Relationships and Connections between Definitions of Metric Proportional Representation

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ABSTRACT

We explore the rich landscape of proportional representation in metric committee selection and reveal notable equivalences and implications (up to approximation factors) among existing fairness notions. We distinguish between *multi-representation* and *single-representation* guarantees.

For multi-representation, we introduce an "umbrella" definition we call *uniform core*, which requires that the q-core notion [12] hold for every q simultaneously. We show that *proportionally representative fairness* [5] (also defined as *mPJR* [20]) and this umbrella definition are equivalent (up to a constant factor), thus connecting two initially different-looking notions. Additionally, we establish that this equivalence class implies a notion of *proportionally representative committee* [19].

We then investigate ordinal proportionality axioms, specifically the RankJR axiom family introduced by Brill and Peters [6], which also inspired the study of metric JR axioms but employs ordinal rankings in lieu of distances. We demonstrate that RankPJR implies mPJR (again, up to a constant factor); however, the converse direction does not hold. An immediate consequence of this analysis is that the output of the (ordinal) Expanding Approvals Rule satisfies all of the aforementioned representation axioms to within a constant factor.

Turning to single-representation guarantees, we establish that three well-known notions, namely, *Proportionally Fair Clustering* [10], *Individual Fairness* [18], and the *Approximate Core* [23], are essentially equivalent.

KEYWORDS

Proportional Representation, Fair Clustering, Committee Selection, Computational Social Choice

ACM Reference Format:
Yusuf Hakan Kalayci and David Kempe. 2026. Relationships and Connections between Definitions of Metric Proportional Representation. In *Proc. of the 25th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2026), Paphos, Cyprus, May 25 – 29, 2026*, IFAAMAS, 9 pages. https://doi.org/10.65109/VKHN5660

1 INTRODUCTION

The classical committee selection problem in a metric space involves voters V and candidates C, all in a common metric. A committee $W \subseteq C$ of size k must be chosen to represent the voters [8, 16, 17, 19, 27]. This setup naturally appears in multiple domains. In clustering, V comprises data points and C feasible cluster centers [7, 10, 18, 23, 25], and if V and C coincide, it becomes a data-summarization problem [13, 24]. In facility location, feasible sites are the candidates, while agents (voters) must be "served" by the chosen facilities [2, 3, 14, 18]. The unifying goal across these domains is to provide a "good" (often proportional) representation of V.

For concreteness, we use the voting terminology, though our results apply equally to clustering and facility location. Many ways of defining "good" representation trace back to the idea of a "core" from cooperative game theory [15]. In essence, any sufficiently large coalition of voters should "weakly prefer" the chosen committee to any "affordable alternative". This is often interpreted as follows: any θ fraction of the population should control some corresponding fraction of the outcome [26]. However, the exact interpretations of "weak preference" and "affordable" can vary.

As a result, the literature contains many definitions, each employing slightly different formalizations while striving to capture the same fairness/representation intuition. Our main goal is to systematically connect these definitions and illustrate how they imply or refine one another, thereby providing a comprehensive map of relationships to guide future research and practice in this area. We categorize these notions along two axes. The first axis distinguishes *single-representation* axioms, which guarantee at least one nearby representative for any large cohesive group, from *multi-representation* axioms, which guarantee a proportionally sized set of representatives. The second axis distinguishes axioms by how "satisfaction" is measured: *non-transferable utility* axioms require that sufficiently many individual voters are each close to some committee member, whereas *transferable utility* axioms bound the aggregate (total) distance, so that well-served voters can compensate for poorly-served ones on average. Fig. 1 provides a visual overview of these six definitions, highlighting the role of coalitions, witness candidates, and committee members in each.

The following three notions exemplify single-representation axioms (all defined in Section 2.2): Proportionally Fair Clustering (PFC) [10], requiring that *someone is always served better* — for any large coalition and alternative candidate, at least one member must be closer to a committee member than to the alternative; Individual Fairness (IF) [18], requiring that *every dense neighborhood is covered* — every voter whose neighborhood has at least n/k voters near some candidate must have a committee member within a proportional distance; and Approximate Core (AC) [23], requiring *bounded total cost* — no large coalition can reduce its total distance to the committee by switching to one alternative. PFC and IF are non-transferable utility axioms, while AC is a transferable one.

Proc. of the 25th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2026), C. Amato, L. Dennis, V. Mascardi, J. Thangarajah (eds.), May 25 – 29, 2026, Paphos, Cyprus. © 2026 International Foundation for Autonomous Agents and Multiagent Systems (www.ifaamas.org). https://doi.org/10.65109/VKHN5660

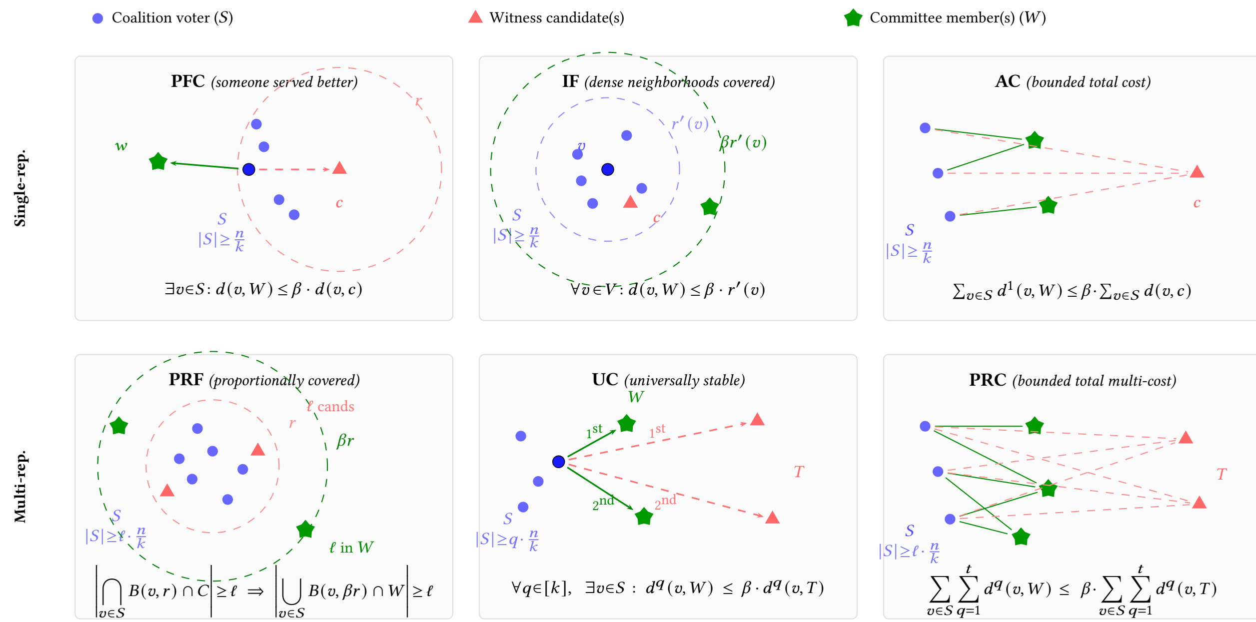

Figure 1: Visual overview of the six main proportional representation definitions. Each panel shows a coalition S (blue circles), witness candidates (red triangles), and committee members in W (green stars). Top row: single-representation axioms guarantee at least one nearby representative. Bottom row: multi-representation axioms guarantee proportionally many. Left two columns: non-transferable utility (individual voter satisfaction). Right column: transferable utility (aggregate cost, satisfaction on average).

