Exploring Relations among Fairness Notions in Discrete Fair Division

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ABSTRACT

Fair allocation of indivisible items among agents is a fundamental and extensively studied problem. However, *fairness* does not have a single universally accepted definition, leading to a variety of competing fairness notions. Some of these notions are considered stronger or more desirable, but they are also more difficult to guarantee. In this work, we examine 22 different notions of fairness and organize them into a hierarchy. Formally, we say that a fairness notion F_1 *implies* another notion F_2 if every F_1-fair allocation is also F_2-fair. We give a near-complete picture of implications among fairness notions: for almost every pair of notions, we either prove an implication or give a counterexample demonstrating that the implication does not hold. Although some of these results are already known, many are new. We examine multiple settings, including the allocation of goods, chores, and mixed manna. We believe this work clarifies the relative strengths and applicability of these notions, providing a foundation for future research in fair division. Moreover, we developed an *inference engine* to automate part of our work. It is available as a user-friendly web application and may have broader applications beyond fair division.

KEYWORDS

Fair Allocation; Fair Division

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1 INTRODUCTION

The problem of fairly allocating items among multiple agents has garnered significant attention in computer science, multi-agent systems, and game theory. It has applications in many real-world scenarios such as dividing inheritance, distributing natural resources among countries or states, allocating public housing, divorce settlements, and assigning research papers to reviewers.

Research in fair division began with the study of *divisible* resources, such as land, water, and cake [41, 42, 44]. *Fairness* was formally defined primarily in two ways: *envy-freeness* (EF) and *proportionality* (PROP). EF ensures that each agent believes they received the best bundle compared to others, while PROP guarantees that each agent's value for their own bundle is at least a $1/n$ fraction of their value of the entire set of items, where n is the number of agents. By the 1980s, both EF and PROP allocations were shown to exist, and algorithms to find them were developed [41, 42, 44].

These positive results no longer hold when the items are indivisible. For instance, when 5 identical goods must be divided among two agents, an EF or PROP allocation is not possible. However, one can still aim for *approximate* fairness; e.g., an allocation where one agent receives 3 goods and the other receives 2 goods is intuitively as fair as possible. However, when we move beyond such simple examples to the general setting where items are heterogeneous and agents have different preferences, formally defining (approximate) fairness becomes much more complex. As a result, many fairness notions have been proposed for the indivisible setting.

The concept of EF was relaxed to a notion called *EF1*[1] (envy-freeness up to one item), and algorithms were designed to guarantee EF1 allocations [20, 36]. A stronger relaxation of EF, called *EFX* (envy-free up to any item), was introduced in [23]. Despite significant efforts, the existence of EFX allocations remains an open problem. Consequently, relaxations of EFX [6, 22, 26] as well as the existence of EFX in special cases [4, 25, 39] have been explored. A similar story played out for relaxations of PROP: The existence of PROP1 (proportional up to one item) was easy to prove [9], but stronger notions like MMS (maximin share) [34] and PROPx (proportional up to any item) [10] were shown to be infeasible. As a result, various relaxations of these notions have been studied [1, 2, 13, 22, 33, 34]. Additionally, enforcing polynomial-time computability, compatibility with efficiency notions (such as Pareto optimality), or other constraints [16, 17, 28, 32] makes these problems more challenging, often necessitating the use of even weaker fairness concepts.

Hence, for the problem of fairly allocating indivisible items, a variety of fairness notions have been proposed, each offering a different level of perceived fairness. We believe that a systematic comparison of these fairness notions is essential for guiding both practical applications and future research in this area. To contribute to this objective, we present a comprehensive analysis of 22 different fairness notions, examined through the lens of *implications*.

Formally, we say that a fairness notion F_1 *implies* another notion F_2 if every F_1-fair allocation is also F_2-fair. Conversely, if we can identify an F_1-fair allocation that is not F_2-fair, we have a *counterexample*, demonstrating that F_1 does not imply F_2. For many well-studied settings in fair allocation (e.g., additive valuations), we give a near-complete picture of the implications among fairness

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[1] We formally define EF1 and other fairness notions in Section 3.

notions. For almost every pair of notions, we either prove that one notion implies the other, or we give a counterexample. These results establish a hierarchy of fairness notions, and the counterexamples highlight the strengths and weaknesses of each notion. See Fig. 1 for the implications in the context of additive goods and chores, and Appendix H in [31] for similar figures in other fair division settings.

The literature on fair division covers a wide range of settings, including distinctions between goods, chores, and mixed manna, as well as varying entitlements (equal vs. unequal) and different classes of valuation functions. Special cases, such as identical valuations or fair division among only two agents, have also been explored. Although our primary focus is on equal entitlements and additive valuations over goods and over chores, we also consider all the other combinations of these aspects of fair division. At first, this leads to a combinatorial explosion of possible settings. However, we address this challenge by encoding our results on implications and non-implications in a machine-readable format, and by implementing an *inference engine* that uses these results to automatically deduce new implications and non-implications.

For instance, if we query the inference engine with "Does epistemic envy-freeness (EEF) imply maximin share (MMS) for goods with additive valuations?", it would give an affirmative answer based on these three results we show in the paper:

(1) EEF implies minimum-EF-share fairness (MEFS).
(2) MEFS implies PROP under subadditive valuations (Lemma 28 in Appendix C of [31]).
(3) PROP implies MMS for superadditive valuations (Lemma 46 in Appendix C of [31]).

In Section 5, we give two more examples of inference that are much more involved, and highlight the inference engine's usefulness.

