Fair Orientations: Proportionality and Equitability

AAAI Track

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ABSTRACT

We study the fair allocation of indivisible items under relevance constraints, where each agent has a set of relevant items and can only receive items that are relevant to them. While the relevance constraint has been studied in recent years, existing work has largely focused on envy-freeness. Our work extends this study to other key fairness criteria — such as proportionality, equitability, and their relaxations — in settings where the items may be goods, chores, or a mixture of both. We complement the literature by presenting a picture of the existence and computational complexity of the considered criteria.

KEYWORDS

Fair division; Graph Orientation; Proportionality; Equitability

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1 INTRODUCTION

Fair division is a fundamental and challenging problem in both AI and economics. It involves allocating resources or tasks among participants while ensuring that participants are treated fairly. There is no single fairness criterion universally applicable. Among the various proposed fairness notions, envy-freeness [41] and proportionality [36] are two of the most prominent and well-studied. Envy-freeness requires that no participant prefers the allocation of another over their own, while proportionality requires that each participant receives a value at least their proportional share, a threshold determined by the total number of participants and agents' valuations.

In the classical setting, any item can be allocated to any participant. However, in practical allocation problems, there are often relevance relationships between resources and participants, which restrict an item from being assigned only to a subset of agents. Relevance captures real-world scenarios where certain items are only accessible to specific agents due to geographic restrictions or demand requirements. For example, in allocating railway maintenance jobs between cities A and B, the jobs are typically managed by the local authorities of each city. Similarly, in sports leagues such as the NBA and NFL, matches are held on the home court of one of the competing teams. Motivated by these examples, we study the allocation of indivisible items where relevance relationships between agents and items are represented via graphs. In this framework, vertices correspond to agents, edges correspond to items, and an agent is incident to an edge if the item is relevant to that agent. An agent can only receive the edges incident to her, and such an outcome is called an *orientation.*

The allocation model with relevance relationships, also referred to as the orientation model in this paper, has been studied in machine scheduling [21] and has recently gained popularity in fair division [4, 17, 19, 29]. Most of these works on the fair division problem focus on envy-freeness, and to the best of our knowledge, the only exception is the recent work by Christodoulou and Mastrakoulis [18], which investigates maximin share (MMS) fairness [13] within the orientation model. However, proportionality, another well-studied fairness notion in the classical setting, has been largely overlooked. Proportionality has been widely adopted as a golden fairness criterion in, for example, cake cutting [16, 34], clustering [15, 30], committee selection [27], participatory budgeting [33], matching [3], sortition [20, 22], and bargaining [26].

In the classic setting, an allocation is called *proportional* (PROP) if, informally, each agent gets at least as much value as they would if all the resources were evenly distributed. That is, each agent receives a value of at least $\frac{1}{n}$ fraction of her total value of all items, where n refers to the number of agents. When the relevance relationship is introduced, we refine the threshold so that each agent expects her relevant items to be evenly distributed among the agents relevant to the underlying item. Another fairness criterion that has not been studied in the orientation model is *equitability* [8, 12, 38, 39], which requires that each agent receives the same level of value.

In this paper, we investigate the notions of proportionality and equitability, as well as their relaxations, within a model that considers relevance relationships between agents and items. Our goal is to provide a picture of the existence of the computational results for these fairness notions.

1.1 Our Contribution

We consider the problem of allocating a set E of m indivisible items among n agents, where items can be either goods (providing positive value) or chores (imposing negative value), or a combination of both. Each agent i is associated with a set of relevant items $E_i \subseteq E$. We say an allocation is an orientation if each agent i receives only

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The full version of this paper is available at http://arxiv.org/abs/2602.18098 [40].

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the items relevant to her. Throughout the paper, we focus on the case where all items are allocated, and all resulting allocations are orientations.

Proportionality. Suppose that agent i has an additive valuation function $v_i(\cdot) : 2^E \to \mathbb{R}$, mapping a subset of items to a real number. The original definition of the proportional share for agent i is $\frac{1}{n}v_i(E)$, which is widely adopted in the literature. To define proportionality in the orientation setting, one may set agent i's value for each irrelevant item to 0, an approach applied in the literature (on envy-based notions, e.g., Deligkas et al. [19]), and use the original definition of the proportional share.

However, in the orientation model, whether irrelevant items should be included in an agent's proportional share is a matter of discretion. For example, if a good e is only relevant to agents i and j, then e would be allocated to one of them. If agent i is aware that e will never be allocated to the other $n-2$ agents, she might be unhappy if her entitlement is merely $\frac{1}{n}$ of e. Instead, she may reasonably expect to receive a $\frac{1}{2}$ share of e. Similarly, if a chore or task e is relevant only to agents i, j, it would be overly optimistic for agent i to assume she needs to undertake only a $\frac{1}{n}$ portion of e, as the task will ultimately be assigned to one of these two agents.

We now propose a refined definition of proportional share based on the relevance of each item. For each item $e \in E$, let n_e be the number of agents to whom e is relevant. The proportional share of i is

$$\mathsf{PROP}_i = \sum_{e \in E_i} \frac{1}{n_e} v_i(e),$$

which means that, for each $e \in E_i$, agent i's entitlement is $\frac{1}{n_e}$ of the value of e. This definition reduces to the original definition if each item e is relevant to all agents. An orientation is called PROP if every agent i's value is at least PROP_i.

Next, we summarize our results, beginning with PROP. First, the PROP orientations do not always exist. We are then interested in the problem of deciding whether an instance admits a PROP orientation and have the following results.

- The decision problem is NP-complete when items are all goods (or all chores) and agents have (1,2) or (-1,-2) bi-valued valuations and the relevance can be represented by simple graphs, i.e., every item is relevant to exactly two agents and $|E_i \cap E_j| \leq 1$ for all i, j.
- When valuations are binary (the absolute marginal value of any item being 1), the decision problem can be answered in polynomial time for arbitrary relevance.