In contrast, multi-representation axioms (defined in Section 2.3) ensure that a coalition of $\ell \cdot n/k$ voters is served by approximately ℓ committee members. The following are commonly studied notions of multi-representation: (1) Proportionally Representative Fairness (PRF) [5] (also defined independently as mPJR [20]), requiring that *cohesive groups are proportionally covered* — any group of $\ell \cdot n/k$ voters clustered near ℓ candidates must have ℓ committee members nearby; (2) the q-core [12], requiring stability against deviations to q-sized alternatives; and (3) Proportionally Representative Committee (PRC) [19], requiring *bounded total multi-cost* — the multi-representation analogue of AC, bounding the aggregate distance to the nearest t committee members. PRF and the q-core are non-transferable, while PRC is a transferable utility axiom.

Our primary goal is to provide a comprehensive overview of how these definitions interrelate. Kellerhals and Peters [20] already established several such results: that PRF implies q-core, and PRF implies PFC (equivalent to IF), all up to constant factors. We build on these foundations and fill in missing links and explore how ordinal-only notions — defined using voters' rankings rather than exact distances — such as RankPJR interact with the other definitions. We hope that this comprehensive map of proportionality relations will serve as a valuable reference for researchers as well as practitioners.

First, we introduce an umbrella definition for the q-core, which we call the Uniform Core (UC) — a *universally stable committee* that satisfies q-core properties for all q simultaneously. We show (in Section 4.1) that UC and PRF (*proportionally covered*) are equivalent up to a constant factor, and that this equivalence class implies PRC (*bounded total multi-cost*).

The converse implication, however, does not hold universally, indicating a hierarchy among multi-representation axioms. The PRC axiom is parameterized by a resource augmentation factor α and an approximation factor β. We show that for most parameter regimes, committees satisfying (α, β)-PRC exist, yet fail to satisfy UC and PRF for universal constants. This separation reflects the transferable-vs.-non-transferable divide: PRC aggregates distances, so many well-served voters can mask the under-representation of a few others, which PRF and UC do not permit.

A similar hierarchy emerges for *single representation* axioms. PFC (*someone served better*) and IF (*dense neighborhoods covered*) are known to be equivalent (up to a constant) [20]. Another pertinent axiom is AC (*bounded total cost*), which, analogously to PRC, incorporates parameters for resource augmentation α and approximation β. We show that for certain parameters, committees can satisfy AC while violating PFC by an arbitrary constant factor. However, we establish that in the absence of resource augmentation, i.e., for the case $\alpha = 1$, AC implies PFC, up to a constant factor. This indicates that AC and PFC/IF represent fundamentally similar fairness guarantees; as with PRC vs. PRF, the gap arises from the transferable utility formulation. These connections, largely building upon existing proof techniques, are presented in detail in Section 4.3.

Next, we study the *RankPJR* definition of Brill and Peters [6] within metric spaces — an *ordinal adaptation* of PRF that uses voters' rankings instead of distances. We refer to it as Ordinal Proportionally Representative Fairness (OPRF), requiring *ranking-based proportional coverage*: if ℓ candidates appear in every coalition member's top-ρ ranking, then ℓ committee members must also be ranked among their top ρ. We show (in Section 4.2) that OPRF implies PRF (up to an approximation), whereas PRF does not imply OPRF.

Finally, we examine two *Expanding Approvals Rule (EAR)* variants (reviewed in Section 3) and their guarantees. In the metric EAR, a threshold r grows from 0 to ∞. When at least n/k uncovered voters lie within distance r of a candidate, that candidate is chosen, and those voters become covered (fully or fractionally). The ordinal EAR uses an integer threshold ρ and selects a candidate if it appears among the closest ρ candidates of at least n/k voters [4]. The output of the metric EAR satisfies PRF (and therefore other proportionality axioms) [5, 20]. Here, we show that it does not satisfy OPRF. Meanwhile, the output of the ordinal EAR satisfies RankPJR+, implying RankPJR [6]; through our chain of results, we show that it satisfies various other proportionality notions as well.

2 NOTIONS OF REPRESENTATION

The committee selection problem is defined over a metric space $(V \cup C, d)$ of $n = |V|$ voters and $m = |C|$ candidates; d is a distance function. The sets V and C may overlap and are assumed finite throughout. The goal is to select a subset $W \subseteq C$ of size k. We write $p = \frac{n}{k}$ for the *Hare quota*. Intuitively, W should represent the set of voters V *proportionally*, and the Hare quota captures the "smallest" fraction of the population who should be "entitled" to their own candidate. Many different definitions of proportionality appear in the literature, and our primary goal is to relate these definitions to each other.

2.1 Notation

Pseudo-metric spaces. Throughout, when we refer to a *metric space*, we actually mean a *pseudo-metric space*, defined by a tuple $(\mathcal{M}, d)$. Here, $\mathcal{M}$ is a set of points, and $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_{\geq 0}$ is a distance function that satisfies symmetry and the triangle inequality. For any point $v \in \mathcal{M}$ and radius $r \geq 0$, we denote by $B(v, r) = \{u \in \mathcal{M} \mid d(v, u) \leq r\}$ the *ball* of radius r centered at v. The minimum distance from a point $v \in \mathcal{M}$ to a set $S \subseteq \mathcal{M}$ is denoted by $d(v, S) = \min_{u \in S} d(v, u)$.

Metric committee selection with ranked ballots. In some instances of metric committee selection, exact pairwise distances are not known. Instead, each voter's preferences are given through *ordinal information*, i.e., *ranked ballots.* Formally, a *ranked ballot profile* consists of a bijection $\pi_v : C \to [m]$ for each voter v. We say that voter v's ballot π_v is *consistent* with the metric d if $\pi_v(c_1) < \pi_v(c_2)$ implies $d(v, c_1) \leq d(v, c_2)$ for all $c_1, c_2 \in C$; we say that the whole profile is consistent with d if each π_v is consistent with d.

