# Proportionality Variations in Repeated Fair Division of Indivisible Goods

## Extended Abstract

Karl Jochen Micheel
MICS, CentraleSupélec, Université Paris-Saclay
Gif-sur-Yvette, France
karl-jochen.micheel@centralesupelec.fr

Anaëlle Wilczynski
MICS, CentraleSupélec, Université Paris-Saclay
Gif-sur-Yvette, France
anaelle.wilczynski@centralesupelec.fr

## ABSTRACT

This article deals with fair division of indivisible goods under additive utilities in a repeated context where the same allocation problem is decided multiple times. In this repeated setting, we propose several variations of the well-established proportionality criterion, by requiring that proportionality should be satisfied on average over specified periods of time. While proportionality can be rarely achieved in a one-shot decision, we identify some configurations where considering several rounds helps reach fairness over time via particular proportionality variations. From a computational point of view, we show that most decision problems related to the existence of allocation sequence satisfying variations of proportionality are hard, but we nevertheless highlight several restrictions where the problems can be solved efficiently and show through experiments that our proposed variations can make fairness more achievable.

## KEYWORDS

Computational social choice; Fair division; Proportionality; Fairness over time

**ACM Reference Format:**
Karl Jochen Micheel and Anaëlle Wilczynski. 2026. Proportionality Variations in Repeated Fair Division of Indivisible Goods: Extended Abstract. In *Proc. of the 25th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2026), Paphos, Cyprus, May 25 – 29, 2026*, IFAAMAS, 3 pages. https://doi.org/10.65109/CYZH1121

## 1 INTRODUCTION

In many situations, a group of people have to divide resources between themselves that cannot be split. This problem is known as *fair division of indivisible goods* [2, 7]. Over the years, many fairness criteria have been proposed in the literature to capture relevant notions of fairness while ensuring practical applicability through efficient computation and guaranteed existence of fair solutions [1]. We focus here on *proportionality* [10], which requires each agent to get a utility at least equal to her utility for the whole set of items divided by the number of agents. Because this is not always achievable when assigning indivisible goods [9], e.g., when distributing one item between two agents, we adopt a different perspective by repeating the allocation process with the same set of agents and items and trying to ensure fairness over time. This is natural to many situations where a set of resources need to be assigned repeatedly to agents, even without the arrival or departure of agents and objects, e.g., when assigning courses to teachers, time slots or projects to workers or medical, or laboratory equipment to researchers. In such contexts, the repetition of the decision process can provide a way to "repair" the unfairness created by past decisions, e.g., when distributing one object to two agents, alternating which agent receives the object allows to reach fairness over time.

We follow here an approach which is similar to previous works [3, 5, 8], by considering temporal notions of fairness which apply on consecutive rounds. More precisely, we focus on the satisfaction of proportionality with respect to predefined periods of time. This is motivated by the fact that, when considering temporal concepts, time is already naturally divided into some periods, e.g., years, months, and weeks. Therefore, specific periods may be naturally taken into consideration by agents for evaluating their satisfaction, or by a central authority computing the allocation to justify its fairness. As several periods of time can be considered together to achieve temporal fairness, this gives rise to particularly relevant structures over consecutive rounds for temporal fairness, e.g., laminar structures where, for example, weeks are included in months which are included in years. To model these concepts, we introduce a general family of temporal proportionality criteria where, given a family of subsets of rounds, the goal is to fulfill the proportionality requirement on average for each defined subset of rounds.

## 2 PROPORTIONALITY VARIATIONS

For any positive integer $k$, let $[k]$ denote the set $\{1, \ldots, k\}$. We are given a set of agents $N = [n]$, a set of items $O = [m]$ and a fixed number of rounds $T \in \mathbb{N}$. For every round $t \in [T]$, each agent $i \in N$ is associated with an additive utility function $v_i^t : 2^O \to \mathbb{R}^+$, where $v_i^t(S) = \sum_{o \in S} v_i^t(\{o\})$, for every $S \subseteq O$. The valuation profile over all rounds is given by $V = (v^1, \ldots, v^T)$, where $v^t = (v_1^t, \ldots, v_n^t)$ is the valuation profile in round $t \in [T]$. An instance of repeated fair division is defined as $\langle N, O, V, T \rangle$.

An allocation $A = (A_1, \ldots, A_n)$ is an ordered partition of $O$, with $A_i$ the bundle of objects assigned to agent $i$, satisfying $A_i \cap A_j = \emptyset$, for every pair of agents $i \neq j$, as well as $\bigcup_{i \in N} A_i = O$. For $t \in [T]$, the allocation in round $t$ is denoted by $A^t = (A_1^t, \ldots, A_n^t)$. An allocation sequence over the $T$ rounds is denoted by $\mathcal{A} = (A^1, \ldots, A^T)$.