1.1 Our Contributions

We establish several implications and counterexamples between fairness notions across various settings. For the most important setting in fair division—additive valuations—we give a near-complete picture of implications for goods, chores, and mixed manna. We also give near-complete picture of implications for submodular, subadditive, and general valuations over goods.

We developed a computer program, called the *inference engine*, to automate part of our work. After proving some (non)implications manually (Section 4), we fed those results into the inference engine, which then generated many additional (non)implications (Fig. 1 and Appendix H of [31]). Our inference engine is implemented as a web application in JavaScript [40]. The engine can be extended beyond fair division (see Section 5), and may have broader applications of independent interest.

Although some of the counterexamples we list (Section 4) are already known, the majority are new. Additionally, we give the simplest possible counterexamples to clearly illustrate the distinctions between fairness notions. This was a challenging task, especially given the large number of counterexamples we show. On the other hand, most implications we prove were previously established, but only for the specific case of additive valuations and equal entitlements. We extend these results to more general settings.

Some fairness notions were originally defined for very specific settings; for example, EFX was introduced in [23] only for additive goods. We extend all fairness notions to the most general fair division setting we consider: mixed manna with unequal entitlements. In some cases, selecting an appropriate definition proved non-trivial, and we explain the insights that motivated our choices.

1.2 Related Work

See [3] for a survey on recent progress and open problems in fair division, where many different fair division settings and fairness notions are considered. [43] is a similar survey for agents with unequal entitlements.

The most common setting in fair division is equally-entitled agents having additive valuations over goods. For this setting, [18] studied implications among 5 fairness notions: CEEI, EF, PROP, MMS, and min-max-share (also called minimum EF share). [5] studies implications between multiplicative approximations of fairness notions. [9] consider implications between EF, PROP, EF1, and PROP1 for mixed manna instead. [24] studies implications for the weighted setting among EF1, PROP1, APS, MNW, and other notions. Over time, as new fairness notions were proposed [8, 11, 14, 22], their connections with other well-established notions were studied. However, the above works only consider a limited number of fairness notions and fair division settings. Our work, on the other hand, aims to be more exhaustive and thus have broader applicability.

1.3 Structure of the Paper

In Section 2, we formally define the fair division problem, describe different fair division settings, and introduce the associated notation. In Section 3, we describe the fairness notions we consider in this paper. In Section 4, we present a summary of our results, i.e., a list of implications and counterexamples between pairs of fairness notions. In Section 5, we describe our inference engine. Section 6 contains concluding remarks and open problems.

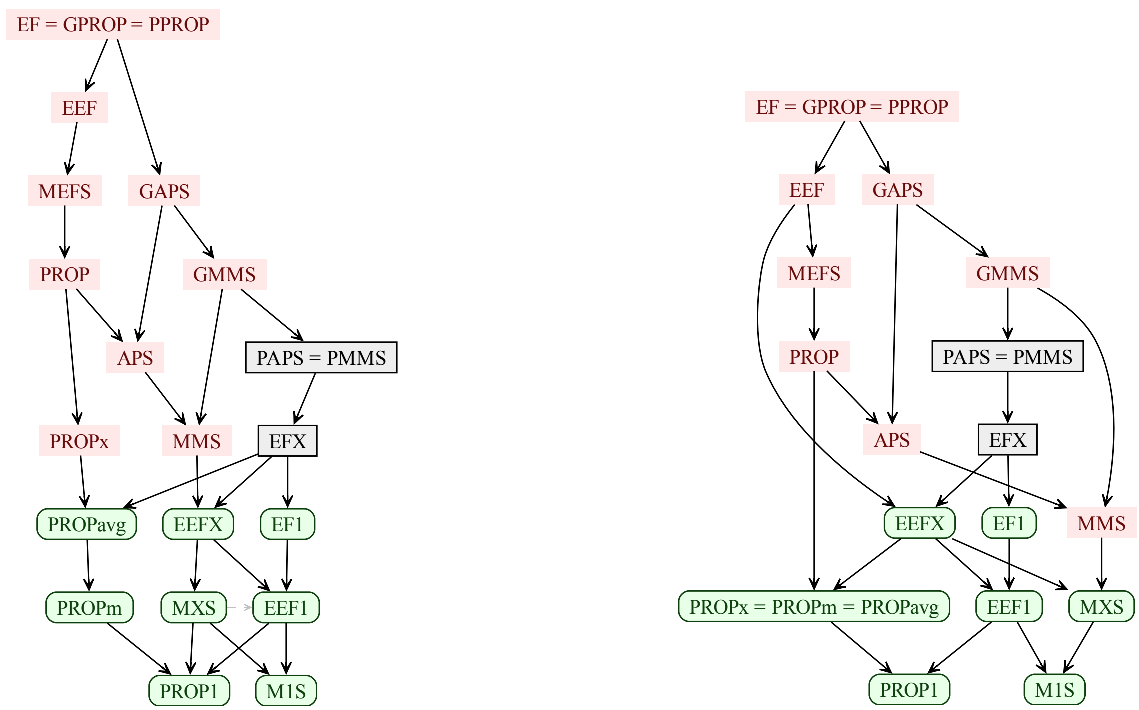
Due to space constraints, all proofs are deferred to appendices in the full version of the paper [31]. Appendices A and B contain details on fair division settings and fairness notions. Appendices C, D, E contain proofs of our (non-)implication results. Appendix F contains results on (in)feasibility of fairness notions. Appendix H contains several implication diagrams, i.e., analogues of Fig. 1 for other fair division settings.