Approval-based committee selection (ABC). In the *approval-based committee selection* (ABC) problem, each voter $v \in V$ provides an *approval set* $A_v \subseteq C$. A committee $W \subseteq C$ is then chosen based on these approval sets. The ABC voting literature features various proportional representation notions. (See a survey of Lackner and Skowron [22] for further details.)

2.2 Single Representation Definitions

We begin by reviewing representation notions that guarantee *single representation.* Chen et al. [10] introduced the following notion of proportionally fair clustering:

Definition 1 (Proportionally Fair Clustering (PFC) [10]). *For $\beta \geq 1$, a committee W of size k is β-*proportionally fair *if for every coalition $S \subseteq V$ with $|S| \geq p = n/k$ and every candidate $c \in C \setminus W$, there exists a voter $v \in S$ such that $\min_{w \in W} d(v, w) \leq \beta \cdot d(v, c)$.*

Chen et al. [10] demonstrated that a $(1 + \sqrt{2})$-proportionally fair committee can be computed in time polynomial in n, m, and k for any finite metric space. Subsequently, Micha and Shah [25] improved this fairness guarantee to 2 specifically for Euclidean spaces with the L^2 distance metric.

Jung et al. [18] introduced a related concept for *data summarization*, i.e., the case $V = C$. Under their definition, β-individual fairness requires that if a voter v has at least p neighbors within distance r (r is referred to as the *neighborhood radius*), then at least one cluster center must lie within distance $\beta \cdot r$ of v. Kellerhals and Peters [20] generalized the definition to the case when $V \subseteq C$ (rather than $V = C$). In order to generalize the definition to arbitrary V, C, we note a key obstacle: even if p voters are close to each other, if there is no nearby candidate, these voters cannot be "entitled" to the inclusion of a close candidate. Thus, any generalization of the neighborhood radius must balance a voter's distance to p nearby voters and a close candidate. There are multiple natural and subtly different ways of capturing the interplay of these two factors, and we choose the following definition:

Definition 2 (Individual Fairness[1] (IF) [18]). *Define the neighborhood radius of a voter v as*

$$r'(v) := \min_{\substack{c \in C \\ S \subseteq V : |S| \geq n/k}} \max_{u \in S} \max\{d(v, u), d(u, c)\}.$$

*A committee W of size k is β-*individually fair *if, for every voter $v \in V$, there exists a committee member $w \in W$ such that $d(v, w) \leq \beta \cdot r'(v)$; that is, a committee member exists within the β multiple of the neighborhood radius.*

The advantage of Definition 2 is that it allows us to generalize the result of Kellerhals and Peters [20] without a loss in the constant factor. A natural alternative would be to consider only the distance from c to v (not all $u \in S$), to consider all pairwise distances in S, or other variations. These alternative definitions lead to very similar results, typically losing an additive 1 or 2 in one or both directions of the implications.

Kellerhals and Peters [20] later also generalized the individual fairness definition to ensure *multiple representation*, requiring that if a voter v has $q \cdot p$ neighbors within distance r, then q centers lie within distance r of v. We omit extending this idea to the general committee setting, as the resulting notion closely parallels existing concepts such as Proportionally Representative Fairness (Definition 6) or an equivalent definition called mPJR (Definition 5).

Next, we review a fairness notion that measures a candidate's cost to a coalition as the sum of distances between coalition voters and the candidate. Fairness notions using total distance face a significant limitation: in some instances, the cost bounds must be relaxed by polynomially large factors. Consequently, these approaches typically allow relaxations of the *representation* constraints for coalitions as well. The first formal definition of such a relaxed fairness notion was introduced by Li et al. [23], as follows.

[1] The term "individual fairness" was popularized by later works improving algorithmic efficiency [9, 28].

Definition 3 (Approximate Core (AC) [23]). *For a resource augmentation parameter* $\alpha \geq 1$, *a committee* W *of size* k *is in the* (α, β)*-core if for every coalition* $S \subseteq V$ *of size at least* $\alpha \cdot \frac{n}{k}$, *the committee* W *satisfies* $\sum_{v \in S} d(v, W) \leq \beta \cdot \min_{c \in C} \sum_{v \in S} d(v, c)$.

2.3 Multiple Representation Definitions

Next, we discuss proportionality definitions which ensure that any coalition is represented by a set of committee members whose size is proportional to the size of the coalition. Aziz et al. [5] formulated the following requirement for metrics extremely cohesive clusters.

Definition 4 (Unanimous Proportionality (UP) [5]). *A committee* W *of size* k *satisfies* unanimous proportionality *if the following holds: for every coalition* $S \subseteq V$ *with* $|S| \geq \ell \cdot \frac{n}{k}$ *such that all voters in* S *are co-located (i.e.,* $d(v, v') = 0$ *for all* $v, v' \in S$*), the committee* W *contains the* ℓ *candidates in* C *closest to* S.

UP serves as a baseline that provides a sharp separation between single-representation and multiple-representation guarantees. Single-representation definitions (PFC, IF, AC) only require that a large cohesive group has *at least one* nearby representative, but they do not guarantee proportionally many. In contrast, multiple-representation definitions (PRF, UC, and the variants below) aim to satisfy UP or an approximation thereof. The counterexamples separating these two classes are rooted in violations of UP.

Aziz et al. [5] and Kellerhals and Peters [20] introduced equivalent representation definitions from slightly different angles. Kellerhals and Peters [20] established a link between metric-space proportional representation and proportionality notions from approval-based committee selection, defining a meta proportional representation axiom, called MetricJR. This axiom translates the family of Justified Representation definitions into metric settings.

Definition 5 (Metric JR Axioms [20]). *Let* Π *be a proportionality axiom defined for approval-based committee selection (ABC). An outcome* W *satisfies* $m\Pi$ *for the committee selection problem in a metric space if for all* $r \in \mathbb{R}_{\geq 0}$, *for the ABC instance in which each* $v \in V$ *has the approval set* $B(v, r) \cap C$, *the outcome* W *satisfies* Π.

