Participation Incentives in Online Cooperative Games

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ABSTRACT

This paper studies cooperative games where coalitions are formed online and the value generated by the grand coalition must be irrevocably distributed among the players at each time step. We investigate the fundamental issue of strategic participation incentives and address these concerns by formalizing participation incentive axioms. Our analysis reveals that existing value-sharing mechanisms fail to meet these criteria. Consequently, we propose a family of equal sharing rules that fulfill these desirable participation incentive axioms. Additionally, we refine our mechanisms under superadditive valuations to ensure individual rationality while preserving the previously established axioms.

KEYWORDS

Participation Incentives; Cooperative Games; Online Mechanism Design

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1 INTRODUCTION

Cooperative game theory studies the behavior of self-interested players in strategic settings when binding agreements among groups are feasible. A central objective is to design allocation rules that fairly distribute the collective benefits among players, thereby ensuring the cooperation incentives. Canonical cooperative games typically assume the static grand coalition containing all the involved players and the focus lies in determining how the total surplus should be allocated. Prominent solution concepts in this setting include the Shapley Value [33], the Banzhaf Index [6], and the Nucleolus [32], among others.

In many real-world contexts, however, coalitions do not emerge instantaneously. Instead, players may join sequentially, with the value generated at each stage requiring irrevocable allocation before the eventual grand coalition can be realized. Consider, for example, the formation of a startup: the venture may begin with a handful of founders, while additional contributors arrive over time, each bringing distinct skills and resources. It is typically infeasible for participants to defer all compensation until the final composition of the coalition is known, and in practice, it may be unclear whether the set of contributors has even stabilized. Such scenarios naturally give rise to **online cooperative games**, where each newly arriving player augments the coalition's value, which must then be distributed among current members. This dynamic setting introduces new strategic considerations, as players may strategically manipulate the timing of their entry into the coalition.

The formal model of online cooperative games with strategic arrival behavior was first introduced by Ge et al. [22]. They study axioms including incentivizing players to join the coalition as early as possible and to remain in the coalition as long as possible, and ex-ante fairness, termed *Shapley-Fairness* (SF), which requires that each player's expected reward under uniformly distributed arrival permutations equals their offline Shapley value.They showed that simply applying Shapley Value or distributing the marginal value to the newly arriving player suffers from a fundamental flaw where players may have incentives to leave the coalition as total rewards can decrease over time, or to delay joining in order to obtain higher marginal contributions. In view of this, they proposed the *Rewarding First Critical Player* (RFC) rule, which rewards the first arriving player who is essential in generating value. Although the RFC rule does not generally incentivize early arrival, it has been shown that in 0–1 games the incentives for early arrival, staying, and Shapley fairness cannot all be satisfied simultaneously, and the RFC rule satisfies these axioms whenever feasible.

In this paper, we focus on participation incentive axioms, including the *Incentive for Early Arrival* (EA) and the *Incentive to Stay* (Stay) axioms, and we also introduce new natural incentive principles. The first axiom is the *Strong Incentive to Stay* (S-Stay), which refines the incentive to stay studied by Ge et al. [22]. Beyond requiring that each player's allocated share be non-decreasing over time, S-Stay also ensures that any player who is essential in generating marginal value with a newly arriving player receives a strictly positive reward. Intuitively, this axiom provides a stronger incentive for players to remain in the coalition, as continued participation can yield additional benefits through collaboration with future arrivals. The second is termed *Incentive for Participation* (Part), which requires whenever a newly joining player contributes positive marginal value to the existing coalition, she must receive an immediate positive reward, capturing an *instant incentive* to participate. The third axiom we consider is the classic participation incentive of Individual Rationality (IR), which states that a player may be discouraged from joining the coalition if she cannot secure a payoff exceeding what she could achieve on her own. In summary, this paper aims to address the following central questions:

> *For the online cooperative game setting, what are the key participation incentive properties? How do the existing rules fare with respect to these properties? Can we design new rules that perform even better with respect to participation properties?*