2 PRELIMINARIES

In the fair division problem, there is a finite set M of items that must be distributed among a finite set N of agents fairly. Formally, we are given as input a *fair division instance* $\mathcal{I} := (N, M, V, w)$. Here $w := (w_i)_{i \in N}$ is a collection of positive numbers that sum to 1, and $V := (v_i)_{i \in N}$ is a collection of functions, where $v_i : 2^M \to \mathbb{R}$ and $v_i(\emptyset) = 0$ for each $i \in N$. v_i is called agent i's *valuation function*, and w_i is called agent i's *entitlement*.

Our task is to find a fair allocation. An *allocation* $A := (A_i)_{i \in N}$ is a collection of pairwise-disjoint subsets of M such that $\bigcup_{i=1}^{n} A_i = M$. The set A_i is called agent i's *bundle* in A.

For any non-negative integer k, define $[k] := \{1, 2, \ldots, k\}$. We generally assume without loss of generality that $N = [n]$ and $M = [m]$. For an agent i and item j, we often write $v_i(j)$ instead of $v_i(\{j\})$ for notational convenience. We denote a fair division instance by



(a) Goods **(b) Chores**

Figure 1: Implications between fairness notions for additive valuations over goods and over chores when agents have equal entitlements. There is a vertex for each fairness notion. Notion F_1 implies notion F_2 iff there is a path from F_1 to F_2 in the graph (except that it is not known whether MXS implies EEF1 for goods). Borderless vertices (red) are infeasible notions, vertices with rounded corners (green) are feasible notions, and the feasibility of the remaining vertices (gray) are open problems. Note that goods and chores have some key differences, e.g., for goods, PROP $\implies$ MMS $\implies$ EEF1 $\implies$ PROP1, but for chores, MMS $\not\Longrightarrow$ PROP1, and PROP $\not\Longrightarrow$ EEF1.

$(N, M, V, -)$ when entitlements are equal. For any function $u : 2^M \to \mathbb{R}$ and sets $S, T \subseteq M$, the *marginal value* of S over T is defined as $u(S \mid T) := u(S \cup T) - u(T)$.

2.1 Fairness Notions

A *fairness notion* F is a function that takes as input a fair division instance $\mathcal{I}$, an allocation A, and an agent i, and outputs either true or false. When $F(\mathcal{I}, A, i)$ is true, we say that allocation A is F*-fair* to agent i, or that agent i is F*-satisfied* by allocation A. Allocation A is said to be F-fair if it is F-fair to every agent.

A fairness notion F is said to be *feasible* if for every fair division instance, there exists an F-fair allocation. We say that a notion F_1 of fairness *implies* another notion F_2 of fairness if every F_1-fair allocation is also an F_2-fair allocation. An allocation A is $(F_1 + F_2)$-fair to an agent i if it is both F_1-fair and F_2-fair to agent i.

In a fair division instance $([n], [m], (v_i)_{i=1}^n, w)$, an allocation A *Pareto-dominates* an allocation B if $v_i(A_i) \geq v_i(B_i)$ for each agent $i \in [n]$, and $v_i(A_i) > v_i(B_i)$ for some agent $i \in [n]$. An allocation is *Pareto-optimal* (PO) if it is not Pareto-dominated by any other allocation. An allocation A is F + PO if it is PO and F-fair.

2.2 Fair Division Settings

We study many fair division settings in this paper. A fair division setting is given by multiple *features*. By picking different values of these features, we get many different fair division settings. We consider 5 features in this paper: (i) whether entitlements are equal, (ii) whether there are only two agents, (iii) whether agents have identical valuations, (iv) valuation function type, (v) marginal values. The first three are self-explanatory. We give an overview of the last two, and defer the details to Appendix A.

Valuation Function Type: This feature indicates how values of different sets of items are related to each other. We consider many popular function types like additive, subadditive, superadditive, submodular, supermodular, and general functions.

Marginal values: This feature indicates the possible marginal values items can have. For an agent $i \in N$, the marginal value of item j over set S is given by $v_i(j \mid S) := v_i(S \cup \{j\}) - v_i(S)$. We consider several popular marginal value types, e.g., non-negative (goods), non-positive (chores), bivalued ($\{a, b\}$), binary ($\{0, 1\}$), negative binary ($\{0, -1\}$).

3 FAIRNESS NOTIONS

We now list all the fairness notions we consider in this paper. Due to the large number of notions we consider, we describe them briefly. Details like motivation and examples can be found in the papers we cite for each notion, or in the survey papers cited in Section 1.2.

3.1 Envy-Based Notions

Definition 1 (EF). *Let $\mathcal{I} := ([n], [m], (v_i)_{i=1}^n, w)$ be a fair division instance. In an allocation A, an agent $i \in [n]$* envies *another agent $j \in [n] \setminus \{i\}$ if $v_i(A_i)/w_i < v_i(A_j)/w_j$. Agent i is* envy-free *in A (or A is EF-fair to i) if she doesn't envy any other agent in A.*

For unequal entitlements, most papers use the term WEF instead of EF. But we use the term EF in this paper to emphasize that unequal entitlements is a property of the fair division setting, not the fairness notion. It is easy to see that EF allocations may not exist, so several relaxations have been studied. Two of the most popular relaxations of EF are EF1 (envy-free up to one item) [20, 36], and EFX (envy-free up to any item) [23].

Definition 2 (EF1). *Let $\mathcal{I} := ([n], [m], (v_i)_{i=1}^n, w)$ be a fair division instance. In $\mathcal{I}$, an allocation A is EF1-fair to agent i if for every other agent j, either i does not envy j, or $v_i(A_i)/w_i \ge v_i(A_j \setminus \{g\})/w_j$ for some $g \in A_j$, or $v_i(A_i \setminus \{c\})/w_i \ge v_i(A_j)/w_j$ for some $c \in A_i$.*

Equivalently, A is EF1-fair to i if for every other agent j and some $S \subseteq A_i \cup A_j$ such that $|S| \le 1$, we have $v_i(A_i \setminus S)/w_i \ge v_i(A_j \setminus S)/w_j$.