A specific version of these metric JR axioms, namely, *mPJR*, was introduced independently by Aziz et al. [5] under the name *Proportionally Representative Fairness*:

Definition 6 (Proportionally Representative Fairness (PRF) [5]). *A committee* W *satisfies* β*-PRF if for every* $\ell \in [k]$, *every coalition* $S \subseteq V$ *with* $|S| \geq \ell \cdot p$, *and every* $r \geq 0$*: if at least* ℓ *candidates in* C *lie within distance* r *of every voter in* S, *then*

$$\left| \bigcup_{v \in S} B(v, \beta \cdot r) \cap W \right| \geq \ell.$$

Aziz et al. [5] originally introduced two variants of PRF: an unconstrained version for the case $V \subseteq C$, where r is the diameter of the coalition S, and a constrained version for the general setting, where r is the maximum distance between a voter in S and a candidate in the witness set of ℓ candidates. Here, we state the constrained (and thus more general) version; the representation condition is the same in both cases. The original definitions do not include an approximation factor β, as the Expanding Approval Rule achieves 1-PRF. We introduce the β-parameterized version in order to explore the relationships of PRF with other proportionality concepts. Kellerhals and Peters [20] show that mPJR and PRF (constrained) are equivalent, and we use these two terms interchangeably throughout.

Ebadian and Micha [12] introduced another proportionality notion with a *multiple representation* guarantee, comparing the distance to the q^{th} closest committee representative with that of a proportional-size alternative. The q^{th} smallest distance from a voter v to a (candidate) set T is defined as

$$d^q(v, T) = \begin{cases} \min_{T' \subseteq T, |T'| = q} \max_{c \in T'} d(v, c) & \text{when } |T| \geq q \\ \infty & \text{otherwise.} \end{cases}$$

Definition 7 (q-core [12]). *For* $\beta \geq 1$, *a committee* $W \subseteq C$ *is in the* β*-*q*-core if for every candidate subset* $T \subseteq C$,

$$|\{v \in V \mid d^q(v, W) > \beta \cdot d^q(v, T)\}| < |T| \cdot p.$$

An equivalent condition is the following: a committee $W \subseteq C$ *is in the* β*-*q*-core if for every coalition* $S \subseteq V$ *with* $|S| \geq \ell \cdot p$ *and every candidate set* $T \subseteq C$ *with* $|T| = \ell$, *there exists a voter* $v \in S$ *such that* $d^q(v, W) \leq \beta \cdot d^q(v, T)$.

We are particularly interested in winner sets which are simultaneously in the q-core for all q, and define an umbrella notion which we call the *uniform core*.

Definition 8 (Uniform Core). *A committee* $W \subseteq C$ *of size* k *is in the* β*-uniform core if* W *is in the* β*-*q*-core for all* $1 \leq q \leq k$.

Similar to the definition of the Approximate Core (Definition 3), multi-representation axioms using cost notions measuring total distance between coalition members and candidate sets employ a *bicriteria relaxation*. These axioms incorporate two parameters: a resource augmentation parameter $\alpha \geq 1$ and an approximation factor β. The resource augmentation parameter α relaxes the representation constraint, requiring only t representatives for coalitions of size $\alpha \cdot t \cdot \lceil \frac{n}{k} \rceil$.

Definition 9 (Proportionally Representative Committee [19]). *For a resource augmentation parameter* $\alpha \geq 1$, *a committee* $W \subseteq C$ *is* (α, β)-proportionally representative *if, for every coalition* $S \subseteq V$ *with* $|S| \geq \alpha \cdot t \cdot \lceil \frac{n}{k} \rceil$ *and every set* $T \subseteq C$ *of size* t, *we have*

$$\sum_{v \in S} \sum_{q=1}^{t} d^q(v, W) \leq \beta \cdot \sum_{v \in S} \sum_{q=1}^{t} d^q(v, T).$$

2.4 Multiple Representation Definitions with Ordinal Information

In some scenarios, in particular for the application of elections, the mechanism will not have access to the metric space, but only to rankings submitted by the voters. The rankings are frequently assumed [1, 8, 11, 16] to be an expression of a latent metric space of candidates and voters, in the sense defined in Section 2.1.

In such cases, the mechanism can typically only *explicitly* strive to satisfy guarantees expressed in terms of the *rankings*, even though implicitly, guarantees for underlying metric spaces from which the rankings are derived are desired — the deterioration in achievable guarantees due to the move from full information to (ordinal) rankings is the focus of the area of *metric distortion*.

A notable class of proportionality concepts which relies solely on ordinal information, and which was introduced outside the metric committee selection problem, is the *RankJR* family of axioms [6]. While RankJR is conceptually similar to the metric JR framework of Kellerhals and Peters [20], it differs in its approach by defining each voter's approval set as their top ρ candidates based on ranking, rather than using explicit distance thresholds.

Definition 10 (Rank JR Axioms [6]). *Consider a committee selection problem with ranked ballots $(\pi_v)_{v\in V}$. Let Π be a proportionality axiom defined for approval-based committee selection (ABC). An outcome W for the ranked problem satisfies RankΠ if, for every $\rho \in [m]$, W satisfies Π for the ABC voting instance in which each voter $v \in V$ approves their top ρ candidates in π_v, i.e., $\pi_v^{-1}([\rho])$.*

Analogously to mPJR/PRF, we call RankPJR *Ordinal Proportionally Representative Fairness*:

Definition 11 (Ordinal Proportionally Representative Fairness (OPRF)). *A committee W satisfies OPRF if for every $\ell \in [k]$, every coalition $S \subseteq V$ with $|S| \geq \ell \cdot p$, and every $\rho \in [m]$: if at least ℓ candidates in C appear in the top ρ of every voter's ranking in S, then*

$$\left|\bigcup_{v\in S} \pi_v^{-1}([\rho]) \cap W\right| \geq \ell.$$

3 ALGORITHMS

We review two variants of the *Expanding Approvals Rule (EAR)*, which finds proportionally representative committees in metric settings. Originally introduced by Aziz and Lee [4] in an ordinal context, the EAR concept was later adapted to the metric setting [5, 12, 20], where the distance function is known. In this paper, we refer to these two versions as the *ordinal EAR* and the *metric EAR*.

The EAR algorithm begins by assigning each voter a weight of 1. Voters use these weights to "purchase" candidates whom they "approve". Over time, the notion of approval expands: For metric EAR, a threshold distance r gradually increases from 0 to the maximum voter-candidate distance; each voter approves all candidates within distance r. For ordinal EAR, a rank threshold ρ gradually increases from 1 to $|C|$, and each voter approves their top ρ candidates.

Whenever the total weight of voters approving an unselected candidate c reaches the threshold of n/k, c is added to the committee. The weights of the approving voters are then reduced by a total of n/k, with this reduction distributed arbitrarily among them.