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1.1 Our Contribution

In this paper, we focus on participation incentive axioms in online cooperative games with strategic arrivals. Beyond the Incentive to Stay (Stay) and Incentive for Early Arrival (EA) axioms studied by Ge et al. [22], we introduce three new axioms, S-Stay, Part, and IR, which capture natural and practical concerns in online value-sharing design. We first show that existing sharing rules, including the DMC, SV, and RFC rules, fail to satisfy these incentive axioms, motivating the development of new rules that fulfill all participation incentives. To this end, we propose a class of "Equal Sharing" rules, where at each stage, a subset of players equally share the new marginal contribution. We establish sufficient conditions under which these rules satisfy S-Stay, Part, and EA, and introduce two representative rules, *Marginal Equal Share* (MES) and *Non-Dummy Marginal Equal Share* (NDMES) rules. We further explore a greedy-based variant, termed *Upward Lexicographic Marginal Equal Share* (ULMES) rule, and extend it to eULMES rule via the game decomposition framework from Ge et al. [22]. Finally, we investigate the IR axiom under superadditive valuation functions and refine our proposed rules to ensure IR satisfaction while preserving all previously held axiomatic properties. Table 1 summarizes both existing and newly proposed rules with respect to their satisfaction of the studied axioms. All omitted proofs are provided in the Appendix due to space constraints.

Rules	IR*	**Part**	EA	Stay	**S-Stay**	**OD**	SF	Poly-time
DMC	✓	✓	–	✓	–	✓	✓	✓
SV	✓	✓	✓	–	–	✓	✓	✓
eRFC	–	–	–	✓	–	✓	✓	–†
MES	–	✓	✓	✓	✓	–	–	✓
NDMES	–	✓	✓	✓	✓	✓	–	–‡
ULMES	–	✓	–	✓	✓	✓	–	✓
eULMES	–	✓	✓	✓	✓	✓	–	–†
IR-eULMES	✓	✓	✓	✓	✓	✓	–	–†

Table 1: Summary of results: axioms and rules in bold are newly presented in this paper. †: Poly-time in 0-1 online cooperative games; ‡: Poly-time with subadditive valuation. *: IR axiom is considered in superadditive valuation.

1.2 Related Work

Cooperative Games. Cooperative game theory, originating from the last century [33, 35], is a significant branch of game theory that studies scenarios where players can benefit by forming coalitions and making collective decisions. One of the key problems in this area is how to distribute the value created by coalitions among players, considering axiomatic characterizations (e.g., stability, consistency, etc.). The Shapley Value [33] initiated the research, laying the foundation for a series of subsequent works. von Neumann et al. [35] first proposed the core concept for cooperative games. In the context of transferable utility cooperative games, Shubik [34] studied market games, while Aumann and Maschler [3] investigated cooperative bargaining scenarios. Schmeidler [32] first introduced the concept of the nucleolus, and Roth and Sotomayor [31] bridged cooperative game theory with practical matching markets. There is also a line of research focusing on cooperative games with hedonic preferences [1, 4, 5, 12, 16]. Further details about classic cooperative game theory can be found in several books (see, e.g., [11, 15]).

Online cooperative games study the games in an online manner where players arrive in a random order and the coalition formation decision should be made without any knowledge regarding the players arriving in the future. Our paper is closely related to the work by Ge et al. [22], which was the first to study online cooperative games with consideration of strategic arrivals. Recently, Zhang et al. [36] explores the cost sharing game in the context of online strategic arrivals and propose the Shapley-fair shuffle cost sharing mechanisms. Zhang et al. [37] consider the online cooperative game model where each arriving player can choose to create new coalition or join an existing coalition and design value-sharing policies to optimize the competitive ratio with respect to social welfare. Another branch studying cooperative game in an online manner, mainly concerning on hedonic games, focuses on addressing approximation to the social welfare and stability [13, 19]. The biggest difference from the aforementioned online cooperative game is that it typically assume that players reveal their preferences truthfully without incentive to misreport. Moreover, Flammini et al. [19] studied the online coalition structure generation problem, while Bullinger and Romen [13] investigated online coalition formation with random arrival. An online or dynamic perspective has also been applied to matching and hedonic games (see, e.g., [10, 14, 17]).