Definition 3 (EFX). *Let $\mathcal{I} := ([n], [m], (v_i)_{i=1}^n, w)$ be a fair division instance. In $\mathcal{I}$, an allocation A is EFX-fair to agent i if for each $j \in [n] \setminus \{i\}$, either i doesn't envy j, or both of the following hold:*

(1) $$\frac{v_i(A_i)}{w_i} \ge \frac{\max(\{v_i(A_j \setminus S) : S \subseteq A_j \text{ and } v_i(S \mid A_i) > 0\})}{w_j}.$$

(2) $$\frac{\min(\{v_i(A_i \setminus S) : S \subseteq A_i \text{ and } v_i(S \mid A_i \setminus S) < 0\})}{w_i} \ge \frac{v_i(A_j)}{w_j}.$$

Definition 3 looks different from the original definition of EFX given by [23], and also differs from EFX_0, an alternative definition of EFX studied by many papers [25, 26, 39]. However, we show that Definition 3 is equivalent to the original definition of EFX when valuations are submodular, and equivalent to EFX_0 when marginals are (strictly) positive or negative. Moreover, in Appendix B.1, we explain why Definition 3 makes sense and why it is better than EFX_0 and the original definition of EFX.

3.2 Proportionality-Based Notions

Definition 4 (PROP). *Let $\mathcal{I} := ([n], [m], (v_i)_{i=1}^n, w)$ be a fair division instance. For $\mathcal{I}$, agent i's* proportional share *is $w_i \cdot v_i([m])$. Allocation A is* proportional *(PROP) if $v_i(A_i) \ge w_i \cdot v_i([m])$.*

A popular relaxation of PROP is PROP1 (proportional up to one item), where each agent believes her bundle is better than the proportional share after taking some good or giving away some chore. See Definition 5 for a formal definition.

Definition 5 (PROP1 [27]). *Let $\mathcal{I} := ([n], [m], (v_i)_{i=1}^n, w)$ be a fair division instance. In $\mathcal{I}$, an allocation A is PROP1-fair to agent i if either $v_i(A_i) \ge w_i \cdot v_i([m])$, or $v_i(A_i \cup \{g\}) > w_i \cdot v_i([m])$ for some $g \in [m] \setminus A_i$, or $v_i(A_i \setminus \{c\}) > w_i \cdot v_i([m])$ for some $c \in A_i$.*

Note that Definition 5 uses strict inequalities, whereas most papers do not. This definition is from [30], and we use it to make PROP1 a slightly stronger notion without altering its fundamental properties. See Appendix B.4 for details.

PROPx (PROP up to any item) [10, 35], PROPavg (PROP up to the min-avg good) [33], and PROPm (PROP up to the minimax item) [12] are strengthenings of PROP1. Due to space limitations, we formally describe them in Appendices B.5 and B.6. Our definitions are a little different from the original definitions because they also work for mixed manna, and the original definitions of PROPavg and PROPm have minor errors.

3.3 Maximin Share and AnyPrice Share

For equal entitlements, an agent's maximin share (MMS) [20] is the maximum value she can obtain by partitioning the goods into n bundles and picking the worst one. An allocation is MMS-fair to her if her bundle's value is at least her maximin share. See Appendix B.2 for a more formal definition.

MMS has inspired similar notions for the unequal entitlements case. Weighted MMS (WMMS) [29] and pessimistic share (pessShare) [11] are well-known extensions of MMS, i.e., they are equivalent to MMS for equal entitlements. AnyPrice Share (APS) [11], on the other hand, is a strictly stronger notion. See Appendices B.2 and B.3 for details.

3.4 Derived Notions

New fairness notions can be obtained by systematically modifying existing notions. We start with two related concepts, epistemic fairness [8, 22] and minimum fair share [22].

Definition 6 (epistemic fairness). *Let F be a fairness notion. An allocation A is* epistemic-F-fair *to an agent i if there exists another allocation B that is F-fair to agent i and $B_i = A_i$. B is called agent i's* epistemic-F-certificate *for A.*

Definition 7 (minimum fair share). *For a fair division instance $\mathcal{I} := ([n], [m], (v_i)_{i=1}^n, w)$ and fairness notion F, let $\mathcal{A}(\mathcal{I}, F, i)$ be the set of allocations that are F-fair to agent i. Then A is minimum-F-share-fair to agent i if there exists an allocation $B \in \mathcal{A}(\mathcal{I}, F, i)$ such that $v_i(A_i) \ge v_i(B_i)$. Then B is called agent i's* minimum-F-share-certificate *for A. Equivalently, an allocation A is* minimum-F-share-fair *to agent i if $v_i(A_i)$ is at least her minimum-F-share, defined as*

$$\text{minFS}(\mathcal{I}, F, i) := \min_{X \in \mathcal{A}(\mathcal{I}, F, i)} v_i(X_i).$$

We now describe pairwise [23] and groupwise [14] fairness.