A closely related algorithm called Greedy Capture (GC) was proposed by Chen et al. [10] and Micha and Shah [25]. Like metric EAR, GC increases a threshold r over time, but differs in two ways. First, GC uses binary weights (0 or 1) to track whether a voter is *covered* by the committee: a candidate is added when it is approved by at least $\lceil n/k \rceil$ *uncovered* voters. Second, while EAR only reduces weights at the moment a new candidate is selected, GC marks voters as covered *throughout* the process, whenever they come within distance r of an already-chosen committee member; that is, after c is chosen, additional voters may be "assigned" to c.

Another variant of GC was introduced by Kalayci et al. [19] and called *Truncated Greedy Capture (TGC)*: TGC covers voters precisely at the moment a new candidate is selected, rather than marking them covered at any point in the process. TGC is a special case of the Metric EAR algorithm which defines $p := \lceil \frac{n}{k} \rceil$ instead of $p = \frac{n}{k}$, thereby ensuring that all weight decreases are integral.

We now describe the ordinal variant of the Expanding Approvals Rule (EAR). This version, which was the original one introduced by Aziz and Lee [4], constructs the approval sets from the voters' rankings rather than a metric. More specifically, the ordinal EAR uses an integer parameter ρ: in iteration ρ, each voter's approval set consists of their top ρ preferred candidates. Whenever the total weight of voters approving any candidate c (not already in the committee) reaches n/k, Ordinal EAR adds c to the committee and reduces the approving voters' weight by a total of n/k.

Algorithm 1 Metric Expanding Approvals Rule

Input: Metric space (V, C, d), committee size k
Output: Committee W
  Let $W \leftarrow \emptyset$ be the selected committee.
  Set $\sigma(v) = 1$ for each $v \in V$.
  Let $D = \max_{v\in V, c\in C} d(v, c)$.
  Define $p := \frac{n}{k}$.
  for r growing continuously from 0 to D **do**
    Let $N_c := \{v \in V \mid d(v, c) \leq r\}$ for all $c \in C$.
    while there exists $c \in C \setminus W$ such that $\sigma(N_c) \geq p$ **do**
      Select such a candidate c arbitrarily.
      Update the committee $W \leftarrow W \cup \{c\}$.
      Subtract total weight of p from the weights $\sigma(v)$ of voters v in N_c.

Algorithm 2 Ordinal Expanding Approvals Rule

Input: Election $(V, C, (\pi_v)_{v\in V})$, Committee Size k.
Output: Committee W
  Let $W \leftarrow \emptyset$ be the selected committee.
  Set $\sigma(v) = 1$ for each $v \in V$.
  Define $p = \frac{n}{k}$.
  for $\rho = 1$ to $|C|$ **do**
    Let $N_c := \{v \in V \mid \pi_v(c) \leq \rho\}$ for all $c \in C$.
    while there exists $c \in C \setminus W$ such that $\sigma(N_c) \geq p$ **do**
      Select such a c arbitrarily.
      Update the committee $W \leftarrow W \cup \{c\}$.
      Subtract total weight of p from the voters in N_c.

Table 1 gives the best known representation guarantees for both EAR variants, drawing on this and prior work. For AC and PRC, the guarantees reflect achievable tradeoffs between the resource augmentation parameter α and the approximation factor β.

4 RELATIONS

This section investigates the relationships between the various definitions of proportional representation and their logical implications. A comprehensive diagram (Figure 2) highlights both prior and novel connections, illustrating the implications among these definitions and quantifying the approximation guarantees they provide.

Table 1: Proportional representation guarantees for the Metric and Ordinal EAR algorithm.

Notion	Metric EAR	Ordinal EAR
PFC (Def. 1)	$\beta = 1 + \sqrt{2}$ [5, 20]	$\beta = \frac{5+\sqrt{41}}{2} \approx 5.71$ [19]
IF (Def. 2)	$\beta = 2$ [20]	$5 + \sqrt{41} \approx 11.41$ **This work**
AC (Def. 3)	$\beta = \frac{2\alpha}{\alpha-1}$ [20]	$\beta = 1 + \frac{\alpha\cdot(7+\sqrt{41})}{2\cdot(\alpha-1)}$ [19]
PRF (Def. 6)	$\beta = 1$ [5, 20]	$\beta = 3$ **This work**
UC (Def. 8)	$\beta = 5$ [20]	$\beta = 4 + \sqrt{13}$ [21]
OPRF (Def. 11)	Violates **This work**	Satisfies [6]
PRC (Def. 9)	$\beta = 1 + \frac{\alpha\cdot(2+\sqrt{2})}{\alpha-1}$ [19]	$\beta = 1 + \frac{\alpha\cdot(7+\sqrt{41})}{2\cdot(\alpha-1)}$ [19]

4.1 Multi-Representation Axioms

Before proving relations between multi-representation axioms, we first demonstrate a property of the PRF axiom which will be utilized for proving two theorems.

Lemma 4.1. *Let W be a committee satisfying β-PRF, with voter subsets $G_1, \ldots, G_\ell$, candidates $T = \{c_1, \ldots, c_\ell\}$, and voter $v^* \in V$ such that:*

(1) $|\bigcup_{i\in I} G_i| \geq |I| \cdot \frac{n}{k}$ *for all* $I \subseteq [\ell]$, *and*
(2) $d(v^*, c_i) \geq d(v, c_i)$ *for all* $v \in G_i$ *and* $i \in [\ell]$.

Then there exist distinct committee members $w_1, \ldots, w_\ell \in W$ such that $d(v^*, w_i) \leq (3\cdot\beta + 2)\cdot d(v^*, c_i)$ for all $i \in [\ell]$.

Proof. Index $c_1, \ldots, c_\ell$ by increasing distance from v^*, so that $d(v^*, c_i) \leq d(v^*, c_{i+1})$. Fix any $q \in [\ell]$. For any indices $i, j \leq q$ and voter $v_i \in G_i$, the triangle inequality gives:

$$d(v_i, c_j) \leq d(v_i, c_i) + d(v^*, c_i) + d(v^*, c_j).$$

Since $d(v_i, c_i) \leq d(v^*, c_i)$ and $j \leq q$, all terms are bounded by $d^q(v^*, T) = d(v^*, c_q)$.

Define the coalition $S' := \bigcup_{i\leq q} G_i$. All candidates c_i with $i \leq q$ have distance at most $3 \cdot d^q(v^*, T)$ from voters in S'. By the β-PRF condition, at least q committee members in W must have distance at most $3\beta \cdot d^q(v^*, T)$ to S'. Call this set W_q, i.e., $W_q = \{w \in W \mid d(w, S') \leq 3\beta \cdot d^q(v^*, T)\}$.

