Strengthening Proportionality in Temporal Voting

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ABSTRACT

We study proportional representation in the framework of temporal voting with approval ballots. Prior work adapted basic proportional representation concepts—justified representation (JR), proportional JR (PJR), and extended JR (EJR)—from the multiwinner setting to the temporal setting. Our work introduces and examines ways of going beyond EJR. Specifically, we consider stronger variants of JR, PJR, and EJR, and introduce temporal adaptations of more demanding multiwinner axioms, such as EJR+, full JR (FJR), full proportional JR (FPJR), and core stability. For each of these concepts, we investigate its existence and study its relationship to existing notions, thereby establishing a rich hierarchy of proportionality concepts. Notably, we show that two of our proposed axioms—EJR+ and FJR—strengthen EJR while remaining satisfiable in every temporal election.

KEYWORDS

Temporal Elections, Proportionality, Approval Voting

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1 INTRODUCTION

Proportional representation is a fundamental principle in the design of fair decision-making mechanisms, particularly in the framework of *multiwinner voting*, where the goal is to select a representative group of candidates based on the preferences of the electorate. For approval ballots, this principle is instantiated as the axiom of *justified representation (JR)* and its extensions, such as PJR/EJR/FJR/EJR+ and the core [2, 10, 38, 44]. These notions are motivated by classical social choice contexts, from electing diverse panels and boards to ensuring the presence of minority voices in political bodies [33, 42]. More recently, they were shown to be relevant for machine learning (ML) systems that incorporate human feedback, optimize group-level utility, or seek fairness across populations. Examples include increasing diversity in recommendation systems and social media [28, 43, 47], ensuring balanced representation in blockchain governance [4, 12, 13], and offering representation guarantees in democratic processes augmented with large language models [5, 27].

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However, beyond static selection tasks, many ML applications involve decision-making over multiple rounds, with preferences that may change over time. For example, curriculum learning constructs sequences of training tasks for reinforcement learning agents; temporal fairness guarantees can prevent the curriculum from overemphasizing early-stage objectives at the expense of later goals [48, 49]. Similarly, streaming services and content recommendation systems periodically refresh their offerings: enforcing proportionality across time ensures that under-served genres or user communities are not systematically sidelined across updates [16]. In the domain of generative AI, blending outputs from multiple models—each with its own inductive biases—calls for proportional merging strategies that preserve diversity of generated content over time [37].

In these applications, proportional representation needs to be ensured not just in a single round, but across the entire time horizon. This calls for an adaptation of the JR axioms to the temporal setting.

Recently, Bulteau et al. [11], Chandak et al. [15], and Elkind et al. [24] considered temporal voting with approval ballots, where the goal is to select one candidate per round. They adapted some of the more established justified representation axioms (namely, JR, PJR and EJR) to this setting, formulated temporal variants of popular multiwinner rules, e.g., Proportional Approval Voting [30], Method of Equal Shares [39], or Greedy Cohesive Rule [6], and checked whether their outputs satisfy temporal JR axioms, as well as considered the complexity of verifying whether an outcome satisfies a given axiom. However, they did not attempt to extend more demanding proportionality axioms such as EJR+, FJR, and core stability to the temporal setting.

1.1 Our Contributions

In this work, we aim to identify the most ambitious JR-style axioms that can be satisfied in temporal voting, and, more broadly, to create a comprehensive map of the landscape of temporal JR axioms.

In Section 3, we focus on the EJR+ axiom [10]. We propose an adaptation of this axiom to the temporal setting that preserves the three main attractive features of its multiwinner variant: strengthening EJR, being verifiable in polynomial time, and being satisfiable by a polynomial-time computable voting rule. Our analysis in this section highlights the difficulties of extending proportionality axioms from the multiwinner setting to the temporal setting: the most

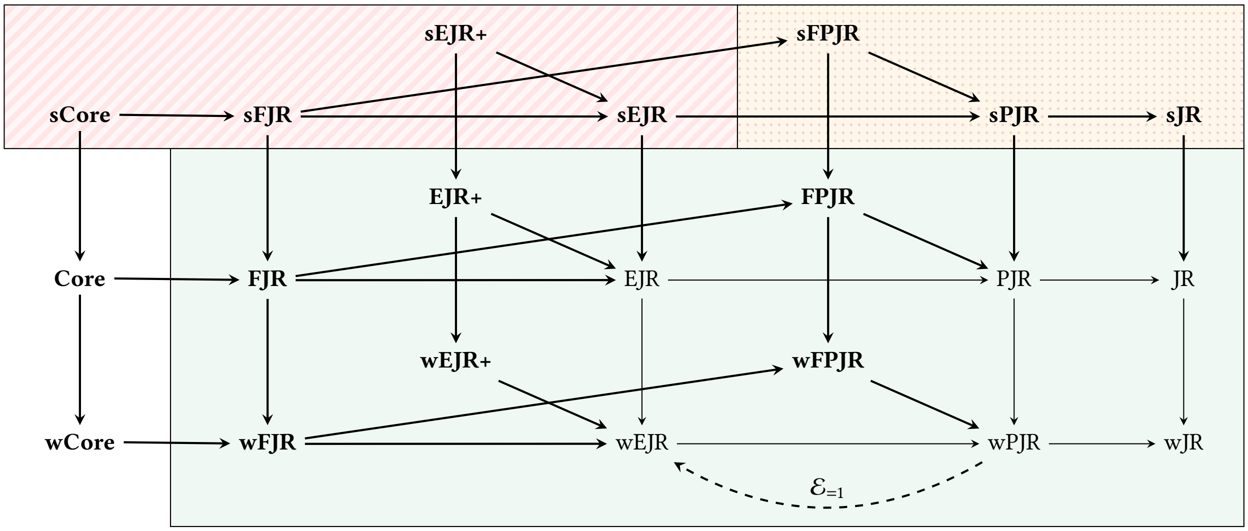

Figure 1: Axioms considered in our paper. A solid arrow from axiom A to axiom B means that A implies B; an absence of a path from A to B means that the implication does not hold (even if each voter approves exactly one candidate per round, denoted by $\mathcal{E}_{=1}$). A dashed arrow means implication for $\mathcal{E}_{=1}$, but not in general. Thick arrows denote implications established in this paper. The axioms on the green plain background can be satisfied in every election. The axioms on the yellow dotted background can be satisfied if each voter approves at least one candidate per round and the number of rounds is divisible by the number of voters, but not in general. If even in this special case an axiom is not satisfiable, then it is on the red striped background. For axioms on the white background, satisfiability remains open. The axioms introduced in this paper are written in bold.

straightforward variant of temporal EJR+ turns out to be unsatisfiable in general, and we explain why this is the case by uncovering a hidden layer of our axiomatic hierarchy.

In Section 4, we pursue the same approach for the FJR axiom [38], and show that an outcome satisfying temporal FJR always exists and can be found by a modification of the Greedy Cohesive Rule.

In Section 5, we look at the recently proposed notion of full proportional justified representation (FPJR) [29] that combines the FJR notion of group cohesion with a PJR-style collective guarantee[1]. We adapt it to our setting and show that if the number of voters divides the number of rounds and each voter approves at least one candidate per round, an outcome that satisfies a strong version of FPJR (sFPJR) can be found in polynomial time by the Serial Dictatorship Rule. Importantly, EJR+ and FJR are stronger than EJR, and sFPJR is incomparable with EJR and the other two axioms. Thus, each of our results advances the frontier of satisfiable proportionality guarantees in the temporal setting.