Definition 8 (restricting, pairwise fairness, and groupwise fairness). *Let $\mathcal{I} := (N, M, (v_i)_{i \in N}, w)$ be a fair division instance and A be an allocation. For a subset $S \subseteq N$ of agents, where $|S| \ge 2$, let* restrict$(\mathcal{I}, A, S)$ *be the pair $(\mathcal{I}^{(S)}, A^{(S)})$, where allocation $A^{(S)} := (A_j)_{j \in S}$, instance $\mathcal{I}^{(S)} := (S, \bigcup_{j \in S} A_j, (v_j)_{j \in S}, \widehat{w})$, and weights $\widehat{w}_j := w_j / \sum_{k \in S} w_k$.*

A is pairwise-F-fair *to agent i if for all $j \in N \setminus \{i\}$, $A^{(\{i,j\})}$ is F-fair to i in the instance $\mathcal{I}^{(\{i,j\})}$. A is* groupwise-F-fair *to agent i if for all $S \subseteq N \setminus \{i\}$, $A^{(\{i\} \cup S)}$ is F-fair to i in the instance $\mathcal{I}^{(\{i\} \cup S)}$.*

In this paper, we consider these derived notions: Epistemic envy-freeness (EEF), epistemic EFX (EEFX), epistemic EF1 (EEF1), minimum EF share (MEFS), minimum EFX share (MXS), minimum EF1 share (M1S), pairwise proportionality (PPROP), pairwise MMS (PMMS), pairwise APS (PAPS), groupwise proportionality (GPROP), groupwise MMS (GMMS), groupwise APS (GAPS).

4 SUMMARY OF RESULTS

4.1 Manually-Proved Results

We first prove several implications and non-implications among fairness notions manually, i.e., without using the inference engine. We summarize implications in Table 2 (page 7), and defer the proofs to Appendix C. We state several non-implications in Table 3 (page 8), and defer the proofs to Appendix D.

In Appendix C.8, we prove additional implications when marginals are $\{0,1\}$ or $\{-1,0\}$. In Appendix C.9, we prove additional implications for unit-demand valuations. In Appendix E, we prove several non-implications for non-additive valuations. We list results regarding the feasibility and infeasibility of fairness notions in Appendix F.

4.2 Automatically-Inferred Results

After proving implications and counterexamples manually, we feed them into our inference engine, which uses them to infer many more results. These results give us a near-complete picture of implications, similar to Fig. 1, for many different fair division settings. We summarize these results in Table 1.

Table 1: Fair division settings for which we have near-complete pictures of implications. For each of these settings, we also have near-complete pictures for the special case of only two agents and the special case of identical valuations. The last column states the number of unresolved (non-)implications for each setting. We only list settings with at most 3 unresolved implications, whereas there are $\binom{22}{2} = 110$ pairs of fairness notions. The figures referenced in this table can be found in Appendix H.

valuation	marginals	entitlements	figures	open problems
additive	goods	equal	Figures 1a and 9	1
additive	chores	equal	Figures 1b and 9	0
additive	mixed	equal	Figures 4 and 9	0
submodular	goods	equal	Figures 10 and 11	1
subadditive	goods	equal	Figure 13	1
general	goods	equal	Figures 14 and 15	0
additive	goods	unequal	Figures 6 and 7	3
additive	chores	unequal	Figure 8	3
additive	mixed	unequal	–	3
subadditive	goods	unequal	–	2
general	goods	unequal	–	2
additive	$\{-1,0\}$	equal	Figure 5	0
additive	$\{0,1\}$	equal	Figure 5	0
submodular	$\{0,1\}$	equal	Figure 12	2
subadditive	$\{0,1\}$	equal	–	2
general	$\{0,1\}$	equal	–	2

5 INFERENCE ENGINE

We wrote a computer program, called the *inference engine*. We initialize it with a list of implications and counterexamples (Section 4.1), and then we repeatedly query it with a fair division setting. For each query, it infers additional implications and counterexamples using a method similar to transitive closure.

Although the inference procedure is simple, the engine can still make non-trivial inferences. We present two such examples:

(1) EEFX does not imply PROPavg (for equal entitlements over additive goods). This follows from a chain of (non-)implications: EEFX is implied by APS (Lemmas 49 and 42), APS does not imply PROPm (Example 89), and PROPm is implied by PROPavg. Finding such a chain can be difficult if one is not intimately aware of all implications and counterexamples. The engine helps uncover many such non-obvious insights.

(2) APS does not imply EEF1 for unequal entitlements (over additive goods). To prove this, one cannot find a chain as in the previous example, since it doesn't exist. However, for the simpler case of binary valuations among two agents, APS is equivalent to PROP1 (Lemmas 57 and 56), EEF1 is equivalent to M1S (Lemmas 52 and 22), and PROP1 does not imply M1S for this simpler case (Example 93).

For each fair division setting that we query the engine with, it summarizes its inferred results as a directed acyclic graph (DAG). See Fig. 1 and Appendix H for examples of the program's output for various important fair division settings. The program is implemented as a web application; see Fig. 2 for the program's screenshot. Its source is available on Github [40].

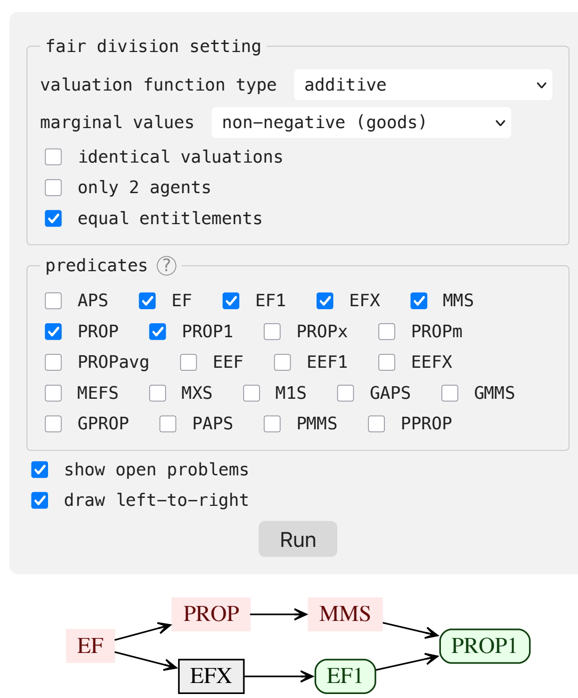


Figure 2: Screenshot from the inference engine's web interface for fair division.