For any $w \in W_q$, consider the closest voter; let $i \in [\ell]$ be such that this voter is in G_i, and call the voter v_i. Then, we observe that

$$\begin{aligned} d(v^*, w) &\leq d(w, v_i) + d(v_i, c_i) + d(c_i, v^*) \\ &\leq 3\beta \cdot d^q(v^*, T) + d(v^*, c_i) + d(v^*, c_i), \end{aligned}$$

which is also bounded by $(3\beta + 2) \cdot d^q(v^*, T)$.

For each $q \leq \ell$, we can find sets W_q with $|W_q| = q$, where each candidate is within distance $(3\beta+2)\cdot d^q(v^*, T)$ of v^*. We iteratively select w_i from $W_i \setminus \{w_1, \ldots, w_{i-1}\}$ to satisfy the inequality for each $i \leq \ell$. □

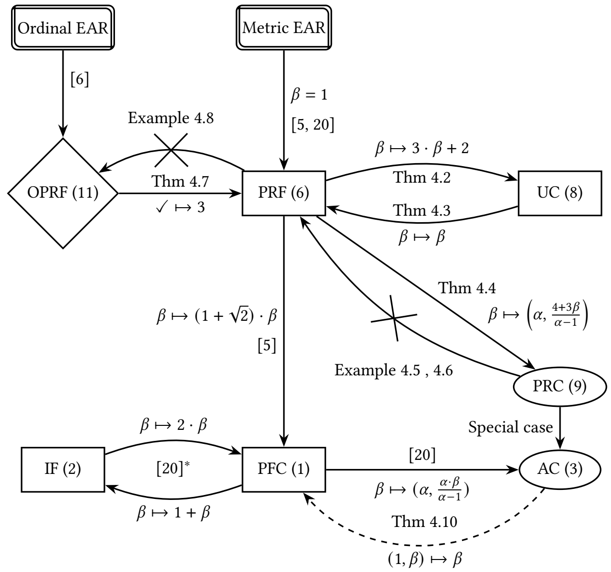

Figure 2: The figure presents relationships between proportional representation definitions: the diamond indicates an ordinal axiom, ellipses represent axioms utilizing cost notions measuring total distance between coalition members and candidate sets, rectangles denote the remaining axioms, and double-line borders mark algorithms. Solid arrows with a cross indicate counterexamples, solid arrows without a cross show implications (a label $X \mapsto Y$ denotes that satisfying the source axiom with parameter X implies satisfying the target axiom with parameter Y), and dashed arrows highlight essential implications where source parameters may not hold for universal constants. All relations are transitive. In a concurrent study, Kellerhals and Peters [21] demonstrated that OPRF implies UC. This implication has been omitted from the figure for legibility, as it can be derived through the existing transitive relations.

4.1.1 *PRF and UC are Equivalent.*

Now, we are ready to show that the definitions of Proportional Representative Fairness (PRF) and the Uniform Core condition (UC) are equivalent up to a constant approximation factor.

Theorem 4.2. *If W satisfies β-PRF then it is also in the $(3\beta+2)$-UC.*

A special case of this theorem (namely, that 1-PRF implies 5-UC) was proved by Kellerhals and Peters [20]. Here, we present an alternative proof for a more general form of the theorem.

Proof. Suppose that W satisfies β-PRF, and let $q \leq k$. Consider any coalition $S \subseteq V$ of size at least $\ell \cdot \frac{n}{k}$ for some $\ell \geq q$, and let $T \subseteq C$ be any set of ℓ candidates. We will show that W satisfies the condition of Definition 7, by exhibiting a voter $v \in S$ with $d^q(v, W) \leq (3\beta+2)\cdot d^q(v, T)$. We define an ordering on the candidates $c_i \in T$ based on when they are added by a run of a modified version $\mathcal{A}$ of the metric EAR procedure. Specifically, $\mathcal{A}$ gets a set of voters, a set of candidates, the committee size, and p (modified Hare quota) as input and runs with the following changes/specifications:

(i) The set of voters is S, the set of candidates is T, and the desired committee size is ℓ. (ii) p is still defined to be $\frac{n}{k}$, even though the voter set S is typically not equal to V, and the committee size is ℓ, not k. (iii) Whenever $\mathcal{A}$ selects a candidate c for the committee, it subtracts weight from voters in increasing order of their distances from c, continuing until only the farthest voters from c have non-zero weight.

Because the desired committee size is $\ell = |T|$, $\mathcal{A}$ must select all candidates in T. Let $c_1, \ldots, c_\ell$ be the order in which the candidates are chosen. Let G_i be the set of voters whose weight is reduced when c_i is added to the committee; call this set the *neighborhood* of c_i. Notice that some voters might belong to more than one neighborhood G_i. However, for any subset $I \subseteq [\ell]$, $|\bigcup_{i\in I} G_i| \geq |I| \cdot \frac{n}{k}$, as the total weight decrease for I by $\mathcal{A}$ will be at least $|I| \cdot \frac{n}{k}$, and each voter started with weight 1.

Let v^* be the voter in G_ℓ that is farthest from c_ℓ. We first prove that $d(v^*, c_i) \geq d(v, c_i)$ for all $v \in G_i$. Thereto, notice that because v^* had its weight decreased for the last candidate c_ℓ added, it was eligible to have its weight decreased for each of the prior candidates as well. For each i, either v^* did not have its weight decreased at all, or it was the last voter in G_i to have weight decreased. The reason is that v^* had weight remaining until the last added candidate c_ℓ, and $\mathcal{A}$ would not start decreasing weight for another voter $v \in G_i$ until all the weight for v^* had been decreased. As a result, in either case, v^* is at least as far from c_i as every $v \in G_i$, for each $i \leq \ell$.

We can therefore apply Lemma 4.1, which shows that there exist distinct committee members $w_1, \ldots, w_\ell$ such that $d(v^*, w_i) \leq (3\beta + 2) \cdot d(v^*, c_i)$. The voter v^* is now the witness for the fact that W is in the $(3\beta + 2)$-q-core. By applying this argument for all q, we complete the proof. □

Theorem 4.3. *If W is in the β-uniform core, it also satisfies β-PRF.*

Proof. Fix $\ell \in [k]$, a coalition $S \subseteq V$ with $|S| \geq \ell \cdot \frac{n}{k}$, and $r \geq 0$. Suppose that at least ℓ candidates lie within distance r of every voter in S, and let T be any set of ℓ such candidates.

Because W is in the β-uniform core, in particular it is in the β-ℓ-core. Since $|S| \geq \ell \cdot \frac{n}{k}$ and $|T| = \ell$, there exists a voter $v^* \in S$ with $d^\ell(v^*, W) \leq \beta \cdot d^\ell(v^*, T)$.

Since $v^* \in S$ and every candidate in T lies within distance r of every voter in S, we have $d^\ell(v^*, T) \leq r$.

