solution; if* $OPT_{-\infty}(\mathbf{x}) = 1 - x_n$, *then* $y = 1$ *is an optimal solution; if* $OPT_{-\infty}(\mathbf{x}) = \frac{x_{k+1} - x_k}{2}$ *for some* k, *then* $\frac{x_k + x_{k+1}}{2}$ *is an optimal solution.*

For deterministic mechanisms, the approximation ratio is unbounded, as shown by Feigenbaum and Sethuraman [12]. The good news is that randomized mechanisms can achieve a bounded approximation guarantee. However, no 2-candidate randomized mechanism suffices: whenever the two candidates are $\{a, b\}$, the ratio remains unbounded for the profile (a, b) as the minimum utility is 0. By the same reasoning, no k-candidate randomized mechanism can provide a bounded approximation. We show that the uniform distribution has a sub-linear approximation.

Theorem 5. *Mechanism 2 is* $O(\sqrt{n})$*-approximation for the minimum utility.*

Proof. For any profile $\mathbf{x}$, Mechanism 2 always returns the uniform distribution over interval $[0, 1]$. The minimum utility is

$$\begin{aligned}\mathrm{ALG} &= \int_0^1 \min_{i \in N} |y - x_i| \cdot dy \\ &= x_1 \cdot \frac{x_1}{2} + \sum_{i=1}^{n-1} (x_{i+1} - x_i) \cdot \frac{x_{i+1} - x_i}{4} + (1 - x_n) \cdot \frac{1 - x_n}{2}.\end{aligned}$$

We consider three cases based on the optimum.

Case 1: $\mathrm{OPT}_{-\infty}(\mathbf{x}) = x_1$. By Cauchy-Schwarz Inequality,

$$\sum_{i=1}^{n-1} (x_{i+1} - x_i)^2 + (1 - x_n)^2 \ge \frac{1}{n}\left(1 - x_n + \sum_{i=1}^{n-1} (x_{i+1} - x_i)\right)^2$$

$$= \frac{(1-x_1)^2}{n}.$$

Therefore, the ratio is

$$\frac{\mathrm{OPT}_{-\infty}}{\mathrm{ALG}} \le \frac{x_1}{\frac{x_1^2}{2} + \frac{(1-x_1)^2}{4n}} = \frac{4n}{2nx_1 + \frac{(1-x_1)^2}{x_1}} \le \frac{4n}{2nx_1 + \frac{1}{x_1} - 2}$$
$$\le \frac{4n}{2\sqrt{2n} - 2} = O(\sqrt{n}).$$

Case 2: $\mathrm{OPT}_{-\infty}(\mathbf{x}) = 1 - x_n$. By symmetry, the proof is the same as in Case 1 and is omitted.

Case 3: $\mathrm{OPT}_{-\infty}(\mathbf{x}) = \frac{x_{k+1}-x_k}{2}$ for some k. According to Cauchy-Schwarz Inequality, we have

$$x_1^2 + \sum_{i=1}^{k-1}(x_{i+1}-x_i)^2 + \sum_{i=k+1}^{n-1}(x_{i+1}-x_i)^2 + (1-x_n)^2$$
$$\ge \frac{1}{n}\left(x_1 + \sum_{i=1}^{k-1}(x_{i+1}-x_i) + \sum_{i=k+1}^{n-1}(x_{i+1}-x_i) + (1-x_n)\right)^2$$
$$= \frac{(x_k + 1 - x_{k+1})^2}{n}.$$

Denote by $a = x_{k+1} - x_k \in [0,1]$. We have

$$\mathrm{ALG} \ge \frac{(1-a)^2}{4n} + \frac{x_1^2 + (1-x_n)^2}{4} + \frac{a^2}{4} \ge \frac{(1-a)^2}{4n} + \frac{a^2}{4}.$$