We then consider the up to one or any relaxation of PROP. Informally, an orientation is PROP up to one (resp., any) item, abbreviated as PROP1 (resp., PROPX), if each agent would satisfy PROP after hypothetically adding or removing one (resp., any) item from her bundle. For the weaker criterion of PROP1, we show that

- PROP1 and fractional Pareto optimality (fPO) orientations exist and can be computed in polynomial time for mixed items and arbitrary relevance.
- For goods and multigraphs, there is always a PROP1 orientation (called strongly PROP1) that is also an MMS fair allocation, which in turn implies the existence of MMS allocations in the multigraph model, as proved in [18].

This result strengthens the unweighted result in [6], where each item can be allocated to every agent. In the full version of the paper [40], we also propose a stronger version of PROP1, called SPROP1, which cannot be guaranteed to exist for additive valuations without relevance constraints. However, such an orientation does exist in the multigraph orientation model. We also consider PROPX, of which the results are more negative.

- PROPX orientations do not always exist.
- It is NP-complete to decide whether an instance admits a PROPX orientation, even for simple graphs containing only goods (or only chores) and agents with (0,1)- or (0,-1)-binary valuations.

These results show a sharp contrast to the unconstrained setting, where PROPX allocations always exist for chores [5, 31].

Equitability. An allocation is equitable (EQ) if the subjective values of all agents are the same. For indivisible items, EQ allocations do not always exist, and hence, we consider relaxations such as EQ1 and EQX. Their formal definitions are provided in the next section. In contrast to the results in the unconstrained setting, where EQ1 and EQX allocations are guaranteed to exist, we prove that

- EQ1/EQX orientations do not always exist.
- It is NP-complete to decide whether an instance admits an EQ, EQ1, or EQX orientation, even when the relevance can be represented through simple graphs.

Envy-freeness. Although envy-freeness has been studied within the orientation model in the literature, its relaxation, envy-free up to one item (EF1), remains poorly understood for chores. To address this gap, we complement the existing results by examining EF1 orientations for chores. Different from proportionality and equitability, envy-based fairness criteria are intrapersonal; that is, each agent has an evaluation on other agents' bundles so that whether envy exists becomes verifiable. To ensure that envy-based notions are well-defined, a common assumption in existing work is that the value of any irrelevant item is zero for every agent [19, 44]. In this paper, we adopt this assumption when examining envy-based notions of fairness.

Our results show that, in stark contrast to goods for which an EF1 orientation always exists even in the orientation model [19], EF1 orientations may not exist for chores, even when the relevance relationships are represented by simple graphs. Moreover, for such simple graphs, we characterize the necessary and sufficient conditions that guarantee the existence of EF1 orientations and show that it can be decided in polynomial time whether a given instance admits one. However, when relevance is modeled using multigraphs, i.e., each item is relevant to two agents, and multiple items can be relevant to a pair of agents, the decision problem becomes NP-complete.

1.2 Other Related Work

We refer to surveys [2, 32] for comprehensive coverage of recent results on the fair allocation of indivisible items, and [37] for various constraints that have been studied. In the following, we recall the most relevant work to our paper. Without constraints, EF1 and PROP1 allocations always exist and can be computed in polynomial time for goods, chores, and the mixture of goods and chores [11].

Furthermore, PROP1 and Pareto optimal are known to be compatible in the mixed setting [6, 9]. PROPX allocations exist for chores but not for goods [5, 6, 31]. The existence of EFX allocation remains unknown, except for several special cases [14].

Initiated by Christodoulou et al. [17], EFX orientations of relevant items have been extensively studied. Afshinmehr et al. [1] and Hsu [24] extended the study to multi-graphs, and Zhou et al. [44] and Hsu and King [25] generalized the results to chores, the mixture of goods and chores, and other variants of the EFX concept. Zeng and Mehta [43] characterized the graph structures for which EFX orientations always exist. In contrast to EFX, when the items are goods, Deligkas et al. [19] proved that an EF1 orientation always exists even in the general hypergraph orientation model. Without orientation constraints, Amanatidis et al. [4], Kaviani et al. [28, 29] studied the computation of approximate EFX allocations in multi-graphs, where an edge may be allocated to non-incident edges. Recently, Christodoulou and Mastrakoulis [18] examined exact and approximate MMS allocations with additive and more general valuation functions.

2 PRELIMINARIES

2.1 The Orientation Model

For any $k \in \mathbb{N}^+$, let $[k] = \{1, \ldots, k\}$. We study the model of allocating a set $E = \{e_1, \ldots, e_m\}$ of m indivisible items to a set $N = \{1, \ldots, n\}$ of n agents. Each agent i has an additive valuation function $v_i : 2^E \to \mathbb{R}$, that is, for any $S \subseteq E$, $v_i(S) = \sum_{e \in S} v_i(\{e\})$. For ease of notation, we write $v_i(e)$ instead of $v_i(\{e\})$. We call a valuation *binary* if $v_i(e) \in \{0, a\}$ for all items $e \in E$, where $a = 1$ or -1. For any $S \subseteq E$, let $|S|$ be the number of items in S. An item is a good (resp., a chore) for an agent if it yields a non-negative (resp., non-positive) value. If the value of an item is zero for an agent, then the item can be a good or a chore. In this paper, we say an instance is a *goods-instance* (resp., *chores-instance*) if all items are goods (resp., chores) for all agents. We call it a *mixed-instance* if an item can yield a positive value for one agent but a negative value for another.

In the general *orientation model*, each agent i has a non-empty set of *relevant* items $E_i \subseteq E$. For each item $e \in E$, let $N_e = \{i \in N \mid e \in E_i\}$ be the set of agents to whom e is relevant, and let $n_e = |N_e|$. We assume that N_e is non-empty for every e, as otherwise, this item can be removed. The orientation model is general and incorporates the classic unconstrained setting, where $E_i = E$ and $N_e = N$ for all $i \in N$ and $e \in E$. Throughout the paper, we assume $v_i(e) = 0$ for all i and $e \in E \setminus E_i$.