Let $W' = \{w \in W \mid d(v^*, w) \leq d^\ell(v^*, W)\}$. By definition, $|W'| \geq \ell$, and for all $w \in W'$: $d(v^*, w) \leq d^\ell(v^*, W) \leq \beta \cdot d^\ell(v^*, T) \leq \beta \cdot r$. Since $v^* \in S$, this witnesses that W satisfies β-PRF. □

4.1.2 Relations between PRF and PRC.

Theorem 4.4. *If W satisfies β-PRF, then for every $\alpha > 1$, it also satisfies $\left(\alpha,\ 1 + \frac{(3\beta+3)\alpha}{\alpha-1}\right)$-PRC.*

Proof. Suppose that W satisfies β-PRF. Let $\alpha > 1$ be fixed. Consider a set of voters $S \subseteq V$ with $|S| \geq \alpha \cdot \ell \cdot \lceil \frac{n}{k} \rceil$ for some $\ell \geq 1$.

Let $C_S \subseteq C$ be a collection of ℓ candidates minimizing the total pairwise distances to S, i.e., $C_S \in \operatorname{argmin}_{C' \subseteq C, |C'|=\ell} \sum_{v\in S} \sum_{c\in C'} d(v, c)$.

We consider the outcome of a run of the modified metric EAR algorithm $\mathcal{A}$ with voter set S, candidates C_S, committee size ℓ, and $p = n/k$.

$\mathcal{A}$ will choose all candidates in C_S; let $c_1, \ldots, c_\ell$ be the order in which candidates are chosen. Define $\sigma(v, c_i)$ as the amount of weight reduced from voter v when candidate c_i is added to the committee, and $\rho(v) = 1 - \sum_{i=1}^{\ell} \sigma(v, c_i)$ as the remaining weight of voter v after all candidates in C_S are added to the committee. Define $G_i = \{v \mid \sigma(v, c_i) > 0\}$ to be the set of voters whose weight is reduced when c_i is added to the committee. Recall that the total weight reduced by each candidate c_i, i.e., $\sum_v \sigma(v, c_i)$, is exactly n/k.

Select $v^* \in \operatorname{argmin}_{v:\rho(v)>0}(\sum_{c\in C_S} d(v^*, c))$, i.e., minimizing the total distance to C_S among all v with positive $\rho(v)$. Notice that such a v^* always exists because $|S| > \ell \cdot n/k$, so some voters must not have had their weight reduced to 0. Observe that for any subset $I \subseteq [\ell]$, $|\bigcup_{i\in I} G_i| \geq |I| \cdot \frac{n}{k}$ and $d(v^*, c_i) \geq d(v, c_i)$ for each $v \in G_i$, by the same argument as in the proof of Theorem 4.2. Therefore, we can apply Lemma 4.1 to obtain committee members $W' = \{w_1, \ldots, w_\ell\}$ satisfying $d(v^*, w_j) \leq (3\beta + 2) \cdot d(v^*, c_j)$.

For any $v \in S$ and $j \in [\ell]$, the triangle inequality gives:

$$\begin{aligned} d(v, w_j) &\leq d(v, c_j) + d(c_j, v^*) + d(v^*, w_j) \\ &\leq d(v, c_j) + d(v^*, c_j) + (3\beta + 2) \cdot d(v^*, c_j) \\ &= d(v, c_j) + (3\beta + 3) \cdot d(v^*, c_j). \end{aligned}$$

Summing over all $v \in S$ and $j \in [\ell]$:

$$\sum_{v\in S} \sum_{j=1}^{\ell} d(v, w_j) \leq \sum_{v\in S} \sum_{j=1}^{\ell} d(v, c_j) + (3\beta + 3) \cdot |S| \cdot \sum_{j=1}^{\ell} d(v^*, c_j).$$

On the other hand, for the total pairwise distances between S and C_S:

$$\sum_{v\in S} \sum_{i=1}^{\ell} d(v, c_i) \geq \sum_{v\in S} \sum_{i=1}^{\ell} \rho(v) \cdot d(v, c_i) \geq \left(|S| - \frac{\ell \cdot n}{k}\right) \sum_{i=1}^{\ell} d(v^*, c_i),$$

by the definition of v^*. Since $|S| - \ell \cdot \frac{n}{k} \geq |S| - \frac{|S|}{\alpha} = |S| \cdot \frac{\alpha-1}{\alpha}$, we have:

$$|S| \cdot \sum_{j=1}^{\ell} d(v^*, c_j) \leq \frac{\alpha}{\alpha - 1} \cdot \sum_{v\in S} \sum_{j=1}^{\ell} d(v, c_j).$$

Substituting back:

$$\sum_{v\in S} \sum_{j=1}^{\ell} d(v, w_j) \leq \left(1 + \frac{(3\beta + 3)\alpha}{\alpha - 1}\right) \cdot \sum_{v\in S} \sum_{j=1}^{\ell} d(v, c_j).$$

Since $\sum_{q=1}^{\ell} d^q(v, W) \leq \sum_{j=1}^{\ell} d(v, w_j)$ (as $W' \subseteq W$) and C_S minimizes total pairwise distances, this establishes that W satisfies $\left(\alpha,\ 1 + \frac{(3\beta+3)\alpha}{\alpha-1}\right)$-PRC. □

While PRF and UC are equivalent up to constants, PRC is strictly weaker than PRF. We demonstrate this with two examples, showing, respectively, that for any $\alpha \geq 2$ or $\beta > 1$, there exist committees satisfying (α, β)-PRC but violating β'-PRF for any constant β'.

Example 4.5. *Fix any $\alpha \geq 2$. Consider $n = k^2$ voters partitioned into k disjoint sets $V_1, \ldots, V_k$, each comprising k voters. Define a metric where voters within the same V_i have distance 0, and voters in different V_i have distance 1. Let $C = V = \bigcup_{i=1}^{k} V_i$ be the candidate set, and consider a committee R with two voters from V_1 and one voter from each of $V_2, \ldots, V_{k-1}$.*

We first demonstrate that R is an $(\alpha, 1)$-Proportionally Representative Committee (PRC). Let S be an arbitrary coalition of size $\alpha \cdot k \cdot t$

for some $t \geq 1$. For each voter $v \in S$, define $D_v^t(R) = \sum_{q=1}^{t} d^q(v, R)$ as the sum of distances to the t closest candidates in R. For at least $|S| - k$ voters, $D_v^t(R)$ will be at most $t - 1$, because all but at most k voters belong to sets V_i that have at least one representative in R. Thus, $\sum_{v \in S} D_v^t(R) \leq (t-1) \cdot (|S| - k) + t \cdot k = (t-1) \cdot |S| + k$. On the other hand, for any subset T of candidates with size t, the total distance between voters in S and candidates in T is at least $|S| \cdot t - t \cdot k$, since at most $t \cdot k$ voter-candidate pairs can have distance 0. The latter exceeds the former one by $|S| - (t+1) \cdot k$ which is non-negative since $|S| \geq \alpha \cdot t \cdot k \geq 2 \cdot t \cdot k \geq (t+1) \cdot k$. Because this holds for every coalition S, R satisfies $(\alpha, 1)$-PRC.