Therefore, the ratio is

$$\frac{\mathrm{OPT}_{-\infty}}{\mathrm{ALG}} \le \frac{\frac{a}{2}}{\frac{(1-a)^2}{4n} + \frac{a^2}{4}} = \frac{2n}{na + \frac{(1-a)^2}{a}}$$
$$\le \frac{2n}{na + \frac{1}{a} - 2} \le \frac{2n}{2\sqrt{n} - 2} = O(\sqrt{n}).$$

□

The following example shows that the analysis of Mechanism 2 is asymptotically tight. Let $x_1 = \frac{1}{\sqrt{n}}$, and let $x_2, \ldots, x_n$ evenly partition the interval $[x_1, 1]$. The optimal minimum utility is $\frac{1}{\sqrt{n}}$, while $\mathrm{ALG} = \frac{1}{2n} + \frac{n+1}{4n^2} \cdot \left(1 - \frac{1}{\sqrt{n}}\right)^2$. Hence the approximation ratio is $\Omega(\sqrt{n})$.

As for the inapproximability of randomized mechanisms, [12] proves a lower bound of 1.5 in the limit $n \to \infty$. This does not directly apply when n is finite, particularly for small n. Below we provide a lower bound for the case of two agents.

Theorem 6. *No randomized SP mechanism has an approximation ratio better than* 1.026 *for the minimum utility when* $n = 2$.

3.3 Geometric Mean

When $p \to 0^+$, the unnormalized su_p becomes unbounded, and we consider the normalized L_p social utility instead, that is, the geometric mean of utilities. Formally, the objective is to maximize $\sqrt[n]{\prod_{i\in N} u(x_i, y)}$, which is equivalent to maximizing the Nash welfare $\prod_{i\in N} u(x_i, y)$. First we show that, the approximation ratio is unbounded for all deterministic mechanisms.

Theorem 7. *No deterministic SP mechanism has bounded approximation ratio for the geometric mean of utilities.*

Proof. Recall that any deterministic SP mechanism has at most two candidates by Lemma 1. For the profile where the agents are located at these candidates, then the social utility of the mechanism is 0, while the optimal social utility is positive. □

Hence we consider randomized mechanisms and show that the uniform distribution (Mechanism 2) is a constant approximation.

Theorem 8. *Mechanism 2 is a* $(\sqrt{2}+1)$*-approximation for the geometric mean of utilities.*

Proof. We only need to prove that, for each agent, the individual utility induced by the mechanism is a $(\sqrt{2}+1)$-approximation of their best possible utility. For each agent i with $x_i \ge \frac{1}{2}$, the best possible utility is x_i, and her utility under the mechanism is

$$u(x_i, \mathcal{U}([0,1])) = \int_0^1 |y - x_i| dy = \int_0^{x_i} (x_i - y) dy + \int_{x_i}^1 (y - x_i) dy$$
$$= x_i \frac{x_i}{2} + (1-x_i)\frac{1-x_i}{2} = \frac{2x_i^2 - 2x_i + 1}{2}.$$

The ratio is

$$\frac{x_i}{u(x_i, \mathcal{U}([0,1]))} = \frac{x_i}{\frac{2x_i^2 - 2x_i + 1}{2}} = \frac{2}{2x_i + \frac{1}{x_i} - 2} \le \frac{2}{2\sqrt{2} - 2} = \sqrt{2} + 1.$$

For an agent i with $x_i < \frac{1}{2}$, the symmetric proof gives the ratio. □

Theorem 9. *No randomized SP mechanism has an approximation ratio better than* $\sqrt{\frac{6}{5}}$ *for the geometric mean of utilities.*