In Section 6 we complete the picture of temporal proportionality notions by formulating temporal core stability axioms.

For all axioms that we define, we establish that they satisfy the expected implications (e.g., temporal FJR implies temporal EJR, just like in the multiwinner setting); for cases where implications do not hold, we provide concrete counterexamples. Thus, our work establishes a rich hierarchy of proportionality concepts in temporal voting, illustrated in Figure 1.

[1]For the standard multiwinner setting, this concept was put forward by Kalayci et al. [29] in their AAMAS'25 paper and, concurrently and independently, by the first author of this paper, in his MSc thesis, submitted in August 2024 [40]

1.2 Related Work

Proportional representation in temporal voting was first studied by Bulteau et al. [11], who extended two notions of proportional representation—JR and PJR—to the temporal setting. They showed that a JR outcome can be computed in polynomial time, even for the most demanding version of JR among those they considered. For PJR, they proved the existence of an outcome satisfying the axiom, but their constructive proof relies on an exponential-time algorithm. This work was extended by Chandak et al. [15], who introduced a temporal variant of EJR and showed that several multiwinner voting rules that satisfy PJR or EJR in the standard model can be adapted to the temporal setting so as to satisfy the corresponding temporal variants of PJR and EJR. Subsequently, Elkind et al. [24] studied the complexity of verifying whether an outcome satisfies various temporal proportionality axioms.

Elkind et al. [22] focused on the tradeoffs between the social welfare and other desiderata in the temporal setting, such as strategyproofness and a weaker form of proportionality. Elkind et al. [23] considered similar questions for the setting where candidates are undesirable. A similar model has also been explored by Lackner [31] and Lackner and Maly [32] under the *perpetual voting* framework, where they consider temporal extensions of multiwinner voting rules and their axiomatic properties. We refer the reader to Elkind et al. [25] for a systematic review of recent work on temporal voting and related models. An adjacent line of work looks into sequential committee elections where an entire committee is elected at each round [7, 8, 16, 20, 50]. These works focus on constraining the extent of changes to the committees chosen across rounds.

Multiple other subareas are similar in spirit to the temporal voting framework, but with additional constraints and restrictions. In *apportionment with approval preferences*, the goal is to allocate the seats of a fixed-size committee to parties based on voters' (approval) preferences over the parties [9, 19]. This is equivalent to a restricted setting of temporal voting with static voter preferences. In *fair scheduling*, each agent's preference is a permutation, and so is the outcome [21, 36]. Our model differs from this setting in that we allow each project to be chosen more than once (both in the agents' preferences and in the outcome). *Fair public decision-making* is similar to the temporal voting model, but the focus is typically on weaker forms of proportionality [1, 18, 26, 46].

Another relevant work is that of Masařík et al. [34], who considered a more general voting model incorporating feasibility constraints, and proposed proportionality axioms for this model. We include a detailed discussion of the connection between their concepts and some of ours in the full version of our work [41].

Other works in *online fair division* study proportionality in settings where items are allocated privately, in contrast to our setting, where the selected candidates are shared publicly [17, 35].

All omitted proofs appear in the full version of our paper [41].

2 PRELIMINARIES

For every natural number $k \in \mathbb{N}$, we let $[k] = \{1, 2, \dots, k\}$. A *temporal election* is a tuple $E = (C, N, \ell, A)$, where C is the set of *candidates*, N is the set of n *voters*, ℓ is the number of *rounds*, and $A = (\mathbf{a}_i)_{i \in N}$ is an *approval profile*, where $\mathbf{a}_i = (a_{i,1}, a_{i,2}, \dots, a_{i,\ell})$ is the *ballot* of voter $i \in N$, i.e., a list of ℓ subsets of C; in round $r \in [\ell]$, voter i *approves* candidates in $a_{i,r} \subseteq C$. We denote the set of all temporal elections by $\mathcal{E}$ and the set of temporal elections in which in every round every voter approves at least (resp. exactly) one candidate by $\mathcal{E}_{\geq 1}$ (resp. $\mathcal{E}_{=1}$). We say that voters from a subset $S \subseteq N$ *agree* in a round $r \in [\ell]$ if there exists a candidate that they all approve in this round, i.e., $\bigcap_{i \in S} a_{i,r} \neq \emptyset$.

An *outcome* $\mathbf{o} = (o_1, o_2, \dots, o_\ell) \in C^\ell$ is a sequence of candidates chosen in each round. For a subset of rounds $R \subseteq [\ell]$ and an outcome $\mathbf{o}$, we write $\mathbf{o}_R = (o_r)_{r \in R}$ to denote the *suboutcome* with respect to R. The *satisfaction* of a subset of voters $S \subseteq N$ from a suboutcome $\mathbf{o}_R$ is $\mathrm{sat}_S(\mathbf{o}_R) = |\{r \in R : o_r \in \bigcup_{i \in S} a_{i,r}\}|$, i.e., the number of rounds in R in which the selected candidate is approved by at least one voter from S. If $S = \{i\}$ for some $i \in N$, we drop the brackets and write $\mathrm{sat}_i(\mathbf{o}_R)$. A *voting rule* f is a function that for every election E outputs a non-empty set of outcomes $f(E)$.

Proportionality Axioms from Prior Work

The temporal voting setting can be seen as an extension of *multiwinner voting*, where, in a single round of voting, voters in N report approvals $(a_i)_{i \in N}$ over candidates in C, and the goal is to select k winning candidates forming a *committee* $W \subseteq C$. The satisfaction of a group of voters S is then defined as the number of candidates in W approved by at least one member of S.

Intuitively, proportionality in this context is achieved when for each $0 < \alpha \leq 1$ it holds that a group of voters of size $\lfloor \alpha n \rfloor$ with similar preferences is able to appoint at least $\lfloor \alpha k \rfloor$ candidates. Accordingly, key proportionality axioms for multiwinner voting [2, 44] require W to provide each group of voters S that agree on $t \geq 1$ candidates (in the sense that $|\bigcap_{i \in S} a_i| \geq t$) with a level of satisfaction that depends on t and $|S|$. The weakest such axiom, *justified representation (JR^{mw})*, requires that the satisfaction of S is at least 1 if $k \cdot |S|/n \geq 1$. *Proportional justified representation (PJR^{mw})* strengthens this guarantee to $\min(t, \lfloor k \cdot |S|/n \rfloor)$, i.e., the smaller of (a) the number of candidates that S agrees on and (b) the number of seats that S deserves due to its size. Since the guarantee is expressed in terms of group satisfaction, PJR^{mw} may deem S to be collectively satisfied even when the members of S receive very little satisfaction individually. This cannot occur under *extended justified representation* (EJR^{mw}), which requires that at least one voter in S approves $\min(t, \lfloor k \cdot |S|/n \rfloor)$ members of W.

Bulteau et al. [11] and Chandak et al. [15] propose two versions of each of these axioms for the temporal setting. We will now present these axioms using the terminology of Elkind et al. [24].[2] Note that the temporal setting is more restrictive than the multiwinner setting, as we cannot select two candidates in the same round.