Online Mechanism Design. In the online cooperative game model where players can strategically manipulate their arrival time, the arrival time is private information and the problem can be viewed as a dynamic mechanism design problem. Mechanism design in dynamic environments focuses on problems involving multiple players with private information, where the goal is to elicit this private information while making decisions without knowledge of future events. There is a vast body of work considering mechanism design in the online manner [2, 7, 8, 27]. Lavi and Nisan [26] initiated the study of truthful online auctions in dynamic environments. Later, Friedman and Parkes [20] coined the concept of online mechanism design. Some works [28, 29] discussed the state-of-the-art VCG mechanism in dynamic online settings. Online matching has also been a hot topic in dynamic algorithm design [18, 21, 24, 25]. Moreover, there is a wide literature on solutions for different sequential mechanism design problems, including scheduling [30], online combinatorial optimization [9, 23].

2 PRELIMINARY

2.1 Model

An online cooperative game (OCG) G is a triple (N, v, π), where $N := \{1, 2, \ldots, n\}$ is the set of n players, $v : 2^N \to \mathbb{R}_+$ is the valuation function mapping a subset of players to a non-negative real number, and $\pi \in \Pi(N)$ is a permutation of N representing the arrival order of all the players over all possible permutations $\Pi(N)$. Given any subset $S \subseteq N$, S creates a coalition with value $v(S)$. In this paper, we focus on *normalized* and *monotone* general valuation function: (1). *Normalized*: $v(\emptyset) = 0$; (2). *Monotone*: $S \subseteq T \subseteq N$, $v(T) \geq v(S)$. A valuation function $v : 2^N \to \mathbb{R}$ is *submodular* if for any S, T s.t. $S \subseteq T \subseteq N$, we have $v(S) + v(T) \geq v(S \cup T) + v(S \cap T)$; A valuation function $v : 2^N \to \mathbb{R}$ is *subadditive* if for any S, T s.t. $S \subseteq T \subseteq N, v(S)+v(T) \geq v(S \cup T)$; A valuation function $v : 2^N \to \mathbb{R}$ is *superadditive* if for any S, T s.t. $S \cap T = \emptyset, v(S) + v(T) \leq v(S \cup T)$.

Given a permutation π, for any pair of players $i, j \in N$, let $i \prec_\pi j$ denote that player i arrives earlier than player j, according to the permutation π. An *online value-sharing rule* ϕ maps the game $G = (N, v, \pi)$ to an n-tuple $\phi(G) = (\phi(G, 1), \ldots, \phi(G, n))$, where $\phi(G, i)$ denotes the value assigned to player i. For any player $i \in N$, we assume $\phi(G, i) \geq 0$ and $\sum_{i \in N} \phi(G, i) = v(N)$.

We now introduce two significant definitions that will serve as the foundation for the axioms and sharing rules. The first is the notion of a **prefix subgame**. A subset $S \subseteq N$ is called a *prefix* of the arriving order π if S consists of the first $|S|$ players to arrive. In this case, we write $S \sqsubseteq \pi$. Given an OCG $G = (N, v, \pi)$ and a prefix $S \sqsubseteq \pi$, the prefix subgame is defined as $G^S = (S, v_{|S}, \pi_{|S})$, where $v_{|S}$ is the valuation function restricted to coalitions $C \subseteq S$, and $\pi_{|S}$ denotes the arrival order of players in S. For any player i in the arriving order π, we denote by $N_{\pi|i}$ the prefix consisting of all players who arrive no later than i (including i). The corresponding prefix subgame is then $G^{N_{\pi|i}} = (N_{\pi|i}, v_{|N_{\pi|i}}, \pi_{|N_{\pi|i}})$. When the context is clear, we simplify this notation by writing $G^i = (N_{|i}, v_{|i}, \pi_{|i})$. We next define the **Simple Online Cooperative Game** (SOCG) with a restrictive 0-1 valuation function. In an SOCG, there is a *pivotal* player such that, upon her arrival, the value of the coalition jumps from 0 to 1. Given that the valuation function is monotone, the grand coalition value remains at 1 after all players have arrived, i.e., $v(N) = 1$. Formally, an OCG $G = (N, v, \pi)$ is an SOCG if the valuation function $v(\cdot)$ satisfies: $\forall\, S \subseteq N, v(S) \in \{0, 1\}$ and $v(\emptyset) = 0, v(N) = 1$.