We propose to adapt proportionality to the context of repeated fair division by considering the satisfaction of proportionality on average over some predefined periods of time.

*Proc. of the 25th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2026), C. Amato, L. Dennis, V. Mascardi, J. Thangarajah (eds.), May 25 – 29, 2026, Paphos, Cyprus.* © 2026 International Foundation for Autonomous Agents and Multiagent Systems (www.ifaamas.org). https://doi.org/10.65109/CYZH1121

**Definition 1** ($\mathscr{S}$-proportionality)**.** *For a given family $\mathscr{S}$ of subsets of $[T]$, an allocation sequence $\mathcal{A} = (A^1, \dots, A^T)$ satisfies $\mathscr{S}$-proportionality if, for every $S \in \mathscr{S}$, and every agent $i \in N$, it holds that $\sum_{t \in S} v_i^t(A_i^t) \geq \frac{\sum_{t \in S} v_i^t(O)}{n}$.*

By definition, when a proportional allocation exists in each round, $\mathscr{S}$-proportionality is satisfiable for every family $\mathscr{S}$.

We will highlight particular relevant structures for the family of subsets over which proportionality is defined. First, we can consider subsets which do not overlap, e.g., to represent the division of the days according to a fixed given scale like months or weeks. For example, one could simply consider a partition of the rounds. The most extreme case of this type of structure is to consider the period of all rounds as a whole, which corresponds to "proportionality overall" established by Igarashi et al. [6].

**Definition 2** (Global proportionality)**.** *An allocation sequence $\mathcal{A}$ satisfies global proportionality if it satisfies $\mathscr{S}$-proportionality for $\mathscr{S}$ composed only of the grand coalition of rounds, i.e., $\mathscr{S} = \{[T]\}$.*

Alternatively, one could aim to ensure to the agents a good level of satisfaction at each step, by considering that the agents have a memory bounded to the last $k$ rounds.

**Definition 3** ($k$-consecutive proportionality)**.** *An allocation sequence $\mathcal{A}$ satisfies $k$-consecutive proportionality if it satisfies $\mathscr{S}$-proportionality for $\mathscr{S}$ defined as the set of all possible subsets of consecutive rounds of size $k$, i.e., $\mathscr{S} := \{\{t, \dots, t+k-1\} : t \in [T-k+1]\}$.*

We illustrate below the previously introduced variations.

*Example 2.1.* Consider a repeated fair division instance over $T = 3$ rounds with $n = 2$ agents and $m = 4$ objects, with the following constant and identical agents' valuations:

| Objects | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| Valuations | 11 | 6 | 4 | 3 |

The allocation sequence below on the left is 2-consecutive but not globally proportional, while the sequence below on the right is globally proportional but not 2-consecutive.

| Rounds | 1 | 2 | 3 |
|---|---|---|---|
| Agent 1 | $\{a, c\}$ | $\{b, d\}$ | $\{a, c\}$ |
| Agent 2 | $\{b, d\}$ | $\{a, c\}$ | $\{b, d\}$ |

| Rounds | 1 | 2 | 3 |
|---|---|---|---|
| Agent 1 | $\{a, b\}$ | $\{b, c\}$ | $\{b, d\}$ |
| Agent 2 | $\{c, d\}$ | $\{a, d\}$ | $\{a, c\}$ |

## 3 COMPUTATIONAL RESULTS

For computational results, we mostly focus on global proportionality, our weakest criterion, which gives a good view of what level of satisfiability can be reached. Trivially, if $m \cdot T < n$, then no allocation sequence satisfies global proportionality. However, even if $m \cdot T \geq n$, global proportionality cannot always be satisfied, as already pointed out by Igarashi et al. [6]. Nevertheless, the guarantee of existence can be reached for specific numbers of rounds.

Proposition 1. *For constant valuations, there always exists an allocation sequence satisfying global proportionality for $T = \ell \cdot n$ rounds, for every $\ell \in \mathbb{N}$, where each allocation is EF1; this sequence can be computed in polynomial time.*

The same guarantee holds for $(\ell \cdot n)$-consecutive proportionality, for any $\ell \in \mathbb{N}$, with $\ell \cdot n \leq T$.

We will now investigate the computational complexity of the following existence problem called ExistGlobalProp: *Given a repeated fair division instance $\langle N, O, V, T \rangle$, does there exist a globally proportional allocation sequence?* Since the problem of existence of a proportional allocation is a specific case of ExistGlobalProp, we already know that ExistGlobalProp is hard for $T = 1$ [4]. We show that hardness still holds for other numbers of rounds.

Theorem 3.1. *ExistGlobalProp is NP-complete, for every $T$ rounds with $T > 1$, even under constant and ternary agents' valuations.*

Therefore, the general existence problem for $\mathscr{S}$-proportionality is hard. We further show that ExistGlobalProp can be polynomially reduced to the proportionality existence problem, implying that the latter problem is NP-complete even under ternary valuations. Up to our best knowledge, hardness of proportionality has not been established before for this specific restriction. In contrast, some positive results can be reached when assuming binary valuations.