The weakest version of these axioms only considers subsets of voters that agree in every round.

Definition 2.1. Given a temporal election $E = (C, N, \ell, A)$ and an outcome $\mathbf{o}$, if for every subset of voters $S \subseteq N$ that agree in all ℓ rounds it holds that

- $\mathrm{sat}_S(\mathbf{o}) \geq \min(1, \lfloor \ell \cdot |S|/n \rfloor)$, then $\mathbf{o}$ provides *weak justified representation* (wJR),
- $\mathrm{sat}_S(\mathbf{o}) \geq \lfloor \ell \cdot |S|/n \rfloor$, then $\mathbf{o}$ provides *weak proportional justified representation* (wPJR),
- $\mathrm{sat}_i(\mathbf{o}) \geq \lfloor \ell \cdot |S|/n \rfloor$ for some $i \in S$, then $\mathbf{o}$ provides *weak extended justified representation* (wEJR).

Definition 2.1 offers no guarantees to groups that disagree even in a single round. A more flexible approach is to provide guarantees that scale with the number of rounds where the members agree.

Definition 2.2. Given a temporal election $E = (C, N, \ell, A)$ and an outcome $\mathbf{o}$, if for each $t > 0$ and every subset of voters $S \subseteq N$ that agree in a size-t subset of rounds it holds that

- $\mathrm{sat}_S(\mathbf{o}) \geq \min(1, \lfloor t \cdot |S|/n \rfloor)$, then $\mathbf{o}$ provides *justified representation* (JR),
- $\mathrm{sat}_S(\mathbf{o}) \geq \lfloor t \cdot |S|/n \rfloor$, then $\mathbf{o}$ provides *proportional justified representation* (PJR),
- $\mathrm{sat}_i(\mathbf{o}) \geq \lfloor t \cdot |S|/n \rfloor$ for some $i \in S$, then $\mathbf{o}$ provides *extended justified representation* (EJR).

Chandak et al. [15] show that every temporal election admits an outcome that provides EJR. Also, EJR implies PJR, PJR implies JR, and each axiom implies its weak counterpart (an axiom A_1 implies axiom A_2 if every outcome that provides A_1 also provides A_2).

Example 2.3. Consider an election $E = (C, N, \ell, A)$ with $\ell = 8$ rounds, a set of voters $N = [8]$, a set of candidates $C = \{a, b, c, d\}$, and approval profile depicted in Table 1 (for singleton approval sets, we omit the brackets). Here, according to wJR, every voter should receive a satisfaction of at least 1, as $8 \cdot 1/8 \geq 1$.

Since voters 1 and 2 agree in all rounds (they approve b in the first round and a in the remaining rounds), wPJR requires that they

[2] In the conference version of their paper, Chandak et al. [14] use 'JR/PJR/EJR' for what we call 'weak JR/PJR/EJR' and 'strong JR/PJR/EJR' for what we call 'JR/PJR/EJR', but their journal version [15] uses the same terminology as we do.

Table 1: Ballots of voters in Example 2.3.

Rounds	Voter 1	2	3	4	5	6	7	8
1	$\{a,b\}$	b	c	c	c	c	c	c
2	a	$\{a,b\}$	c	c	c	c	c	c
3	a	a	d	c	c	c	c	c
4	a	a	c	d	c	c	c	c
5	a	a	c	c	d	c	c	c
6	a	a	c	c	c	d	c	c
7	a	a	c	c	c	c	d	c
8	a	a	c	c	c	c	c	d

collectively receive a satisfaction of at least 2. For example, outcome (a, b, d, d, d, d, d, d) satisfies wPJR. However, it does not satisfy wEJR, as neither 1 nor 2 approves the outcomes of two rounds. To satisfy wEJR, we can select, e.g., (b, a, d, d, d, d, d, d).

Neither wPJR nor wEJR guarantee anything beyond wJR to voters in $N' = \{3, \dots, 8\}$, as any pair of voters in N' disagree in some round. On the other hand, each size-4 subgroup of N' agree in some 4 rounds; thus, EJR requires that among each such subgroup, at least one voter receives a satisfaction of $\lfloor 4 \cdot 4/8 \rfloor = 2$. Thus, (b, c, a, d, d, d, d, d) satisfies EJR.

3 EXTENDED JUSTIFIED REPRESENTATION +

In the context of multiwinner voting, Brill and Peters [10] have recently strengthened EJR^{mw} to a new axiom, which they call *Extended Justified Representation + (EJR^{mw}+)*. The goal of this section is to adapt this axiom to the temporal setting. EJR^{mw}+ has three attractive features, which we aim to preserve in our temporal adaptation: (i) it implies EJR^{mw} (while the converse is not true) (ii) a committee that provides EJR^{mw}+ can be found in polynomial time, and (iii) it can be verified in polynomial time whether a given committee provides EJR^{mw}+.

EJR^{mw}+ requires that for every group of voters S that jointly approve a candidate c, either c is included in the winning committee W or there is a voter $i \in S$ whose satisfaction from W is at least $\lfloor k \cdot |S|/n \rfloor$. A direct translation of this axiom to the temporal setting gives us the following definition.

Definition 3.1. Given a temporal election $E = (C, N, \ell, A)$, an outcome $\mathbf{o}$ provides *strong extended justified representation +* (sEJR+) if for every subset of voters $S \subseteq N$ and every round $r \in [\ell]$ with $\bigcap_{i \in S} a_{i,r} \neq \emptyset$ it holds that

(i) $\mathrm{sat}_i(\mathbf{o}) \geq \lfloor \ell \cdot |S|/n \rfloor$ for some $i \in S$, or
(ii) $o_r \in \bigcap_{i \in S} a_{i,r}$.

Note that the second condition is phrased in terms of choosing an outcome from $\bigcap_{i \in S} a_{i,r}$ rather than selecting a specific candidate; this is because there may be several candidates approved by all members of S in round r, but only one of them can be selected. In contrast, in the multiwinner setting, if S agrees on multiple candidates, several of them can be included in the winning committee.

We labeled the concept introduced in Definition 3.1 as 'strong' for two reasons. First, there are elections (even in $\mathcal{E}_{=1}$) with no sEJR+ outcomes. Second, sEJR+ is fundamentally different from JR/PJR/EJR, as defined in Definition 2.2. To show this, we will now define strong JR/PJR/EJR following the approach of Definition 3.1, and explain how the resulting notions differ from JR/PJR/EJR.

Definition 3.2. Given a temporal election $E = (C, N, \ell, A)$ and an outcome $\mathbf{o}$, if for each $t > 0$ and every subset of voters $S \subseteq N$ that agree in a size-t subset of rounds it holds that

- $\mathrm{sat}_S(\mathbf{o}) \geq \min(1, \lfloor \ell \cdot |S|/n \rfloor)$, then $\mathbf{o}$ provides *strong justified representation* (sJR),
- $\mathrm{sat}_S(\mathbf{o}) \geq \min(t, \lfloor \ell \cdot |S|/n \rfloor)$, then $\mathbf{o}$ provides *strong proportional justified representation* (sPJR),
- $\mathrm{sat}_i(\mathbf{o}) \geq \min(t, \lfloor \ell \cdot |S|/n \rfloor)$ for some $i \in S$, then $\mathbf{o}$ provides *strong extended justified representation* (sEJR).