However, R violates β-PRF for any $\beta \geq 1$, as V_k has distance 1 to R, despite voters within V_k having distance 0 to each other.

Theorem 4.5 also separates two single-representation axioms: Approximate Core (AC) and Proportionally Fair Clustering (PFC). R satisfies $(\alpha, 1)$-AC (which is the special case of PRC considering only $t = 1$), but violates β-PFC for any β by the same argument as the violation of β-PRF.

Example 4.6. *Fix any $\beta > 1$. Let $k > \frac{1}{\beta - 1}$, and define an instance with $n = k^2$ voters, and $2k$ candidates $C = C_1 \cup C_2$, with $|C_1| = |C_2| = k$. Candidates in C_1 have distance 1 from all voters, while candidates in C_2 have distance $(\beta - 1) \cdot k$. Note that $(\beta - 1) \cdot k > 1$, so every voter is closer to candidates in C_1 than those in C_2. Consider a committee R consisting of $k - 1$ candidates from C_1 and one candidate from C_2.*

Since $p = n/k = k$, for any coalition S other than $S = V$, we have that $t \leq k - 1$ in the definition of PRC. Because each $v \in S$ has distance 1 to all the selected candidates $c \in R \cap C_1$, of which there are $k - 1$, R is optimal for S. For the grand coalition $S = V$, the cost of R is:

$$\sum_{v \in V} \sum_{i=1}^{k} d^i(v, R) = k^2 \cdot (k-1) \cdot 1 + k^2 \cdot (\beta - 1) \cdot k \leq \beta k^3.$$

The optimal cost — attained by choosing all candidates in C_1 — is k^3, so R satisfies $(1, \beta)$-PRC. However, C_1 contains k candidates at distance 1 from V, so R violates β'-PRF for any $\beta' < (\beta - 1) \cdot k$. By letting $k \to \infty$, we see that β'-PRF can be violated for arbitrarily large constants β'.

4.2 Ordinal Multi-Representation Axioms

In this section, we establish connections among the notions of OPRF, PRF, and UC. Theorem 4.7 (proved in the full version) shows that any committee satisfying OPRF also satisfies 3-PRF.

Theorem 4.7. *If a committee W satisfies OPRF, then W also satisfies 3-PRF.*

By combining our chain of reductions with the results of Brill and Peters [6], who prove that the Ordinal EAR satisfies OPRF, we conclude that OPRF simultaneously guarantees various other proportionality notions. Independently, Kellerhals and Peters [21] recently established that OPRF (also referred to as RankPJR) implies $(4 + \sqrt{13})$-UC.[2]

We next demonstrate that the converse does not hold: a committee satisfying PRF does not necessarily satisfy OPRF.

[2]By combining our Theorem 4.7 and Theorem 4.2, we obtain a worse constant of $11 > 4 + \sqrt{13}$. This is primarily due to chaining two reductions, each of which loses a factor.

Example 4.8. *Consider a line with four voters, v_1, v_2, v_3, v_4, located at coordinates 0, 4, 8, 12, and three candidates, c_1, c_2, c_3, at coordinates $3, 6, 9 - \epsilon$ (for an arbitrarily small $\epsilon > 0$). Let $k = 2$.*

Under the metric EAR algorithm, the committee is $W_m = \{c_2, c_3\}$. However, if we examine the coalition $S = \{v_1, v_2\}$, both voters rank c_1 as their top choice, yet $c_1 \notin W_m$, violating OPRF.

By contrast, the ordinal EAR algorithm selects $W_o = \{c_1, c_3\}$, which does satisfy OPRF for this instance. Moreover, one could argue that $\{c_1, c_3\}$ better represents the voters, since every voter's top choice appears in the committee.

By duplicating candidates in this example, one can show that no α-multiplicative relaxation of OPRF is implied by PRF.

4.3 Single Representation Axioms

This section establishes relationships between IF, PFC, and AC; proofs can be found in the full version. The equivalence of IF and PFC was shown by Kellerhals and Peters [20] for $V \subseteq C$; Theorem 4.9 generalizes this to arbitrary V, C via Definition 2.

Theorem 4.9. *Let W be a committee and $\beta \geq 1$ be a parameter.*

(1) *If W satisfies β-PFC, then W also satisfies $(\beta + 1)$-IF.*
(2) *If W satisfies β-IF, then W also satisfies 2β-PFC.*

We next demonstrate that AC and PFC are essentially equivalent. One direction of this relation was already shown by Kellerhals and Peters [20], who showed that β-PFC implies $(\alpha, \frac{\alpha \cdot \beta}{\alpha - 1})$-AC. Considering the other direction, we note that with resource augmentation parameter $\alpha \geq 2$, AC is actually a weaker condition than PFC; this was shown in Example 4.5.

However, we still demonstrate that $(1, \beta)$-AC also implies β-PFC, implying that AC and PFC have essentially the same objective.

Theorem 4.10. *If a committee is in the $(1, \beta)$-approximate core, then it also satisfies β-PFC.*

5 CONCLUSION

Proportional representation in metric spaces is a crucial task in clustering, data summarization, facility location, and voting. Given the diverse definitions in the literature, this work consolidates existing relationships and provides a map of implications (Figure 2) as a comprehensive reference for researchers and practitioners.

A few minor gaps remain that may be of interest for future work. Our single-representation analysis does not fully resolve the intermediate regime $\alpha \in (1, 2)$ for AC: we showed that $(1, \beta)$-AC implies β-PFC and that AC with $\alpha \geq 2$ can violate PFC by an arbitrary factor, but whether AC in this intermediate range implies any approximate PFC guarantee is left open. Analogously, on the multi-representation side, it would be interesting to know whether PRC for carefully chosen parameter settings implies PRF up to a constant factor, which would mirror the AC–PFC connection in the single-representation setting. Finally, it remains open whether the approximation constants in our reductions are tight; improving them or establishing matching lower bounds would further clarify the boundaries between the various fairness notions.

ACKNOWLEDGMENTS

We thank the anonymous reviewers for their constructive feedback, which significantly improved the paper.

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