Our program is not limited to just fair division. It can be used more broadly for *conditional predicate implications*. A *predicate* is a function whose co-domain is $\mathbb{B} := \{\texttt{true}, \texttt{false}\}$. Given two predicates $\phi_1, \phi_2 : \Omega \to \mathbb{B}$, we say that ϕ_1 *implies* ϕ_2 conditioned on $S \subseteq \Omega$, denoted as $\phi_1 \Longrightarrow_S \phi_2$, if $\phi_1(x) \Longrightarrow \phi_2(x)$ for all $x \in S$. In fair division, Ω is the set of all pairs $(\mathcal{I}, A)$, where $\mathcal{I}$ is a fair division instance and A is an allocation for $\mathcal{I}$. A fair division setting is a subset of Ω, and fairness notions are predicates over Ω.

The inference engine's input is a tuple $(\mathcal{F}, \Phi, I, C)$. $\mathcal{F}$ is a set family over a ground set Ω. Since Ω can be uncountable, we represent sets in $\mathcal{F}$ implicitly (see Appendix G). Moreover, given $S_1, S_2 \in \mathcal{F}$, we should be able to efficiently tell whether $S_1 \subseteq S_2$. Φ is a set of predicates over Ω. I is a set of *conditional implications*, i.e., a set of triples $(\phi_1, \phi_2, S) \in \Phi \times \Phi \times \mathcal{F}$ where $\phi_1 \Longrightarrow_S \phi_2$. C is a set of *conditional counterexamples*, i.e., a set of triples $(\phi_1, \phi_2, S) \in \Phi \times \Phi \times \mathcal{F}$, where $\phi_1(x) \not\Longrightarrow \phi_2(x)$ for some $x \in S$.

We repeatedly query the inference engine with a set $S \in \mathcal{F}$, and it outputs all possible implications and counterexamples conditioned on S, even those not explicitly present in I and C.

The inference engine works in two steps. In step 1, we find all implications conditioned on S. To do this, we simply select implications from I that are conditioned on supersets of S, and compute their transitive closure. In step 2, we find all counterexamples conditioned on S. To do this, for each $(\phi_1, \phi_2, T) \in C$, we first find all implications conditioned on T like in step 1. Next, if $\phi_1 \Longrightarrow_T \phi_1'$ and $\phi_2' \Longrightarrow_T \phi_2$, then we can infer that $\phi_1' \not\Longrightarrow_T \phi_2'$, because otherwise, by transitivity, we get $\phi_1 \Longrightarrow_T \phi_2$. Using this technique, we expand the set of all counterexamples. Then we select counterexamples conditioned on subsets of S.

We can further extend the inference engine to also make inferences about feasibility and infeasibility of fairness notions using data from Tables 7 and 8 in Appendix F. Specifically, if $F_1 \Longrightarrow_S F_2$ and F_1 is feasible for setting S, then F_2 is also feasible for setting S. Contrapositively, if $F_1 \Longrightarrow_S F_2$ and F_2 is infeasible for setting S, then F_1 is infeasible for setting S.

6 CONCLUSION AND OPEN PROBLEMS

We prove several (non-)implications between fairness notions, and for many settings, we give an almost complete picture of implications. We believe this would help inform further research in fair division. This would be especially useful if one wants to extend a fair division result to a stronger notion or a more general setting, or study a weaker notion or a simpler setting for a hard problem.

Our framework can easily accommodate new fairness notions. One just needs to prove a few key implications and counterexamples, and the rest can be inferred. In fact, we originally started with only 18 fairness notions, and gradually expanded the list to 22 notions as we found out about them. When we added PROPavg [33], for example, we only had to add the following 5 results: PROPx $\Longrightarrow$ PROPavg, PROPavg $\Longrightarrow$ PROPm, EFX $\Longrightarrow$ PROPavg, PROPavg $\not\Longrightarrow$ PROPx, and PROPm $\not\Longrightarrow$ PROPavg. The inference engine inferred PROPavg's relationship to the remaining notions.

Figure 1 shows us that the only notions whose feasibility is unknown are EFX and PMMS, and resolving their feasibility is one of fair division's most important problems. For mixed manna, even the existence of MXS allocations is open. For equally-entitled agents having additive valuations over goods or over chores, EF1+PO allocations are known to exist [15, 23, 38], but their efficient computation remains open. Relaxing the problem to EEF1+PO can be a helpful first step.

Here are four interesting open problems regarding implications that we could not resolve:

(1) For additive goods (equal entitlements), does MXS imply EEF1? Note that the implication holds for the special case of two agents (Lemma 22 in Appendix C).
(2) For additive goods (unequal entitlements), does APS imply PROP1? This is open even for two agents.
(3) For submodular goods (equal entitlements), does MXS imply PROP1? This is true for binary marginals (Lemma 65 in Appendix C).
(4) For submodular goods with binary marginals (equal entitlements), does pairwise-MMS imply MMS?

For less-studied settings, like non-additive valuations over chores, many implications are still open.

Another interesting direction is to study implications of the form F_1+PO $\Longrightarrow$ F_2+PO. For additive valuations and equal entitlements, most questions of this form are already resolved. This is because if $F_1 \Longrightarrow F_2$, then F_1+PO $\Longrightarrow$ F_2+PO. On the other hand, most of our counterexamples use identical valuations, where every allocation is trivially PO.