Together with sEJR+, strong JR/PJR/EJR form a natural hierarchy and imply their counterparts from Definition 2.2.

Proposition 3.3. *It holds that:*

(i) *sEJR+ implies sEJR,*
(ii) *sEJR implies sPJR and EJR,*
(iii) *sPJR implies sJR and PJR, and*
(iv) *sJR implies JR.*

However, there exist elections (even in $\mathcal{E}_{=1}$) for which even the weakest of these axioms, i.e., sJR, is impossible to satisfy. By Proposition 3.3, this impossibility extends to sPJR/sEJR/sEJR+.

Proposition 3.4. *There exists an election $E \in \mathcal{E}_{=1}$ such that no outcome provides sJR.*

Proof. Consider an election $E = (C, N, \ell, A)$ with $\ell = 6$ rounds, a set of voters $N = [12]$, and a set of candidates $C = \{x, y, z, c_1, \dots, c_{12}\}$, where each voter approves a single candidate in each round, as specified in the following table:

Rounds	Voter 1	2	3	4	5	6	7	8	9	10	11	12
1	x	x	x	x	y	y	y	y	z	z	z	z
2–6	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}

Here, in the first round, the voters form three cohesive blocks of size four each, and in the next five rounds, every voter supports a different candidate.

Without loss of generality, assume that candidate x is selected in the first round. Let S be a size-2 subset of $\{5, 6, 7, 8\}$. As $\lfloor \ell \cdot |S|/n \rfloor = \lfloor 6 \cdot 2/12 \rfloor = 1$ and voters in S agree on y in the first round, sJR requires that at least one voter in S has positive satisfaction. Thus, at least three voters in $\{5, 6, 7, 8\}$ must have positive satisfaction. By the same argument, at least three voters in $\{9, 10, 11, 12\}$ must have positive satisfaction. Thus, we need to provide positive satisfaction to 6 voters in rounds 2–6, which is clearly impossible, as in each of these rounds we can satisfy at most one voter. Therefore, no outcome can provide sJR. □

The reason why sJR (and hence sEJR+) is hard to satisfy is that it can give strong guarantees to groups of voters based on agreement in just a few rounds. These rounds may be heavily contested, and selecting a candidate to satisfy one group might make it impossible to satisfy another group that agrees over the same rounds (such as groups 1–4, 5–8 and 9–12 in the proof of Proposition 3.4). In contrast, in the multiwinner setting, all k committee slots are equivalent.

We defined strong variants of our axioms in order to explain why Definition 3.1 results in an axiom that, despite being a direct translation of EJR ^{mw}+ to the temporal setting, is not always satisfiable. However, axioms at this level of our hierarchy are also interesting in their own right: e.g., some of them can be satisfied under additional assumptions, as shown in Section 5.

Now, the difference between sEJR and EJR is that the satisfaction guarantee to a group S that agrees in t rounds is $\lfloor \ell \cdot |S|/n \rfloor$ under the former and $\lfloor t \cdot |S|/n \rfloor$ under the latter. Can we modify the definition of sEJR+ in a similar way to obtain an axiom that can always be satisfied? The challenge is that the multiwinner variant of this axiom, EJR ^{mw}+, provides guarantees to voters in S as long as they agree on one candidate, i.e., there is no 't' that can be used to replace ℓ. We would like to preserve this feature, but define the satisfaction guarantee so that it only takes into account the rounds with some level of agreement. To this end, we replace the size of the group $|S|$ with a more 'local' measure, namely, the number of voters in S who agree on a candidate in a given round, and focus on the rounds where this quantity is high.

Definition 3.5. Given a temporal election $E = (C, N, \ell, A)$ and $\sigma \in [n]$, $\tau \in [\ell]$, we say that a subset of voters S is (σ, τ)-*cohesive* if there exists a set of τ rounds $R \subseteq [\ell]$ and a suboutcome $\mathbf{o}_R = (o_r)_{r\in R}$ such that for each $r \in R$ at least σ voters in S approve o_r. We say that an outcome $\mathbf{o}$ provides *extended justified representation + (EJR+)* if for all $\sigma \in [n]$, $\tau \in [\ell]$, every (σ, τ)-cohesive subset of voters S and every round $r \in [\ell]$ with $\bigcap_{i\in S} a_{i,r} \neq \emptyset$ it holds that

(i) $\mathrm{sat}_i(\mathbf{o}) \geq \lfloor \tau \cdot \sigma/n \rfloor$ for some $i \in S$, or
(ii) $o_r \in \bigcap_{i\in S} a_{i,r}$.

Example 3.6. Consider again the election from the proof of Proposition 3.4. The group $S = \{1, 2, 3, 4\}$ is $(4, 1)$-cohesive (as all members of S approve x in round 1) and $(1, 6)$-cohesive (as at least one member of S approves $(x, c_1, \dots, c_1)$), but $4 \cdot 1 < 12$ and $1 \cdot 6 < 12$, so EJR+ does not offer any representation guarantees to S.

To see that EJR+ is stronger than EJR, let us revisit the election in Example 2.3. There, we showed that outcome (b, c, a, d, d, d, d, d) satisfies EJR, giving satisfaction 2 to five voters in $N' = \{3, \dots, 8\}$. However, this outcome does not satisfy EJR+. Indeed, voters in N' jointly approve an unselected candidate, c, in the first round, and N' is $(5, 8)$-cohesive. Thus, EJR+ guarantees satisfaction of $\lfloor 8 \cdot 5/8 \rfloor = 5$ to at least one $i \in N'$. This guarantee seems much more adequate for a large group of voters with very similar preferences.

We will now argue that this is the 'correct' definition of EJR+ for the temporal setting. Our argument is threefold: we show that (a) EJR+ implies EJR and is implied by sEJR+, (b) EJR+ is polynomial-time verifiable, and (c) an EJR+ outcome always exists and can be found in polynomial time.

Proposition 3.7. *It holds that:*

(i) *sEJR+ implies EJR+ and*
(ii) *EJR+ implies EJR.*

Proposition 3.8. *Given a temporal election E and an outcome $\mathbf{o}$, it can be checked in polynomial time whether $\mathbf{o}$ provides EJR+.*

Proof. Consider an arbitrary temporal election $E = (C, N, \ell, A)$ and an outcome $\mathbf{o}$. Our goal is to check if there are integers $\sigma \in [n]$, $\tau \in [\ell]$, a (σ, τ)-cohesive subset of voters S, and a round $r \in [\ell]$ with $\bigcap_{i\in S} a_{i,r} \neq \emptyset$ such that $o_r \notin \bigcap_{i\in S} a_{i,r}$ and for each $i \in S$ it holds that $\mathrm{sat}_i(S) < \lfloor \tau \cdot \sigma/n \rfloor$. Our algorithm makes use of the observation that every superset of a (σ, τ)-cohesive subset of voters is itself (σ, τ)-cohesive.