We are also interested in two structured cases. In the *simple graph* orientation model, $n_e = 2$ for all e and $|E_i \cap E_j| \leq 1$ for all $i \neq j$. That is, the model can be described as a simple graph $G = (N, E)$, where each vertex is an agent and each edge is an item relevant to the two agents incident to this edge. Similarly, in the *multigraph* orientation model, $n_e = 2$ for every e, but the number of items in $E_i \cap E_j$ is not limited. That is, there may be multiple edges between any two agents (vertices). In this paper, when the instance is a graph or multigraph, we use the terminologies vertex i and agent i, and edge e and item e, interchangeably.

2.2 Solution Concepts

In the orientation model, each item can only be allocated to an agent to whom the item is relevant. Formally, an *orientation* is denoted by $\pi = (\pi_1, \ldots, \pi_n)$, where $\pi_i \subseteq E_i$, and for any $i \neq j$, $\pi_i \cap \pi_j = \emptyset$ and $\bigcup_{i \in N} \pi_i = E$. Below, we introduce proportional fairness for mixed instances. Let $\mathrm{PROP}_i = \sum_{e \in E_i} \frac{1}{n_e} \cdot v_i(e)$ be the (refined) proportional share for agent i.

Definition 2.1. An orientation $\pi = (\pi_1, \ldots, \pi_n)$ is proportional (PROP) if for every agent $i \in N$, $v_i(\pi_i) \geq \mathrm{PROP}_i$.

Definition 2.2. An orientation $\pi = (\pi_1, \ldots, \pi_n)$ is proportional up to any item (PROPX) if for every agent $i \in N$, either (1) $v_i(\pi_i) \geq \mathrm{PROP}_i$, or (2) $v_i(\pi_i \cup \{e\}) \geq \mathrm{PROP}_i$ for any $e \in E_i \setminus \pi_i$ such that $v_i(e) \geq 0$ and $v_i(\pi_i \setminus \{e\}) \geq \mathrm{PROP}_i$ for any $e \in \pi_i$ such that $v_i(e) \leq 0$.

Definition 2.3. An orientation $\pi = (\pi_1, \ldots, \pi_n)$ is proportional up to one item (PROP1) if for every agent $i \in N$, one of the following three holds: (1) $v_i(\pi_i) \geq \mathrm{PROP}_i$, (2) $v_i(\pi_i \cup \{e\}) \geq \mathrm{PROP}_i$ for some item $e \in E_i \setminus \pi_i$, or (3) $v_i(\pi_i \setminus \{e\}) \geq \mathrm{PROP}_i$ for some item $e \in \pi_i$.

A PROP orientation satisfies PROPX, which in turn satisfies PROP1. The above definition can be directly applied to goods- and chores- instances. For example, for goods-instances, an orientation is PROPX if for every agent i, $v_i(\pi_i \cup \{e\}) \geq \mathrm{PROP}_i$ for any $e \in E_i \setminus \pi_i$; For chores-instances, an orientation is PROPX if for every agent i, $v_i(\pi_i \setminus \{e\}) \geq \mathrm{PROP}_i$ for any $e \in \pi_i$. Regarding PROP1, the quantifier of e in the prior two definitions is changed to existence.

In this paper, we also consider equitability (EQ) and envy-freeness (EF). An orientation $\pi = (\pi_1, \ldots, \pi_n)$ is equitable (EQ) if $v_i(\pi_i) = v_j(\pi_j)$ for any two agents $i, j \in N$, and envy-free (EF) if $v_i(\pi_i) \geq v_i(\pi_j)$. That is, in EQ orientations, the agents have the same value, while in EF orientations, every agent has the largest value in their own allocations. It is known that exact equitability or envy-freeness is not guaranteed to exist for indivisible items, and hence, we focus on their relaxations[1].

Definition 2.4 (EQX). An orientation $\pi = (\pi_1, \ldots, \pi_n)$ is equitable up to any item (EQX) if for any two agents $i, j \in N$, either (1) $v_i(\pi_i) = v_j(\pi_j)$, or (2) $v_i(\pi_i) \geq v_j(\pi_j \setminus \{e\})$ holds for every item $e \in \pi_j$ with $v_j(e) > 0$, and $v_i(\pi_i \setminus \{e\}) \geq v_j(\pi_j)$ holds for every item $e \in \pi_i$ with $v_i(e) < 0$.

Definition 2.5 (EQ1). An orientation $\pi = (\pi_1, \ldots, \pi_n)$ is equitable up to one item (EQ1) if for any two agents $i, j \in N$, one of the following three holds: (1) $v_i(\pi_i) = v_j(\pi_j)$, (2) there exists $e \in \pi_j$ such that $v_i(\pi_i) \geq v_j(\pi_j \setminus \{e\})$, or (3) there exists $e \in \pi_i$ such that $v_i(\pi_i \setminus \{e\}) \geq v_j(\pi_j)$.

Definition 2.6 (EF1). An orientation $\pi = (\pi_1, \ldots, \pi_n)$ is envy-free up to one item (EF1) if for any two agents $i, j \in N$, one of the following three holds: (1) $v_i(\pi_i) \geq v_i(\pi_j)$, (2) there exists $e \in \pi_j$ such that $v_i(\pi_i) \geq v_i(\pi_j \setminus \{e\})$, or (3) there exists $e \in \pi_i$ such that $v_i(\pi_i \setminus \{e\}) \geq v_i(\pi_j)$.

In the relevance model, an item that is not relevant to an agent is assigned a value of 0 for that agent [19, 44]. While this approach

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[1] EFX has been studied in, e.g., [17, 44], and thus we omit the corresponding discussion in the current paper.

is reasonable for goods, it becomes questionable for chores, as we may wish to allocate items that impose no cost on the agents. In this sense, we argue that proportionality and equitability are more suitable fairness criteria in this context, since their definitions do not depend on an agent's valuation of irrelevant items.

3 PROPORTIONALITY AND ITS RELAXATIONS

In this section, we study PROP fairness and its relaxations, namely PROP1 and PROPX, and present results on their existence and on addressing the computational complexity of the associated decision problems.