Proposition 2. *When the agents have binary valuations, the existence of an allocation sequence satisfying $\mathscr{S}$-proportionality can be decided in polynomial time, as well as constructing one if it exists:*

- *for every family of subsets $\mathscr{S}$ where no subsets overlap,*
- *for every family $\mathscr{S}$ where all subsets have the same size and take consecutive rounds, under constant valuations.*

Note that the second case of Proposition 2 is tight, as we prove the problem is hard when agents' valuations evolve, even when all subsets are consecutive and have the same size.

Beyond binary valuations, we propose another preference restriction, called *mirrored valuations*, which generalizes Borda valuations and ensures further positive results for proportionality variations.

Proposition 3. *Under mirrored valuations with the same valuation scale at each round, there always exists an allocation sequence (computable in polynomial time) which satisfies global proportionality and where each allocation in the sequence is EF1, when:*

- *$n$ is a multiple of $m$ and $T = \frac{2n}{m}$, or*
- *$m = \ell \cdot n$ with $\ell$ odd and $T$ is even, or*
- *$m = \ell \cdot n$ with $\ell$ even (the statement holds for every $\mathscr{S}$-proportionality criterion, as it holds for proportionality).*

## 4 PERSPECTIVES

In this article, we derive several temporal variations for the proportionality criterion depending on which periods of time are taken into consideration for fairness. In general, we cannot guarantee existence and prove that corresponding decision problems are mostly hard. Nevertheless, we have also performed extensive experiments that revealed that, in practice, our variations of proportionality are reachable more often than traditional proportionality. We also identify configurations under specific preference restrictions–most notably, under binary or mirrored valuations–where related decision problems can be solved in polynomial time or where a fair solution is guaranteed to exist. Our introduced notions leave a lot of room for future research by applying different notions of fairness to the introduced families of subsets or even by applying proportionality to different families. Another interesting approach can be approximation of our proposed variations, e.g., satisfying $k$-consecutive proportionality for at least a given number of subsets.

## ACKNOWLEDGMENTS

This work is supported by the ANR project APPLE-PIE (grant ANR-22-CE23-0008-01).

## REFERENCES

[1] Georgios Amanatidis, Haris Aziz, Georgios Birmpas, Aris Filos-Ratsikas, Bo Li, Hervé Moulin, Alexandros A. Voudouris, and Xiaowei Wu. 2023. Fair division of indivisible goods: Recent progress and open questions. *Artificial Intelligence* 322 (2023), 103965.

[2] Sylvain Bouveret, Yann Chevaleyre, and Nicolas Maudet. 2016. Fair Allocation of Indivisible Goods. In *Handbook of Computational Social Choice*, Felix Brandt, Vincent Conitzer, Ulle Endriss, Jérôme Lang, and Ariel D. Procaccia (Eds.). Cambridge University Press, Chapter 12, 284–310.

[3] Benjamin Cookson, Soroush Ebadian, and Nisarg Shah. 2025. Temporal Fair Division. In *Proceedings of the 39th AAAI Conference on Artificial Intelligence (AAAI 2025)*, Vol. 39. 13727–13734.

[4] Stephen Demko and Theodore P. Hill. 1988. Equitable distribution of indivisible objects. *Mathematical Social Sciences* 16, 2 (1988), 145–158.

[5] Edith Elkind, Alexander Lam, Mohamad Latifian, Tzeh Yuan Neoh, and Nicholas Teh. 2025. Temporal Fair Division of Indivisible Items. In *Proceedings of the 24th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2025)*. 676–685.

[6] Ayumi Igarashi, Martin Lackner, Oliviero Nardi, and Arianna Novaro. 2024. Repeated fair allocation of indivisible items. In *Proceedings of the 38th AAAI Conference on Artificial Intelligence (AAAI 2024)*, Vol. 38. 9781–9789.

[7] Jérôme Lang and Jörg Rothe. 2024. Fair division of indivisible goods. In *Economics and Computation: An Introduction to Algorithmic Game Theory, Computational Social Choice, and Fair Division*. Springer, Chapter 9, 605–680.

[8] Karl Jochen Micheel and Anaëlle Wilczynski. 2024. Fairness in repeated house allocation. In *Proceedings of the 27th European Conference on Artificial Intelligence (ECAI 2024)*. 2390–2392.

[9] Hervé Moulin. 2019. Fair division in the internet age. *Annual Review of Economics* 11, 1 (2019), 407–441.

[10] Hugo Steinhaus. 1948. The Problem of Fair Division. *Econometrica* 16, 1 (1948), 101–104.