We go over all possibilities for $r \in [\ell]$ and $c \in C \setminus \{o_r\}$. For a given pair (r, c), we go over all $\lambda \in [\ell]$ and identify the set of all voters $i \in N$ who (i) approve c in round r and (ii) have $\mathrm{sat}_i(\mathbf{o}) < \lambda$; denote this set by $S_{r,c,\lambda}$. If all voters in $S_{r,c,\lambda}$ approve o_r in round r, we disregard this set. Otherwise, for each $q \in [\ell]$ let c_q be a candidate that receives the maximum number of approvals from voters in $S_{r,c,\lambda}$, and let α_q be the number of voters in $S_{r,c,\lambda}$ who approve c_q in round q. We then sort $(\alpha_q)_{q\in[\ell]}$ is non-increasing order; denote the resulting list by $(\alpha'_q)_{q\in[\ell]}$. Finally, we check whether there exists a $q \in [\ell]$ such that $\lfloor q \cdot \alpha'_q/n \rfloor \geq \lambda$. If this is the case, the respective set of q rounds witnesses that $S_{r,c,\lambda}$ is (α'_q, q)-cohesive. Moreover, voters in $S_{r,c,\lambda}$ approve c in round r (so $\bigcap_{i\in S} a_{i,r} \neq \emptyset$, but $o_r \notin \bigcap_{i\in S} a_{i,r}$), yet for each voter $i \in S_{r,c,\lambda}$ its satisfaction is less than $\lambda \leq \lfloor q \cdot \alpha'_q/n \rfloor$, i.e., $S_{r,c,\lambda}$ witnesses that $\mathbf{o}$ violates EJR+. In this case, our algorithm reports that $\mathbf{o}$ does not provide EJR+.

Our algorithm goes over $m\ell^2$ triples (r, c, λ); it performs $O(nm\ell + \ell \log \ell)$ operations for each such triple. Hence, it runs in time polynomial in the input size.

Clearly, if our algorithm reports that EJR+ is violated, it provides an explicit witness. It remains to argue that if EJR+ is violated, our algorithm will be able to detect this. To see this, suppose that the violation of EJR+ is witnessed by a (σ, τ)-cohesive set S such that $\bigcap_{i\in S} a_{i,r} \neq \emptyset$, but $o_r \notin \bigcap_{i\in S} a_{i,r}$, and the satisfaction of each voter in S is less than $\lfloor \sigma \cdot \tau/n \rfloor$. Then there exists a candidate $c \in \bigcap_{i\in S} a_{i,r}$; note that $c \neq o_r$. Let R be the set of rounds witnessing that S is (σ, τ)-cohesive. When our algorithm considers the triple (r, c, λ) with $\lambda = \lfloor \sigma \cdot \tau/n \rfloor$, it constructs the set $S_{r,c,\lambda}$ and by definition it holds that $S \subseteq S_{r,c,\lambda}$. Then, for each $q \in R$ we have $\alpha_q \geq \sigma$, since at least σ voters from S approve the same candidate in round q. Hence, the sequence $(\alpha_q)_{q\in[\ell]}$ contains at least $|R| = \tau$ values that are at least σ, and consequently our algorithm is able to determine that EJR+ is violated. □

To prove that every temporal election has an EJR+ outcome, we show that the temporal variant of the ε-lsPAV rule (introduced by Aziz et al. [3] in the multiwinner setting and adapted to the temporal setting by Chandak et al. [15]) with $\varepsilon = 1/(2\ell^2)$ always outputs EJR+ outcomes and runs in polynomial time.

Definition 3.9. For a temporal election $E = (C, N, \ell, A)$, the *harmonic score* of an outcome $\mathbf{o}$ is defined as $s_H(\mathbf{o}, E) = \sum_{i\in N} \sum_{j=1}^{\mathrm{sat}_i(\mathbf{o})} \frac{1}{j}$. The ε-lsPAV rule outputs all outcomes $\mathbf{o}$ in C^ℓ such that for every outcome $\mathbf{o}' \in C^\ell$ that agrees with $\mathbf{o}$ in $\ell - 1$ rounds it holds that $s_H(\mathbf{o}, E) + \varepsilon \geq s_H(\mathbf{o}', E)$.

Setting $\varepsilon \geq 1/(2\ell^2)$ ensures that ε-lsPAV outputs an outcome in polynomial time (see [3, 15]). Moreover, we now show that ε-lsPAV provides EJR+ if we set $\varepsilon < 1/\ell^2$.

Theorem 3.10. *For every temporal election $E = (C, N, \ell, A)$, each output of ε-lsPAV with $\varepsilon < 1/\ell^2$ provides EJR+.*

The proof considers an outcome for which there is a (σ, τ)-cohesive group S witnessing an EJR+ violation, and shows that in this case we can replace the outcome of some round so as to increase the overall harmonic score by more than $1/\ell^2$. However, our proof strategy diverges from the original lsPAV proof: we make explicit use of the fact that there is a round $r \in [\ell]$ in which all voters in S approve a common candidate and, additionally, $|S| > \sigma$. This is justified because $|S| = \sigma$ would imply an EJR violation, which was proven impossible by Chandak et al. [15].

We note that the output of ε-lsPAV always satisfies condition (i) in Definition 3.5. Thus, we can obtain a stronger, but still satisfiable version of EJR+ by removing condition (ii). However, we decided to keep it for consistency with the multiwinner version of this axiom.[3]

Finally, it is also possible to define weak EJR+, following the approach in Definition 2.1.

Definition 3.11. Given a temporal election $E = (C, N, \ell, A)$, we say that an outcome $\mathbf{o}$ provides *weak extended justified representation + (wEJR+)* if for all $\sigma \in [n]$, every (σ, ℓ)-cohesive subset of voters S and every round $r \in [\ell]$ with $\bigcap_{i \in S} a_{i,r} \neq \emptyset$ it holds that

(i) $\text{sat}_i(\mathbf{o}) \geq \lfloor \ell \cdot \sigma / n \rfloor$ for some $i \in S$, or
(ii) $o_r \in \bigcap_{i \in S} a_{i,r}$.

This definition of wEJR+ fits well into our axiomatic hierarchy.

Proposition 3.12. *It holds that:*

(i) *EJR+ implies wEJR+ and*
(ii) *wEJR+ implies wEJR.*

Moreover, the *Method of Equal Shares (MES)* [39], as defined for the temporal setting by Chandak et al. [15], satisfies wEJR+.

Definition 3.13. The Method of Equal Shares (MES) is an iterative rule: in iteration $r \in [\ell]$ it chooses the outcome in round r. It starts by assigning to each voter $i \in N$ an *initial endowment*, $e_i^0 = \ell/n$, from which, for each selected candidate, the supporters of this candidate will have to pay a total price of 1. In each round $r \in [\ell]$, we say that a candidate c is ρ^r*-affordable* if

$$\sum_{i \in N : c \in a_{i,r}} \min(\rho^r, e_i^{r-1}) \geq 1.$$

In round r, MES selects a candidate o_r that is ρ^r-affordable with the minimum value of ρ^r, and updates the endowment of each voter $i \in N$ as follows: $e_i^r = e_i^{r-1} - \min(\rho^r, e_i^{r-1})$, if $o_r \in a_{i,r}$, and $e_i^r = e_i^{r-1}$, otherwise. If in a given round there is no ρ^r-affordable candidate for any value of ρ^r (in other words, there is no group of voters S that approve a common candidate and their endowments sum up to at least 1), MES *terminates ahead of time* and selects arbitrary candidates in all of the remaining rounds.