3.1 PROP Fairness

Similar to the classic fair division of indivisible items, PROP is not always satisfiable even for simple graph instances. Consider an item and two agents, and the item is relevant and yields positive value to both agents. A natural question is whether one can efficiently compute a PROP orientation when it exists. Unfortunately, the answer is no even for restricted instances. In the following, if agents' total value of their relevant items is identical, we say the valuations are *normalized.*

We derive the reduction from 2P2N-3SAT problem, known to be NP-complete [10, 42]. A 2P2N-3SAT instance contains a Boolean formula in conjunctive normal form consisting of the set of variables $X = \{x_1, \ldots, x_s\}$ and the set of clauses $C = \{C_1, \ldots, C_t\}$. Each variable appears exactly twice as a positive literal and exactly twice as a negative literal in the formula, and each clause contains three distinct literals, i.e., $3t = 4s$. Denote by $L = \bigcup_{j=1}^{s}\{x_j^1, x_j^2, \bar{x}_j^1, \bar{x}_j^2\}$ the set of literals and by $C(\ell)$ the clause that contains the literal ℓ.

Theorem 3.1. *Deciding the existence of PROP orientations is NP-complete, even for simple graphs where (1) all items are goods, valuations are normalized, and $v_i(e) \in \{1, 2\}$ for all i and $e \in E_i$; or all items are chores, valuations are normalized, and $v_i(e) \in \{-1, -2\}$ for all i and $e \in E_i$.*

Proof for the goods-instance. The problem is in NP, as given an orientation, one can verify whether it is PROP or not in polynomial time. Next, we derive the NP-completeness by a reduction from 2P2N-3SAT.

Given a 2P2N-3SAT instance, we create a goods-instance as follows:

- for each clause C_j, create a vertex c_j and 11 dummy vertices $d_j^1, \ldots, d_j^{11}$. Create edge (d_j^1, c_j) with value 1 for d_j^1 and c_j. Moreover, the construction of edges with two endpoints being dummy vertices and their values are illustrated in Figure 1;
- for each clause C_j, create 3 edges based on the following rule: if C_j contains literal x_i^k (resp., $\bar{x}_i^{k'}$), create edge (c_j, i) (resp., $(c_j, \bar{i})$) with value 1 for both incident agents, as illustrated in Figures 1 and 2 (edges without an endpoint).
- for each variable x_i, create two vertices i and $\bar{i}$ and one edge $(i, \bar{i})$ with value 2 to both of them, as illustrated in Figure 2;

The created instance has $2s + 12t$ vertices and $s + 17t$ edges. For any i and $e \in E_i$, $v_i(e) \in \{1, 2\}$ holds. Moreover, each agent has a total value of 4 for her incident edges (normalized valuations), and thus, the proportional share of each agent is 2.

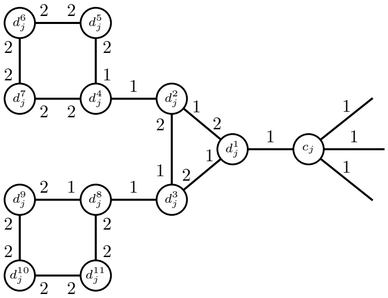

Figure 1: The illustration for goods-instance of the clause gadget for C_j. For each edge, if there is only one label, it represents the value of the edge to both endpoints. If there are two labels, the one closer to a vertex represents that vertex's value for the edge.

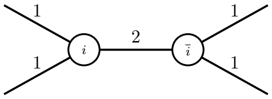

Figure 2: The illustration for goods-instance of the variable gadget for x_i. The label on each edge represents the value of the edge to both endpoints

Suppose that there exists a truth assignment that satisfies all clauses in C. For $c_j, d_j^1, \ldots, d_j^{11}$, observe that we can satisfy each d_j^r with her proportional share even when (d_j^1, c_j) is allocated to c_j; allocate edges in the cycle with 3 vertices d_j^1, d_j^2, d_j^3 in a clockwise manner, and allocate edges in the cycles with 4 vertices in an anticlockwise manner. Edges (d_j^2, d_j^4) and (d_j^3, d_j^8) are arbitrarily allocated to their incident vertices. Thus, we create a partial assignment where each d_j^r receives a value of at least 2 and each c_j receives 1.

For each variable x_i, if x_i is `True`, allocate $(i, \bar{i})$ to vertex i. Then allocate the other two edges incident to i to the vertices corresponding to $C(x_i^1)$ and $C(x_i^2)$; recall that $C(x_i^1)$ refers to the clause that contains literal x_i^1. For vertex $\bar{i}$, allocate her the two edges with value 1. Similarly, if $\bar{x}_i$ is `True`, allocate $(i, \bar{i})$ to vertex $\bar{i}$ and allocate the other two edges incident to $\bar{i}$ to the vertices corresponding to $C(\bar{x}_i^1)$ and $C(\bar{x}_i^2)$, and for vertex i, allocate her the two edges with value 1. At this point, for any $i \in [s]$, both i and $\bar{i}$ receive value 2.

For each vertex c_j, as clause C_j is satisfied, c_j receives one additional incident edge besides (d_j^1, c_j), resulting in a value of at least 2. Therefore, we create a PROP orientation.

Next, for the reverse direction, suppose that there exists a PROP orientation π. We now create a truth assignment of $\{x_i\}_{i \in [t]}$ as follows: if $(i, \bar{i})$ is allocated to vertex i, then set x_i to `True`; and otherwise, if $(i, \bar{i})$ is allocated to vertex $\bar{i}$, set $\bar{x}_i$ to `True`. Such a truth assignment ensures that exactly one of $x_i, \bar{x}_i$ is set to `True`, and hence, the truth assignment is valid. Next, we prove that the assignment satisfies all clauses.