We omitted some fairness notions from our work because they are fundamentally different from the notions we consider. Equitability (EQ) [19], and its relaxations like EQ1 and EQX [3, 32], compare different utility functions with each other. Notions like CEEI [44] and maximum Nash welfare [23] include an aspect of efficiency in addition to fairness. We also didn't study fair division of divisible items [41, 42, 44], or a mix of divisible and indivisible items [37]. Nevertheless, we believe that these notions and settings offer an interesting line of research, and our inference engine (Section 5) can be readily adapted for them.

Some fairness notions and settings don't fit our model (Section 2), so it is unclear how to systematically represent results about them and extend the inference engine (Section 5) for them. Examples of such settings include constrained fair division [16, 17, 28, 32], notions based on social graphs [8], parametrized notions like WEF(x, y) [24], and multiplicative approximations of fairness notions [5]. Specifically, studying multiplicative approximations presents the following challenges:

(1) For all $\alpha \in (0, 1]$, α-EF1 implies $\alpha/(1+\alpha)$-PMMS, and α-PMMS implies $\alpha/(2-\alpha)$-EF1 (Propositions 3.8 and 4.6 in [5]). It's unclear how to properly visually depict such results, since we can have infinite chains like EF1 $\Longrightarrow$ 1/2-PMMS $\Longrightarrow$ 1/3-EF1 $\Longrightarrow$ 1/4-PMMS ....
(2) To infer new implications using existing ones, we need to be able to represent, compose, and evaluate arbitrary functions of approximation ratios. For some results, such representation is non-trivial. For example, MMS $\Longrightarrow$ EEFX (Lemma 42 in Appendix C.6), but $(1-\varepsilon)$-MMS doesn't imply ε-EEFX for $n = 2$ and any $\varepsilon > 0$ (Proposition 4.8 in [5]).

Extending our techniques to these settings and notions would be an interesting line of research.

Table 2: Implications among fairness notions. For conciseness, we write ep-F instead of epistemic-F, min-F-sh instead of minimum-F-share, g-F instead of groupwise-F, and p-F instead of pairwise-F.

	valuation	marginals	identical	n	entitlements		
$F \Rightarrow$ ep-$F \Rightarrow$ min-F-sh	–	–	–	–	–	Remark 17	trivial
g-$F \Longrightarrow F$ + p-F	–	–	–	–	–	Remark 18	trivial
ep-$F \Longrightarrow F$	–	–	–	$n = 2$	–	Remark 17	trivial
(F or p-F) $\Longrightarrow$ g-F	–	–	–	$n = 2$	–	Remark 18	trivial
EF $\Longrightarrow$ EFX+EF1	–	–	–	–	–	Remark 20	trivial
EEF $\Longrightarrow$ EEFX+EEF1	–	–	–	–	–	Remark 20	trivial
MEFS $\Longrightarrow$ MXS+M1S	–	–	–	–	–	Remark 20	trivial
EFX $\Longrightarrow$ EF1*	additive	–	–	–	–	Lemma 21	trivial
EEFX $\Longrightarrow$ EEF1*	additive	–	–	–	–	Lemma 21	trivial
MXS $\Longrightarrow$ M1S*	additive	–	–	–	–	Lemma 21	trivial
MXS $\Longrightarrow$ M1S	–	≤ 0	–	–	–	Lemma 24	**new**
MXS $\Longrightarrow$ M1S	–	dblMono†	–	$n = 2$	–	Lemma 24	**new**
MXS $\Longrightarrow$ EF1	additive	–	–	$n = 2$	–	Lemma 22	**new**
PROP $\Longrightarrow$ PROPx	–	–	–	–	–	–	trivial
PROP $\Longrightarrow$ PROP1	–	–	–	–	–	–	trivial
PROPx $\Longrightarrow$ PROPavg	–	–	–	–	–	Lemma 25	trivial
PROPavg $\Longrightarrow$ PROPm	–	–	–	–	–	–	trivial
PROPm $\Longrightarrow$ PROPx	–	–	–	$n = 2$	–	–	trivial
PROPm $\Longrightarrow$ PROPx	–	chores	–	–	–	–	trivial
PROPm $\Longrightarrow$ PROP1	submodular	–	–	–	–	Lemma 27	folklore
PROPm $\Longrightarrow$ PROP1	–	$> 0, < 0$	–	–	–	Lemma 27	folklore
MEFS $\Longrightarrow$ PROP	subadditive	–	–	–	–	Lemma 28	[18]
EF $\Longrightarrow$ GPROP	subadditive	–	–	–	–	Lemma 29	[18]
PROP $\Longrightarrow$ EF	superadditive	–	yes	–	–	Lemma 30	folklore
PPROP $\Longrightarrow$ EF	superadditive	–	–	–	–	Lemma 31	folklore
PPROP $\Longrightarrow$ GPROP	submodular	–	–	–	–	Lemma 32	**new**
EEF1 $\Longrightarrow$ PROP1	submodular	–	–	–	equal	Lemma 35	**new**§
EEF1 $\Longrightarrow$ PROP1	subadditive	chores	–	–	–	Lemma 34	**new**
EF1 $\Longrightarrow$ PROP1	submodular	–	–	$n = 2$	–	Lemma 36	**new**
EF1 $\Longrightarrow$ PROP1	subadditive	$> 0, < 0$	–	$n = 2$	–	Lemma 36	**new**
EEFX $\Longrightarrow$ PROPx	subadditive	chores	–	–	–	Lemma 37	[35]
EFX $\Longrightarrow$ PROPavg	submodular	goods	–	–	equal	Lemma 38	**new**
EFX $\Longrightarrow$ PROPx	subadditive	–	–	$n = 2$	–	Lemma 39	**new**
MXS $\Longrightarrow$ PROP1	additive	goods	–	–	equal	Lemma 33	[22]
PMMS $\Longrightarrow$ EFX	additive	–	–	–	equal	Lemma 41	folklore
PWMMS $\Longrightarrow$ EFX	–	goods	–	–	–	Lemma 41	folklore
WMMS $\Longrightarrow$ EEFX	–	goods	–	–	–	Lemma 42	[22]
MMS $\Longrightarrow$ MXS	additive	–	–	–	equal	Lemma 43	**new**
MMS $\Longrightarrow$ MXS+M1S	–	goods	–	–	equal	Lemma 44	**new**
PROP $\Longrightarrow$ APS‡	additive	–	–	–	–	Lemma 45	[11]
PROP $\Longrightarrow$ WMMS‡	superadditive	–	–	–	–	Lemma 46	folklore
APS $\Longrightarrow$ MMS‡	–	–	–	–	equal	Lemma 49	[11]
PWMMS $\Longrightarrow$ PAPS	additive	–	–	–	–	Lemma 51	[11]