Note that MES, in contrast to ε-lsPAV, is a semi-online rule, i.e., it does not need to know a priori all of the preferences in all of the rounds—it makes the decisions in a current round based on this and previous rounds and the total number of rounds.

Theorem 3.14. *For every temporal election $E = (C, N, \ell, A)$, each output of MES provides wEJR+.*

The proof is a modification of the proof by Chandak et al. [15] that MES satisfies wEJR.

[3] This strengthening of EJR+ appears to be conceptually different from EJR ^{mw}+. Indeed, one might wonder if this idea can be ported back to the multiwinner setting, resulting in an axiom that is distinct from EJR ^{mw}+. This said, we feel that Definition 3.5 is the 'correct' adaptation of EJR ^{mw}+ to the temporal setting, as it has the same desirable features—satisfiability, efficient computability, and efficient verifiability—as EJR ^{mw}+.

4 FULL JUSTIFIED REPRESENTATION

The next axiom that we adapt to the temporal setting is *full justified representation* (FJR ^{mw}) [38]. Similarly to EJR ^{mw}, it aims to select a size-k committee W that provides guarantees to every subset of voters S in proportion to its size, $|S|$, and the level of agreement. However, when measuring agreement, it does not require all voters in S to approve the same set of candidates. Instead, voters in S go over all subsets of candidates T with $|T| \leq k \cdot |S|/n$ and compute their satisfaction from T as $\kappa_S(T) = \min_{i \in S} |T \cap a_i|$, where a_i is the approval ballot of voter i. FJR ^{mw} then requires that for each choice of T, some voter in S approves at least $\kappa_S(T)$ members of W.

FJR ^{mw} is (strictly) stronger than EJR ^{mw}: intuitively, it provides EJR-like guarantees to a larger collection of voter subsets, so every outcome that provides FJR ^{mw} also provides EJR ^{mw}, but the converse is not true. Moreover, every election has an FJR ^{mw} outcome [38].

To translate FJR ^{mw} to the temporal setting, a natural approach is to iterate over subsets of rounds R (and the best possible selection of outcomes for these rounds) instead of subsets of candidates T. This yields the following definition.

Definition 4.1. Given a temporal election $E = (C, N, \ell, A)$, we say that an outcome $\mathbf{o}$ provides *strong full justified representation* (sFJR) if for every subset of voters $S \subseteq N$, every subset of rounds $R \subseteq [\ell]$ such that $|R| = \lfloor \ell \cdot |S|/n \rfloor$, and every outcome $\mathbf{o}'$, there is a voter $i \in S$ such that $\text{sat}_i(\mathbf{o}) \geq \min_{j \in S} \text{sat}_j(\mathbf{o}'_R)$. Equivalently, for every subset of voters S there is an $i \in S$ such that

$$\text{sat}_i(\mathbf{o}) \geq \max_{R \subseteq [\ell] : |R| = \lfloor \ell \cdot |S|/n \rfloor} \max_{\mathbf{o}'_R \in C^R} \min_{j \in S} \text{sat}_j(\mathbf{o}'_R).$$

However, just as for sEJR+, the direct approach results in a definition that is too demanding; later in this section, we will show that sFJR implies sEJR, which means that some elections have no sFJR outcomes. A simple way to modify this definition is to consider the worst (rather than the best) subset of rounds R for the set S (while still allowing the voters in S to choose outcomes for rounds in R).

Definition 4.2. Given a temporal election $E = (C, N, \ell, A)$, an outcome $\mathbf{o}$ provides *weak full justified representation (wFJR)* if for every subset of voters $S \subseteq N$ there is a subset of rounds $R \subseteq [\ell]$ of size $|R| = \lfloor \ell \cdot |S|/n \rfloor$ such that for every other outcome $\mathbf{o}'$ there is a voter $i \in S$ with $\text{sat}_i(\mathbf{o}) \geq \min_{j \in S} \text{sat}_j(\mathbf{o}'_R)$. Equivalently, for every subset of voters S, there is an $i \in S$ such that

$$\text{sat}_i(\mathbf{o}) \geq \min_{R \subseteq [\ell] : |R| = \lfloor \ell \cdot |S|/n \rfloor} \max_{\mathbf{o}'_R \in C^R} \min_{j \in S} \text{sat}_j(\mathbf{o}'_R).$$

The reason we placed the concept introduced in Definition 4.2 at the 'weak' level of our hierarchy is that requiring a way of jointly reaching a satisfaction threshold (via $\mathbf{o}'_R$) in *all* subsets of rounds of size $\lfloor \ell \cdot |S|/n \rfloor$ is similar in spirit, but weaker than requiring total agreement in all rounds. Indeed, in a moment we will show that wFJR implies wEJR.

Finally, for an FJR axiom in the spirit of EJR/PJR/JR from Definition 2.2, we would like to move away from considering all (subsets of) rounds. To this end, when determining what S 'deserves', we allow S to designate a subset of rounds T, and then we pick the worst subset of rounds R, with size of R dependent on $|T|$ (rather than ℓ). This way, if the voters in S can reach an acceptable compromise over a large set of rounds T, they obtain strong satisfaction guarantees; however, if T is small, they still get a (weaker) guarantee.

Algorithm 1 Greedy Cohesive Rule

Input: A temporal election $E = (C, N, \ell, A)$
1: $\mathbf{o} \leftarrow (c, \dots, c)$ for arbitrary $c \in C$
2: $V \leftarrow N, \quad p \leftarrow 1$
3: **while** $V \neq \emptyset$ **do**
4: pick S_p from $\arg\max_{S \subseteq V} \max_{T \subseteq [\ell]} \mu_S(T)$
5: pick T_p from $\arg\max_{T \subseteq [\ell]} \mu_{S_p}(T)$
6: $V \leftarrow V \setminus S_p, \quad p \leftarrow p + 1$
7: **end while**
8: $\pi \leftarrow$ a permutation of $[p]$ such that $|T_{\pi(1)}| \leq \dots \leq |T_{\pi(p)}|$
9: $\mathcal{T} \leftarrow [\ell]$
10: **for** $q \in [p]$ **do**
11: $i \leftarrow \pi(q)$
12: $R \leftarrow$ arbitrary subset of $\lfloor |T_i| \cdot |S_i|/n \rfloor$ rounds from $T_i \cap \mathcal{T}$
13: pick $\mathbf{o}'_R$ from $\arg\max_{\mathbf{o}''_R \in C^{|R|}} \min_{j \in S_i} \text{sat}_j(\mathbf{o}''_R)$
14: $\mathbf{o}_R \leftarrow \mathbf{o}'_R, \quad \mathcal{T} \leftarrow \mathcal{T} \setminus R$
15: **end for**
16: **return** $\mathbf{o}$

Definition 4.3. Given a temporal election $E = (C, N, \ell, A)$, we say that an outcome $\mathbf{o}$ provides *full justified representation (FJR)* if for every subset of voters $S \subseteq N$ and every subset of rounds $T \subseteq [\ell]$ there exists a subset $R \subseteq T$ with $|R| = \lfloor |T| \cdot |S|/n \rfloor$ such that for every other outcome $\mathbf{o}'$ there is a voter $i \in S$ with $\text{sat}_i(\mathbf{o}) \geq \min_{j \in S} \text{sat}_j(\mathbf{o}'_R)$. Equivalently, for every subset $S \subseteq N$ there is a voter $i \in S$ with $\text{sat}_i(\mathbf{o}) \geq \max_{T \subseteq [\ell]} \mu_S(T)$, where

$$\mu_S(T) = \min_{R \subseteq T : |R| = \lfloor |T| \cdot |S|/n \rfloor} \max_{\mathbf{o}'_R \in C^R} \min_{j \in S} \text{sat}_j(\mathbf{o}'_R).$$

The first argument that our definitions are 'correct' is that all of the expected implications hold.