For a contradiction, suppose that there exists a clause $C_{j'}$ not satisfied. By the created truth assignment, each of the three vertices corresponding to the three literals in $C_{j'}$ does not receive the edge with value 2. Consequently, the three edges connecting $c_{j'}$ to variable vertices must not be allocated to $c_{j'}$ so that the three vertices corresponding to the three literals in $C_{j'}$ can satisfy PROP. Hence, the value of $c_{j'}$ is at most 1, meaning that the orientation is not PROP, deriving the desired contradiction. Therefore, the created truth assignment satisfies all clauses. The proof for the chores-instance can be found in the full version [40]. □

To complement the hardness result, we show that for binary valuations and goods-instance (or chores-instance), one can compute in polynomial time a PROP orientation when it exists.

Theorem 3.2. *For the general orientation model with binary additive valuations, one can in polynomial time determine whether a PROP orientation exists or not, and compute one if it exists.*

Proof Sketch. For the goods-instance, if some item has value zero for all relevant agents, arbitrarily allocate the item to the relevant agents. If some item has value one for exactly one relevant agent, allocate the item to that agent. At this point, let π' be the current partial assignment, and for each i, let a_i be the current value of agent i. Moreover, each of the unallocated items has value one for at least two agents relevant to that item.

Next, create a bipartite graph with two parts of vertices X and Y. For each unallocated item with respect to π', create a vertex in X. For each agent i, create $\lceil \mathsf{PROP}_i \rceil - a_i$ vertices in Y. For each such a vertex, connect it to the vertex $x \in X$ if agent i is relevant to the item corresponding to x and has value one for it. We can show that a PROP orientation exists if and only if there exists a Y-perfect matching in the created bipartite graph. Then by applying the matching algorithm, one can determine whether a PROP orientation exists or not and compute one when it exists in time polynomial in n and m .

The idea is similar for chores-instances. The complete proof can be found in the full version [40]. □

3.2 PROP1 and PROPX Fairness

Motivated by the above impossibility results, we now consider the relaxations of PROP. We begin with PROP1 and show that PROP1 orientations always exist and are compatible with fractional Pareto optimal (fPO). An orientation $\pi = (\pi_1, \ldots, \pi_n)$ *Pareto dominates* another orientation $\pi' = (\pi'_1, \ldots, \pi'_n)$ if for any $i \in [n]$, $v_i(\pi_i) \geq v_i(\pi'_i)$ and at least one inequality is strict. An orientation π is Pareto optimal (PO) if no integral orientation Pareto dominates it, and is fPO if no fractional orientation Pareto dominates it, where in an integral orientation, each item is fully allocated to one agent, and in a fractional orientation, a fraction of item can be assigned to some agent. Formally, in a fractional orientation $\pi = (\pi_1, \ldots, \pi_n)$, $\pi_i = (\pi_{i,e})_{e \in E}$ where $0 \leq \pi_{i,e} \leq 1$ represents the portion of e allocated to agent i.

Theorem 3.3. *For the general orientation model and the mixed-instance, we can compute a PROP1 and fPO orientation in polynomial time.*

Proof Sketch. We start from a fractional proportional orientation created as follows: for any e and any $i \in N_e$, let $\pi_{i,e} = \frac{1}{n_e}$. In this fractional orientation, each agent receives her proportional share. Then we compute another fPO fractional orientation that Pareto improves the initial proportional orientation. Moreover, the new fractional orientation has an acyclic *undirected consumption* graph: a bipartite graph in which vertices on one side are the agents, vertices on the other side are the items, and there is an edge between agent i and item e if and only if $\pi_{i,e} > 0$. To show the existence of such an fPO fractional orientation, we adapt the techniques in [35] to the orientation model. Last, we round the fractional orientation to a PROP1 and fPO integral orientation. The formal proof can be found in the full version [40]. □

We next study PROPX, a notion stricter than PROP1. In sharp contrast, the results for PROPX are mostly negative. Particularly, a PROPX orientation does not always exist, even in the case of simple graphs with binary valuations, and determining whether such an orientation exists is computationally intractable.

Proposition 3.4. *PROPX orientations may not exist, even for simple graphs and binary valuations.*

Proof. Let us begin with the goods-instance and consider the instance illustrated in Figure 3. Fix an arbitrary orientation π, and due to symmetry, assume that $(1, 2)$ is allocated to agent 2. Now we focus on agents 1,3, and 4. To ensure that agent 1 satisfies PROPX, both edges $(1, 3)$ and $(1, 4)$ must be allocated to agent 1. As a consequence, one of agent 3 and agent 4 receives $\emptyset$ and violates PROPX.

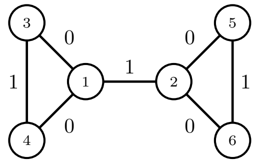

Figure 3: The illustration of the goods-instance where PROPX orientations do not exist. The label on each edge represents the value of the edge to both endpoints

As for chores-instance, we consider the same graph construction and transform agents' valuations into their negatives. By arguments similar to those of goods, one can verify that no orientation is PROPX. □

Theorem 3.5. *For both goods- and chores-instances, deciding the existence of PROPX orientations is NP-complete, even for simple graphs and binary valuations.*

Partial proof for the goods-instance. The decision problem is clearly in NP. We next derive the NP-completeness by a reduction from 2P2N-3SAT.

Given a 2P2N-3SAT instance, we create a goods-instance as follows:

- for each clause C_j, create a vertex c_j and 6 dummy vertices $d_j^1, \ldots, d_j^6$. Create edges (c_j, d_j^1) and (c_j, d_j^4); moreover, create edges with both endpoints being dummy vertices, as illustrated in Figure 4;

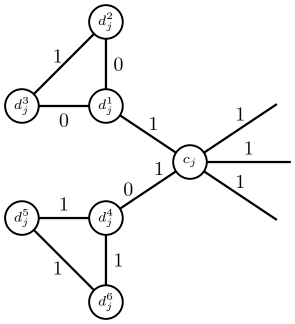

Figure 4: The illustration for goods-instance of the clause gadget for C_j. For each edge, if there is only one label, it represents the value of the edge to both endpoints. If there are two labels, the label closer to a vertex represents that vertex's value for the edge.