Proof. The proof idea mirrors that of Theorem 4. Let f be a randomized SP mechanism. Consider the profile $\mathbf{x}_1 = (\frac{1}{3}, \frac{2}{3})$. For any output y, we have $\left|y - \frac{1}{3}\right| + \left|y - \frac{2}{3}\right| \le 1$. W.l.o.g. assume $\mathbb{E}_{y_1 \sim f(\mathbf{x}_1)}\left[\left|y_1 - \frac{2}{3}\right|\right] \le \frac{1}{2}$. Now we consider a new profile $\mathbf{x}_2 = (\frac{1}{3}, 1)$. By strategyproofness, we must have $\mathbb{E}_{y_2 \sim f(\mathbf{x}_2)}\left[\left|y_2 - \frac{2}{3}\right|\right] \le \frac{1}{2}$, otherwise at profile $\mathbf{x}_1$, agent 2 (whose true location is $\frac{2}{3}$) could beneficially misreport 1. The optimal solution is $y^* = 0$, yielding $\mathrm{OPT}^2 = \frac{1}{3} \cdot 1$.

Next, evaluate the mechanism's social utility under $y_2 \sim f(\mathbf{x}_2)$. We seek the largest possible social utility subject to the constraint $\mathbb{E}\left[\left|y_2 - \frac{2}{3}\right|\right] \le \frac{1}{2}$. Following the analysis for the case $0 < p < 1$ in the proof of Theorem 4, the extreme distribution is attained by placing probability $\frac{3}{4}$ on $y_2 = 0$ and probability $\frac{1}{4}$ on $y_2 = \frac{2}{3}$. Therefore, the approximation ratio satisfies

$$\frac{\mathrm{OPT}^2}{\mathrm{ALG}^2} \ge \frac{\frac{1}{3} \cdot 1}{\frac{3}{4} \cdot \frac{1}{3} \cdot 1 + \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{3}} = \frac{6}{5}.$$

□

4 L_p SOCIAL COSTS

When the facility is obnoxious, agents prefer it to be as far away as possible. Modeling each agent's cost as the remaining segment length—1 minus their distance to the facility—captures this preference: the closer the facility, the higher the cost. In this section, we study the L_p social cost objectives. For $p < 1$, the L_p social cost is a non-norm and non-convex aggregator that overly rewards spreading small costs while tolerating very large costs for a few agents, violating fairness and robustness. Practically, this "risk-seeking" behavior and loss of geometric/algorithmic properties make it ill-suited for cost-minimization objectives. Therefore, this section focuses on L_p social costs with $1 \le p < \infty$, as well as on the limiting case $p = \infty$ (the max-cost). In Section 4.1, we derive tight bounds for deterministic mechanisms. In Section 4.2 we show that randomized mechanisms can improve the approximation ratio for the utilitarian objective ($p = 1$) and establish lower bounds for randomized mechanisms.

4.1 Deterministic Mechanisms

The Majority Vote mechanism (Mechanism 1) performs well for the L_p social cost objectives when $p \ge 1$.

Theorem 10. *Mechanism 1 is a $(2^p + 1)^{\frac{1}{p}}$-approximation for the L_p social cost for every $1 \le p < \infty$. For the L_∞ social cost (max-cost), it is a 2-approximation.*

Proof. When $p = +\infty$, it is the egalitarian objective that minimizes the maximum cost. If $n_1 = 0$ (resp. $n_2 = 0$), the solution $y = 0$ (resp. $y = 1$) returned by the mechanism is clearly optimal for all agents. If both $n_1, n_2 > 0$, the maximum cost induced by the mechanism is no more than 1, while the optimal maximum cost is at least $\frac{1}{2}$. The 2-approximation immediately follows.

When $1 \le p < \infty$, due to symmetry, we only consider the case when $n_1 \le n_2$ and $y = 0$. Note that for each agent i with $x_i \ge \frac{1}{2}$, the solution $y = 0$ is the best possible for them. Thus, we can assume $n_1 = n_2$ without hurting the performance guarantee. We partition the agents into pairs (i, j) with $x_i \in [0, \frac{1}{2}]$ and $x_j \in (\frac{1}{2}, 1]$. It suffices to prove that $y = 0$ is a $(2^p + 1)$-approximation for the p-power of the L_p social cost of these two agents.