Proposition 4.4. *It holds that:*

(i) *sFJR implies sEJR and FJR,*
(ii) *FJR implies EJR and wFJR, and*
(iii) *wFJR implies wEJR.*

In the multiwinner setting, Peters et al. [38] show that the Greedy Cohesive Rule (GCR) [6] always outputs FJR^{mw} outcomes. We will now show that a variant of GCR satisfies FJR in the temporal setting, providing further evidence that our definition of FJR is 'correct'. Our starting point is an adaptation of GCR to the temporal setting by Elkind et al. [24], who show that their variant of the rule satisfies EJR. We modify the algorithm of Elkind et al. [24], and show that our version satisfies FJR.

A formal description of GCR is presented in Algorithm 1. Intuitively, during the first stage (lines 1–7), GCR partitions the voters into sets $S_1, S_2, \dots, S_p$: at each step $q \in [p]$ it picks a subset of voters S_q with the highest 'demand' $\max_T \mu_S(T)$, adds it to the partition, and discards all voters in S_q. During the second stage (lines 8–16), it iterates through the sets $S_1, S_2, \dots, S_p$ in order of the size of the associated subset of rounds T, from smallest to largest; when processing S_q, it selects outcomes for some of the rounds in T_q so as to satisfy the 'demand' of voters in S_q.

Theorem 4.5. *For every temporal election $E = (C, N, \ell, A)$, every output of GCR provides FJR.*

Proof. By construction, the sets $S_1, \dots, S_p$ are pairwise disjoint: for each $q \in [p]$, when we select S_q, we remove voters in S_q from V, and therefore no S_i with $i > q$ can contain a voter from S_q. Let us now show that during the second stage of the algorithm (lines 8–16), when we consider the set S_i, the set of rounds T_i contains $\lfloor |T_i| \cdot |S_i|/n \rfloor$ rounds whose outcomes have not been set in previous iterations. For readability, we assume that $\pi(q) = q$ for all $q \in [p]$.

Suppose for a contradiction that this is not the case, and let $q \in [p]$ be the first index with fewer than $\lfloor |T_q| \cdot |S_q|/n \rfloor$ rounds from T_q available. This means that strictly more than $|T_q| - \lfloor |T_q| \cdot |S_q|/n \rfloor$ of the rounds in T_q have been taken up in previous iterations. Thus,

$$\sum_{i=1}^{q} \left\lfloor \frac{|T_i| \cdot |S_i|}{n} \right\rfloor > |T_q|.$$

Since $|T_q| \geq |T_i|$ for all $i \leq q$, this inequality implies

$$\frac{|T_q|}{n} \cdot \sum_{i=1}^{q} |S_i| \geq \sum_{i=1}^{q} \frac{|T_i| \cdot |S_i|}{n} \geq \sum_{i=1}^{q} \left\lfloor \frac{|T_i| \cdot |S_i|}{n} \right\rfloor > |T_q|.$$

Hence, $\sum_{i=1}^{q} |S_i| > n$, a contradiction with the fact that the sets $S_1, \dots, S_q$ are pairwise disjoint.

Thus, when the algorithm processes S_i, it is presented with $\lfloor |T_i| \cdot |S_i|/n \rfloor$ rounds from T_i and selects outcomes for these rounds so as to maximize the minimum satisfaction of voters in S_i (line 13). Hence, the satisfaction of every $j \in S_i$ from $\mathbf{o}$ is at least $\mu_{S_i}(T_i)$. By the choice of T_i it means that j's satisfaction is at least $\max_{T \subseteq [\ell]} \mu_{S_i}(T)$, i.e., the FJR condition is satisfied for all voters in S_i. In particular, none of the sets $S_1, \dots, S_p$ can be a witness that FJR is violated.

It remains to show that no other subset of voters S can witness FJR violation. Fix $S \subseteq N$. Since $S_1, \dots, S_p$ form a partition of N, S must intersect with at least one of them. Let $i = \min\{q : S_q \cap S \neq \emptyset\}$, and let j be some voter in $S \cap S_i$. As $j \in S_i$, her satisfaction from $\mathbf{o}$ is at least $\max_{T \subseteq [\ell]} \mu_{S_i}(T)$. On the other hand, since Algorithm 1 picked S_i over S in line 4, we have $\max_{T' \subseteq [\ell]} \mu_S(T') \leq \max_{T' \subseteq [\ell]} \mu_{S_i}(T')$. Hence, j's satisfaction from $\mathbf{o}$ is at least $\max_{T' \subseteq [\ell]} \mu_S(T')$. As $j \in S$, the set S cannot be a witness that FJR is violated. □

The running time of GCR is not polynomial in the input size, so it remains open whether an FJR outcome can be computed in polynomial time; this problem is also open in the multiwinner case.

Remark 1. *The fact that there always exists an outcome that provides FJR is also implied by Theorem 11 of Masařík et al. [34], because our FJR is implied by their analogous notion in a more general setting. We discuss this connection in detail in the full version [41].*

5 FULL PROPORTIONAL JR

Under FJR, the guarantee provided to each group S is of the form 'some voter in S has high satisfaction', i.e., it is a lower bound on $\max_{i \in S} \text{sat}_i(\mathbf{o})$; this approach is also used in the definition of EJR. In contrast, the guarantee provided by PJR is of the form 'collectively, voters in S have high satisfaction', i.e., it is a lower bound on $\text{sat}_S(\mathbf{o})$. One can combine the FJR approach to deciding what each group deserves with a PJR-style collective guarantee; we refer to the resulting notion as FPJR. For the multiwinner setting, FPJR was proposed by Kalayci et al. [29], and, concurrently and independently, in the MSc thesis of the first author [40]. As with other notions, we define strong FPJR, FPJR, and weak FPJR.