- for each clause C_j, create three more edges: if C_j includes literal x_i^k (resp., $\bar{x}_i^{k'}$), add an edge (c_j, i) (resp. $(c_j, \bar{i})$ with value 1 for both incident agents, as illustrated in Figure 4.
- for each variable x_i, create two vertices i and $\bar{i}$, and an edge $(i, \bar{i})$ with value 1 for both $i, \bar{i}$. Moreover, apart from $(i, \bar{i})$, each i and $\bar{i}$ values all other incident edges at 0.

The created instance has $2s + 7t$ vertices and $s + 11t$ edges; moreover, each agent's valuation is binary. We first claim that in a PROPX orientation (if it exists), for each c_j, edge (c_j, d_j^1) must be allocated to d_j^1. For a contradiction, assume that this is not the case. We now focus on agents d_j^1, d_j^2, d_j^3. Without loss of generality, assume edge (d_j^2, d_j^3) is allocated to d_j^2 in the orientation. Then in order to make d_j^3 satisfy PROPX, we need to allocate (d_j^1, d_j^3) to her, which then makes d_j^1 violate PROPX, as d_j^1 has value zero in the orientation and edge (d_j^1, d_j^3) (with value zero) is not allocated to her but her proportional share is $\frac{1}{2}$. One can verify that there exists a truth assignment satisfying all clauses if and only if there exists a PROPX orientation. The formal arguments and the proof for chores instance can be found in in the full version [40]. ◻

4 EQUITABILITY AND ITS RELAXATIONS

In this section, we focus on equitability-related notions that have been widely studied in the unconstrained setting [8, 23], but remain underdeveloped in the orientation model.

In the unconstrained setting, EQX/1 allocations can be achieved through the greedy algorithm. However, our results indicate that neither EQ1 nor EQX is guaranteed to be satisfiable in the orientation model. Even worse, deciding their existence is NP-complete. For completeness, we also provide the complexity results on the notion of EQ.

Theorem 4.1. *Determining whether an EQ orientation exists on a simple graph in both goods- and chores-instance is NP-complete.*

The proof of the theorem can be found in the full version [40].

Proposition 4.2. *An EQ1 orientation is not guaranteed to exist even for simple graphs.*

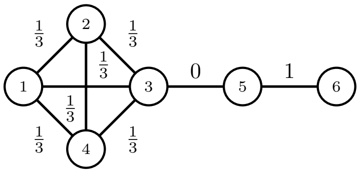

Figure 5: The illustration of the goods-instance where EQ1 does not exist. The label on each edge represents the value of the edge to both endpoints.

Proof. Consider the example with identical valuations shown in Figure 5.In any orientation, one of the agents 5 and 6 receives a value of 0; suppose that this agent is 5. However, as there are 6 edges among 1, 2, 3, and 4, by the pigeonhole principle, at least one of them will receive two or more edges. Suppose this agent is 1, then we have $v_1(\pi_1 \setminus \{e\}) = \frac{1}{3} > v_5(\pi_5) = 0$ for all $e \in \pi_1$, which means that EQ1 does not hold between 1 and 5. As for the chores-instance, we use the same graph while changing each agent's value on each edge to the corresponding negative value. In that case, we have $v_1(\pi_1 \setminus \{e\}) = -\frac{1}{3} < v_5(\pi_5) = 0$ for all $e \in \pi_1$, which violates EQ1. ◻

Next, we study the problem of deciding the existence of EQ1 orientations for a given instance. We show that this problem is computationally intractable.

Theorem 4.3. *Determining whether an EQ1 orientation exists on a simple graph in both goods- and chores-instance is NP-complete.*

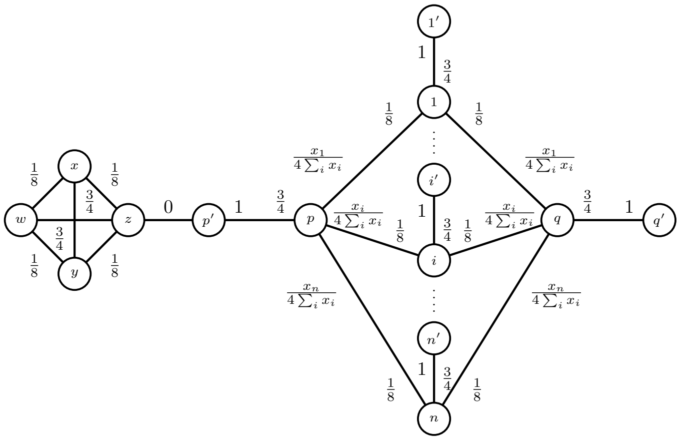

Figure 6: Illustration of the goods-instance for reduction from EQ1. For the edge with one label, the label is the value of that edge for both endpoints. If an edge has two labels, the label closer to a vertex represents that vertex's value for the edge.

Proof. The decision problem is in NP as given an orientation, one can decide whether it is EQ1 in polynomial time. We derive the reduction from the Partition problem and begin with the goods-instance.

We create a goods-instance of a simple graph, as illustrated in Figure 6:

- create vertices $1, \dots, n$ and vertices $1', \dots, n'$, and for each $i \in [n]$, create an edge (i, i');
- create vertices p, p', q, q', and create edges (p, p') and (q, q'). Moreover, for each $i \in [n]$, create edges (p, i) and (q, i);
- create vertices w, x, y, z and edges such that these four vertices form K_4 (i.e., a complete graph with 4 vertices). Moreover, create edge (z, p').

We define the agents' valuations as follows:

- for each agent $i \in [n]$, $v_i((i, i')) = \frac{3}{4}$, $v_{i'}((i, i')) = 1$, $v_i((p, i)) = v_i((q, i)) = \frac{1}{8}$, and $v_p((p, i)) = v_q((q, i)) = \frac{x_i}{4 \sum_i x_i}$;
- $v_p((p, p')) = v_q((q, q')) = \frac{3}{4}$, $v_{p'}((p, p')) = v'_q((q, q')) = 1$, and $v_{p'}((z, p')) = 0$;
- $v_w((w, z)) = v_z((w, z)) = v_x((x, y)) = v_y((x, y)) = \frac{3}{4}$, and $v_a((a, b)) = v_b((a, b)) = \frac{1}{8}$ for each pair of vertices $(a, b) \in \{(w, x), (x, z), (z, y), (y, w)\}$.