* These results hold for additional settings. See Lemma 21 in Appendix C.2 for details.

† A function $v : 2^M \to \mathbb{R}$ is *doubly monotone* if $M = G \cup C$, and $\forall R \subseteq M$, we have $v(g \mid R) \geq 0$ for all $g \in G \setminus R$ and $v(c \mid R) \leq 0$ for all $c \in C \setminus R$.

‡ In addition to $F_1 \Longrightarrow F_2$, we also get p-$F_1 \Longrightarrow$ p-F_2 and g-$F_1 \Longrightarrow$ g-F_2.

§ [9] proved this for additive valuations. Recently, [7] proved this independently for submodular valuations.

Table 3: Non-implications among fairness notions (additive valuations).

	valuation	marginals	identical	n	entitlements		
APS+PROPx $\not\Rightarrow$ PROP	$m = 1$	$1, -1$	yes	any	equal	Example 71	trivial
APS+PROPx $\not\Rightarrow$ EF1	additive	1	yes	$n \geq 3$	equal	Lemma 72	folklore
APS+EEFX $\not\Rightarrow$ EF1	additive	-1	yes	$n \geq 3$	equal	Lemma 73	folklore
EEF $\not\Rightarrow$ EF1	additive	bival*	no	$n = 3$	equal	Lemma 74	**new**
PROP $\not\Rightarrow$ MEFS	additive	> 0	no	$n = 3$	equal	Example 75	**new**
PROP $\not\Rightarrow$ MEFS	additive	< 0	no	$n = 3$	equal	Example 76	**new**
MEFS $\not\Rightarrow$ EEF	additive	> 0	no	$n = 3$	equal	Lemma 77	**new**
MEFS $\not\Rightarrow$ EEF1	additive	< 0 bival	no	$n = 3$	equal	Lemma 79	**new**
EFX $\not\Rightarrow$ MMS	additive	bival*	yes	$n = 2$	equal	Example 80	folklore
EF1 $\not\Rightarrow$ MXS or PROPx	additive	bival*	yes	$n = 2$	equal	Example 81	**new**
PROPx $\not\Rightarrow$ M1S	additive	bival*	yes	$n = 2$	equal	Lemma 82	**new**
MXS $\not\Rightarrow$ PROPx	additive	bival*	yes	$n = 2$	equal	Example 83	[21]
M1S $\not\Rightarrow$ PROP1	additive	bival*	yes	$n = 2$	equal	Lemma 84	**new**
GAPS $\not\Rightarrow$ PROPx	additive	> 0 bival	yes	$n = 3$	equal	Example 85	**new**
GMMS $\not\Rightarrow$ APS	additive	$> 0, < 0$	yes	$n = 3$	equal	Example 87	[11]
PMMS $\not\Rightarrow$ MMS	additive	$> 0, < 0$	yes	$n = 3$	equal	Example 88	[23]
APS $\not\Rightarrow$ PROPm	additive	> 0	yes	$n = 3$	equal	Example 89	**new**
APS $\not\Rightarrow$ PROP1	additive	< 0 bival	yes	$n = 3$	equal	Example 90	**new**
GAPS $\not\Rightarrow$ PROPm	additive	mixed bival	yes	$n = 3$	equal	Lemma 91	**new**
PROPm $\not\Rightarrow$ PROPavg	additive	> 0 bival	yes	$n = 3$	equal	Example 92	**new**
PROP1 $\not\Rightarrow$ M1S	additive	$-1, 1$	yes	$n = 2$	unequal	Example 93	**new**
GAPS $\not\Rightarrow$ PROPx	additive	bival*	yes	$n = 2$	unequal	Lemma 94	**new**
EF1 $\not\Rightarrow$ EFX	additive	$\{-1, 1\}$	yes	$n = 2$	unequal	Example 95	**new**
WMMS $\not\Rightarrow$ M1S	additive	-1	yes	$n = 2$	unequal	Example 96	**new**
EFX $\not\Rightarrow$ WMMS	additive	-1	yes	$n = 2$	unequal	Example 96	**new**
GWMMS $\not\Rightarrow$ PROP1	additive	1	yes	$n = 3$	unequal	Lemma 97	**new**
GWMMS $\not\Rightarrow$ PROP1	additive	-1	yes	$n = 3$	unequal	Lemma 98	**new**
PROP $\not\Rightarrow$ M1S	additive	≤ 0 bival	no	$n = 3$	unequal	Lemma 99	**new**

* Result holds for both positive bivalued marginals and negative bivalued marginals.

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