Definition 5.1. Given a temporal election $E = (C, N, \ell, A)$, an outcome $\mathbf{o}$ provides *strong full proportional justified representation (sFPJR)* (resp., *full proportional justified representation (FPJR)*, or *weak full proportional justified representation (wFPJR)*) if for every $S \subseteq N$ we have $\mathrm{sat}_S(\mathbf{o}) \geq \eta^s(S)$ (resp., $\mathrm{sat}_S(\mathbf{o}) \geq \eta(S)$, or $\mathrm{sat}_S(\mathbf{o}) \geq \eta^w(S)$), where

$$\eta^s(S) = \max_{R\subseteq[\ell]:|R|=\lfloor \ell\cdot|S|/n\rfloor} \max_{\mathbf{o}'_R\in C^R} \min_{i\in S} \mathrm{sat}_i(\mathbf{o}'_R),$$

$$\eta(S) = \max_{T\subseteq[\ell]} \min_{R\subseteq T:|R|=\lfloor |T|\cdot|S|/n\rfloor} \max_{\mathbf{o}'_R\in C^R} \min_{i\in S} \mathrm{sat}_i(\mathbf{o}'_R),$$

$$\eta^w(S) = \min_{R\subseteq[\ell]:|R|=\lfloor \ell\cdot|S|/n\rfloor} \max_{\mathbf{o}'_R\in C^R} \min_{i\in S} \mathrm{sat}_i(\mathbf{o}'_R).$$

Similarly to EJR, FPJR is implied by FJR and implies PJR on each level of our axiomatic hierarchy.

Proposition 5.2. *It holds that:*

(i) *sFJR implies sFPJR, which implies sPJR and FPJR,*
(ii) *FJR implies FPJR, which implies PJR and wFPJR, and*
(iii) *wFJR implies wFPJR, which implies wPJR.*

Propositions 5.2 and 3.4 imply that some elections have no sFPJR outcomes. However, sFPJR turns out to be satisfiable for elections in $\mathcal{E}_{\geq 1}$ as long as the number of rounds ℓ is divisible by the number of voters n. Specifically, we show that in this case the simple *Serial Dictatorship Rule* (SDR) [45] outputs sFPJR outcomes.

SDR operates by iterating through the rounds in $[\ell]$ while cycling through voters in N; in each round $r \in [\ell]$ it picks an outcome approved by the current voter, i.e., it selects an arbitrary $o_r \in a_{r \bmod n, r}$ (for elections in $\mathcal{E}_{\geq 1}$ the set $a_{r \bmod n, r}$ cannot be empty, so such o_r has to exist).

Theorem 5.3. *For every n-voter ℓ-round temporal election $E \in \mathcal{E}_{\geq 1}$ such that $n|\ell$, every output of SDR provides sFPJR.*

Proof. Consider an election $E = (C, N, \ell, A) \in \mathcal{E}_{\geq 1}$ with $n|\ell$. Let SDR return the outcome $\mathbf{o}$, and fix a subset of voters S. Since each voter selects the outcome of exactly ℓ/n rounds, we have $\mathrm{sat}_S(\mathbf{o}) \geq \ell \cdot |S|/n$. Also, for every $R \subseteq [\ell]$, every $\mathbf{o}'_R$ and every $i \in N$ we have $\mathrm{sat}_i(\mathbf{o}'_R) \leq |R|$ and hence $\eta^s(S) \leq \lfloor \ell \cdot |S|/n \rfloor$. Thus, $\mathrm{sat}_S(\mathbf{o}) \geq \eta^s(S)$, which is what we wanted to prove. □

Example 5.4. In the election from Example 2.3, SDR can output (a, b, d, d, d, d, d, d), which, by Theorem 5.3, satisfies sFPJR, as the number of rounds is equal to the number of voters. On the other hand, this outcome does not satisfy wEJR.

Interestingly, if an election with $n|\ell$ is additionally in $\mathcal{E}_{=1}$, then the outputs of SDR also provide wEJR. This is because for elections from $\mathcal{E}_{=1}$ we can show that wPJR (which by Proposition 5.2 is implied by sFPJR) is equivalent to wEJR.

Proposition 5.5. *For every temporal election $E \in \mathcal{E}_{=1}$, it holds that wPJR implies wEJR.*

Finally, we note that sFPJR is the best we can hope for in terms of satisfiability of the strong variants of our axioms: there are instances in $\mathcal{E}_{=1}$ where the number of voters divides the number of rounds, but sEJR (and hence every axiom that implies it) cannot be satisfied.

Proposition 5.6. *There exists an n-voter, ℓ-round temporal election $E \in \mathcal{E}_{=1}$ such that $n|\ell$, but no outcome $\mathbf{o}$ provides sEJR for E.*

6 CORE STABILITY

To complete the analysis of proportionality notions in the temporal setting, we consider the classic concept of *core stability*, which originates in cooperative game theory. Core stability for multiwinner voting was defined by Aziz et al. [2], and it remains open if all multiwinner elections admit outcomes in the core. This concept captures resistance to deviations: a subset of voters can deviate by selecting outcomes in a subset of rounds whose size is proportional to the size of the group, and an outcome is stable if there is no deviation that benefits all members of the group. Strong core stability, core stability and weak core stability differ in how the voters are allowed to choose this subset of rounds.

Definition 6.1. Given a temporal election $E = (C, N, \ell, A)$, an outcome $\mathbf{o}$ provides

- *strong core stability (sCore)* if for each $S \subseteq N$, each $R \subseteq [\ell]$ with $|R| = \lfloor \ell \cdot |S|/n \rfloor$ and each $\mathbf{o}'_R \in C^R$ there is an $i \in S$ with $\mathrm{sat}_i(\mathbf{o}) \geq \mathrm{sat}_i(\mathbf{o}'_R)$;
- *core stability (Core)* if for each $S \subseteq N$ and each $T \subseteq [\ell]$ there is $R \subseteq T$ with $|R| = \lfloor |T| \cdot |S|/n \rfloor$ such that for each $\mathbf{o}'_R \in C^R$ there is an $i \in S$ with $\mathrm{sat}_i(\mathbf{o}) \geq \mathrm{sat}_i(\mathbf{o}'_R)$;
- *weak core stability (wCore)* if for each $S \subseteq N$ there is an $R \subseteq [\ell]$ with $|R| = \lfloor \ell \cdot |S|/n \rfloor$ such that for each $\mathbf{o}'_R \in C^R$ there is an $i \in S$ with $\mathrm{sat}_i(\mathbf{o}) \geq \mathrm{sat}_i(\mathbf{o}'_R)$.

In the multiwinner voting setting, every outcome in the core provides FJR, which extends to our setting as well.

Proposition 6.2. *It holds that:*

(i) *sCore implies sFJR and Core,*
(ii) *Core implies FJR and wCore, and*
(iii) *wCore implies wFJR.*

The satisfiability of Core and wCore remains open, while Proposition 5.6 implies that sCore is unsatisfiable even in $\mathcal{E}_{=1}$ when $n|\ell$.

7 SEPARATION RESULTS

Finally, we show that, apart from the implications already established, no further implication holds among the axioms in Figure 1. To rule out all remaining potential implications, we identify twelve specific pairs of axioms for which we construct outcomes that satisfy one notion but not the other. These counterexamples exhaust all possible implication relationships among the axioms we study.

Proposition 7.1. *For any two axioms A, B in Figure 1, A implies B if and only if there exists a path (a sequence of arrows) from A to B.*

8 CONCLUSION

We have completed the task of defining suitable temporal variants of all known JR-style multiwinner voting axioms, providing a comprehensive multilayer hierarchy. Moreover, by showing the satisfiability of EJR+, FJR, and, under certain conditions, sFPJR, we have pushed the envelope of achievable proportionality guarantees in temporal voting.

In addition to the open questions raised throughout the paper, promising directions for future work include exploring domain restrictions under which proportionality guarantees become feasible, and extending these fairness notions to broader settings such as participatory budgeting.

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