We first claim that in an EQ1 orientation π, each agent has a value of at least $\frac{1}{8}$. Let us focus on w, x, y, z, and in any orientation (and hence in π), there must be an agent receiving two edges, with one having a value of $\frac{1}{8}$ and another having a value of $\frac{3}{4}$. Thus, after removing the edge with value $\frac{3}{4}$, that agent still has value $\frac{1}{8}$. As a result, in an EQ1 orientation π, each agent should receive a value of at least $\frac{1}{8}$. Then edges (p', p) and (q', q) must be allocated to p' and q' respectively, and for each $i \in [n]$, edge (i', i) should be allocated to i'. Moreover, for each $i \in [n]$, vertex i should receive at least one (indeed exactly one) of (i, p) and (i, q). Then the total value of the edges that can be allocated to p and q is $\frac{1}{4}$, which makes in π, the value of p and q should be $\frac{1}{8}$. Thus, there exists a subset of $\{x_1, \dots, x_n\}$ with the total value T. Therefore, there exists an EQ1 orientation if and only if the Partition instance is a yes-instance.

As for the chores-instance, we consider the same graph creation but negate agents' valuations. Similarly, in any orientation, there must be an agent among w, x, y, z receiving at least one edge with value $-\frac{1}{8}$ and one edge with value $-\frac{3}{4}$. Thus, in an EQ1 orientation π, every agent should receive a value of at most $-\frac{1}{8}$. Then edges (p', p) and (q', q) must be allocated to p' and q', respectively. Moreover, for each $i \in [n]$, edge (i', i) should be allocated to i'. Then one can verify that the unallocated edges can make both p and q have a value of no greater than $-\frac{1}{8}$ if and only if the Partition instance is a yes-instance. Therefore, we establish an equivalence between the existence of EQ1 orientation and a yes-instance of the Partition problem. □

The non-existence of EQ1 orientations directly implies that EQX orientations are not guaranteed to exist, as EQX is a notion stricter than EQ1, while the computational hardness of the EQ1 decision problem does not directly extend to EQX. Below, we establish the hardness of the decision problem concerning EQX.

Theorem 4.4. *Determining whether an EQX orientation exists on a simple graph in both goods- and chores-instances is NP-complete, even for instances with additive valuations.*

Partial proof for goods-instance. The decision problem is clearly in NP. We derive the reduction from the Partition problem: whether a set of integers $x_1, \dots, x_n$ can be divided into two subsets with equal sums.

We create a goods-instance of a simple graph, as illustrated in Figure 7:

- create vertices $1, \dots, n$ and vertices p, q; for each $j \in [n] \cup \{p, q\}$, create two dummy vertices k_j, k'_j;
- for each $j \in [n] \cup \{p, q\}$, create edges (j, k_j), (j, k'_j), (k_j, k'_j). Moreover, for each $j \in [n]$, create edges (j, p) and (j, q).

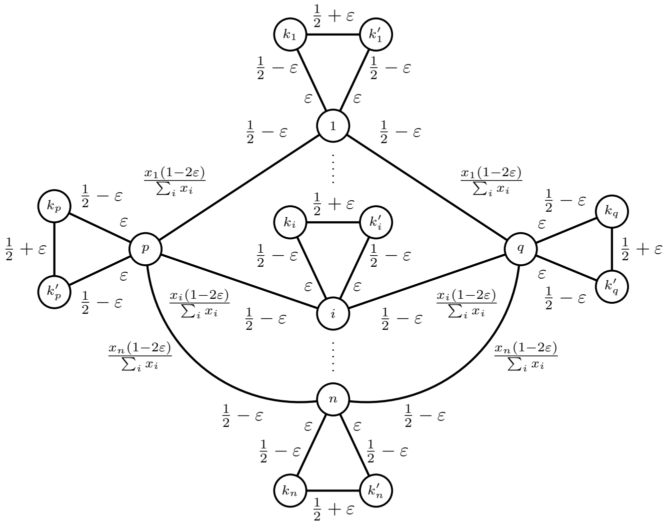

Figure 7: Illustration of the goods-instance for reduction from EQX. The label closer to a vertex represents that vertex's value for the edge.

We define the agents' valuations as follows:

- for each $j \in [n] \cup \{p, q\}$, let $v_j((j, k_j)) = v_j((j, k'_j)) = \varepsilon$, $v_{k_j}((j, k_j)) = v_{k'_j}((j, k'_j)) = \frac{1}{2} - \varepsilon$, and $v_{k_j}((k_j, k'_j)) = v_{k'_j}((k_j, k'_j)) = \frac{1}{2} + \varepsilon$;
- for each $i \in [n]$, let $v_i((p, i)) = v_i((q, i)) = \frac{1}{2} - \varepsilon$;
- for p and q, let $v_p((p, i)) = v_q((q, i)) = \frac{x_i}{\sum_i x_i}(1 - 2\varepsilon)$ for all $i \in [n]$,

where $\varepsilon > 0$ is arbitrarily small. As the total value of each agent is 1, the created instance is normalized.

We claim that in an EQX orientation π, each $j \in [n] \cup \{p, q\}$ must receive exactly one of the edges (j, k_j) and (j, k'_j). Suppose not. If j receives neither (j, k_j) nor (j, k'_j), then EQX is violated between k_j and k'_j; assume without loss of generality that (k_j, k'_j) is allocated to k_j, then k'_j violates EQX when compared to k_j as

$$v_{k_j}(\pi_{k_j} \setminus \{(p, k_j)\}) = \frac{1}{2} + \varepsilon > v_{k'_j}(\pi_{k'_j}) = v_{k'_j}((j, k'_j)) = \frac{1}{2} - \varepsilon.$$

If j receives both (j, k_j) and (j, k'_j), then one of k_j and k'_j receives no edges and violates EQX when compared to j.