The p-power of the mechanism's social cost for (i, j) is

$$A^p = (1 - x_i)^p + (1 - x_j)^p \le 1 + \frac{1}{2^p}.$$

Then, denote by O^p the p-power of the social cost of (i, j) under the optimal solution y^*. We only need to prove the following inequality:

$$O^p \ge \frac{1}{2^p} = \frac{1 + \frac{1}{2^p}}{2^p + 1} \ge \frac{\text{ALG}^p}{2^p + 1}.$$

If $y^* \le x_i$, then

$$O^p = (1 - (x_i - y^*))^p + (1 - (x_j - y^*))^p \ge \frac{1}{2^p} + 0 = \frac{1}{2^p}.$$

If $y^* \ge x_j$, it is symmetric and the proof is omitted. If $y^* \in (x_i, x_j)$, then

$$O^p = (1 - (y^* - x_i))^p + (1 - (x_j - y^*))^p \ge 2 \cdot \left(1 - \frac{x_j - x_i}{2}\right)^p \ge 2 \cdot \frac{1}{2^p} > \frac{1}{2^p},$$

where the first inequality is because the function $g(x) = x^p + (c_0 - x)^p$ with constants $c_0 > 0$ and $p \ge 1$ reaches the minimum value when $x = \frac{c_0}{2}$.

Therefore, we have $A^p \le (2^p+1) \cdot O^p$ for each pair (i, j). Summing up the p-power of the social cost over all pairs, we obtain $\text{ALG}^p \le (2^p + 1) \cdot \text{OPT}^p$, establishing the approximation ratio. □

The following lower bound result indicates that Majority Vote is indeed the best possible for deterministic mechanisms.

Theorem 11. *No deterministic SP mechanism has an approximation ratio better than $(2^p + 1)^{1/p}$ for the L_p social cost for any $1 \le p < \infty$. No deterministic SP mechanism has an approximation ratio better than 2 for the L_∞ social cost.*

4.2 Randomized Mechanisms

For randomized mechanisms, we show that Mechanism 3 (which returns 0 and 1 with specific probabilities) is 2-approximation for both the utilitarian ($p = 1$) and egalitarian ($p = +\infty$) objectives. In particular, when $p = 1$ it improves the bound 3 for deterministic mechanisms.

Theorem 12. *Mechanism 3 is a 2-approximation for the L_p social cost with both $p = 1$ and $p = +\infty$.*

After conducting numerical experiments, we conjecture that Mechanism 3 is indeed a 2-approximation for any $p \ge 1$. Finally, we present lower bounds for randomized strategyproof mechanisms.

Theorem 13. *No randomized SP mechanism has an approximation ratio better than $(\frac{5}{4})^{1/p}$ for the L_p social cost for any $1 \le p < \infty$. No randomized SP mechanism has an approximation ratio better than 1.008 for the L_∞ social cost.*

Furthermore, we establish lower bounds for *two-candidate randomized* mechanisms [21]: there exist two candidates $\{a, b\}$ such that, for any profile $\mathbf{x}$, the mechanism outputs $y = a$ with probability $P_0(\mathbf{x})$ and $y = b$ with probability $1 - P_0(\mathbf{x})$. We prove in that no such mechanism can achieve an approximation ratio better than $(2^{p-1} + 1)^{1/p}$ for any $p \ge 1$.

5 CONCLUSION

We provide a unified analysis of the approximation guarantees of various (group) SP mechanisms for locating a single obnoxious facility on a bounded unit interval to approximately optimize the family of L_p-aggregated utility and cost objectives for $p \in (-\infty, \infty)$. For L_p-aggregated utility and cost objectives, we provide upper and lower bounds on the approximation ratios of any deterministic and randomized (group) SP mechanisms. Our bounds for deterministic (group) SP mechanisms are tight, while our bounds for randomized (group) SP mechanisms have gaps. However, randomized (group) SP mechanisms can often achieve better approximation ratios.

Future work includes tightening gaps for randomized (group) SP mechanisms, extending to higher-dimensional or networked domains, and incorporating richer fairness and robustness notions.

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