Given the above claim, we further show that in an EQX orientation π, each agent $i \in [n]$ must receive exactly one of the edges (p, i) and (q, i). Suppose not. If i receives both (p, i) and (q, i), then k_i and k'_i violate EQX when compared to i, as after removing the edge with value ε for i (such an edge exists due to the above claim), the value of i is still $1 - 2\varepsilon > \frac{1}{2} + \varepsilon$. If i receives neither (p, i) nor (q, i), then i violates EQX when compared to p, as after removing the edge with value ε for p, the value of p is greater than 2ε.

One can verify that an EQX orientation exists if and only if the Partition instance is a yes-instance. Suppose that the Partition instance is a yes-instance, and the solution consists of I_1 and I_2. For each $i \in [n]$, if $i \in I_1$, allocate (i, p) to p and if $i \in I_2$, allocate (i, q) to q. Next, for each $i \in [n]$, allocate i the other incident edge with a value of $\frac{1}{2} - \varepsilon$ for her. For each $j \in [n] \cup \{p, q\}$, allocate (k_j, j) to j, (k'_j, j) to k'_j, and (k'_j, k_j) to k_j. At this point, all edges are allocated, and the orientation is EQX, as I_1 and I_2 are a solution to the Partition instance.

For the reverse direction, suppose that there exists an EQX orientation π. Let $S_p := \{i \in [n] \mid (i, p) \in \pi_p\}$ and $S_q := \{i \in [n] \mid (i, q) \in \pi_q\}$. We prove that S_p and S_q must form a solution to the Partition instance. First, as each i receives exactly one from edges (i, p) and (i, q), it holds that $S_p \cup S_q = [n]$. Assume for the contradiction that S_p and S_q do not form a solution, and then, assume $\sum_{i \in S_p} x_i < \sum_{i \in S_q} x_i$. Moreover, as x_i's are integers, we have $\sum_{i \in S_p} x_i \leq T - 1$, and thus,

$$v_p(\pi_p) \leq \frac{T-1}{2T}(1 - 2\varepsilon) + \varepsilon = \frac{1}{2} + \frac{\varepsilon}{T} - \frac{1}{2T} < \frac{1}{2} - \varepsilon,$$

where the last inequality transition is derived from $\varepsilon \ll \frac{1}{T}$. For any $i \in [n]$, as π_i contains an edge with value ε and an edge with value $\frac{1}{2} - \varepsilon$, agent p violates EQX when compared to i, contradicting that π is an EQX orientation.

The proof for the chores-instance is in the full version [40]. □

We remark that all hardness results in this section hold for normalized instances, where the total values of items for agents are identical.

5 EF1 ORIENTATIONS FOR CHORES

In this section, we focus on the EF1 orientation for chores that have not been covered in the literature. For goods, it is known that an EF1 orientation exists in the general orientation model [19]. However, for chores, EF1 orientations may not exist, even for simple graphs.

Theorem 5.1. *For the chores-instance with simple graphs, EF1 orientations do not always exist. For simple graphs, one can compute an EF1 orientation in polynomial time when such an orientation exists.*

We say an edge is *objectively negative* if it results in negative values for both endpoints. We present a necessary and sufficient condition for an instance with simple graphs to admit EF1 orientations.

Proposition 5.2. *In a chores-instance, there is an EF1 orientation on a simple graph if and only if the number of objectively negative edges is not greater than the number of vertices in the graph.*

Proof. We begin with a claim that if there exists an EF1 orientation π, then each agent should receive at most one objectively negative edge. Suppose not, and assume that i receives two objectively negative edges (i, j) and (i, k) in π, then i will violate EF1 when compared to j, as after removing any edge in π, the value of i is negative, but i's value for π_j is zero (as (i, j) is in π_i, then $\pi_j \cap E_i = \emptyset$); recall that i values e at zero for all $e \notin E_i$. Thus, based on this claim, if there exists an EF1 orientation, the number of objectively negative edges is at most the number of vertices.

For the "if" direction, when the number of objectively negative edges is at most the number of vertices, we can compute an EF1 orientation. First, allocate all edges to an incident agent who has zero values on them. Then, if there are edges remaining unallocated, they must be objectively negative; they will form one or more components. If a component formed by the objectively negative edges contains no cycle, i.e., is a tree, we can allocate every edge downwards to the corresponding child vertex by selecting an arbitrary vertex as the root of the tree. If a component contains a cycle, we can allocate every edge in the cycle to an endpoint in the same direction, and the remaining edges will form a tree. Repeat this process until all edges are allocated. The procedures terminate in at most m rounds, as in each round, at least one edge is allocated. □

The characterization above indicates that if, in a simple graph, the number of objectively negative edges exceeds the number of vertices, then no EF1 orientation will exist. Such an instance clearly exists, demonstrating that EF1 orientations do not always exist for chores. Moreover, the characterization directly leads to a polynomial-time algorithm for deciding the existence of EF1 orientations for simple graphs and for computing one when it exists.

On the contrary, the decision problem becomes computationally intractable for multigraphs, with the formal proof in the full version [40].

Theorem 5.3. *For the chores-instance, determining whether an EF1 orientation exists in a multigraph is NP-complete.*

6 CONCLUSION

In this paper, we studied proportionality, equitability, and their relaxations in the fair orientation problem and presented a picture of existence and computational results on these fairness notions. We also addressed the previously unexplored question of EF1 orientations for chores. We hope these results contribute to a deeper understanding of these widely studied fairness notions in the orientation model.

Looking forward, as the hardness results in Section 4 are derived from Partition, it remains unknown whether pseudo-polynomial-time algorithms exist. For proportionality, it is worthwhile to consider PROPm [7], a notion lying between PROP1 and PROPX, and to investigate whether PROPm orientations always exist. The non-existence example for PROPX presented in this paper does not imply the non-existence of PROPm orientations.

From a broader perspective, developing new fairness notions applicable in the general orientation model, with no restrictions on the number of relevant agents per item, is a promising and valuable direction. Our work has provided some negative results regarding the existing fairness notions and can serve as a foundation for the development of new fairness concepts.

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