2.2 Incentive Axioms from Ge et al. [22]

We revisit some existing axioms introduced by Ge et al. [22]. The first property is termed *Incentive to Stay* [1], which guarantees that each player's shared value is non-decreasing as more players arrive. This encourages the arrived players staying in the grand coalition for more potential rewards.

Definition 2.1 (Incentive to Stay (Stay)). An online value-sharing rule ϕ_G satisfies *incentive to stay (Stay)* if given an OCG $G = (N, v, \pi)$, for any two prefix subgames $G^S = (S, v_{|S}, \pi_{|S})$ and $G^T = (T, v_{|T}, \pi_{|T})$, where $T \subseteq S$, and $T, S \sqsubseteq \pi$, every player $q \in T$ satisfies $\phi(G^T, q) \leq \phi(G^S, q)$.

Stay is the first natural participation incentive axiom studied by Ge et al. [22]. Next, we revisit another participation incentive axiom called *Incentive for Early Arrival.* Recall that in the online cooperative game model, the arrival time of each agent is treated as private information. So players might strategically choose to delay their arrival for extra benefits. The axiom of EA requires that for every player i, when fixing the arrival order of all other players, arriving as early as possible is a dominant strategy for player i.

Definition 2.2 (Incentive for Early Arrival (EA)). An online value-sharing rule $\phi(G)$ satisfies EA if, for any two OCGs $G = (N, v, \pi)$ and $G' = (N, v, \pi')$, for each player i, it always holds $\phi(G, i) \geq \phi(G', i)$ whenever $\pi_{|N \setminus \{i\}} = \pi'_{|N \setminus \{i\}}$ and $N_{\pi|i} \subset N_{\pi'|i}$.

Before introducing the next axiom called *Shapley-Fairness*, we first introduce the concept of marginal contribution and the classic Shapley Value. Given an OCG $G = (N, v, \pi)$, for each player $i \in N$, we define the *marginal contribution* of i to a coalition S in G as $\mathrm{MC}(G, S, i) = v(S \cup \{i\}) - v(S)$. Based on the definition of marginal contribution, we introduce the Shapley Value.

Definition 2.3 (Shapley Value [33] (SV)). Given an OCG $G = (N, v, \pi)$, each player i's Shapley Value is

$$\mathrm{SV}(G, i) = \frac{1}{|N|!} \sum_{S \subseteq N \setminus \{i\}} |S|! \cdot (|N| - |S| - 1)! \cdot \mathrm{MC}(G, S, i).$$

The Shapley value assigns to each player their average marginal contribution across all possible coalitions. It ensures that players are rewarded fairly based on how much they add to the value of any coalition they join. A follow-up definition, termed *Shapley-Fairness* extends the Shapley Value in online cooperative games.

Definition 2.4 (Shapley-Fairness (SF)). Given an OCG $G = (N, v, \pi)$, an online value-sharing rule $\phi(G)$ is said to satisfy *Shapley-Fairness (SF)* if, for each player $i \in N$,

$$\frac{1}{|N|!} \sum_{\pi \in \Pi(N)} \phi(G, i) \;=\; \mathrm{SV}(G, i),$$

Intuitively, *SF* requires that, for any online coalition game (OCG) G, if all arrival orders of the players are equally likely, then the expected payoff of each player coincides with her Shapley value. However, Shapley-Fairness only guarantees fairness in expectation across permutations; for a fixed arrival order, the payoffs assigned by an SF rule need not match the Shapley values exactly. In online settings, by contrast, only one arrival order is realized, and the total value must be allocated sequentially as players appear. This makes SF overly restrictive and motivates relaxing the requirement for two key reasons. First, incentivizing early arrivals necessarily reduces the shares available to later players, since the total value of the grand coalition is fixed; Shapley-Fairness, being insensitive to arrival order, fails to capture this trade-off and therefore cannot serve as a participation-incentive axiom. Second, beyond its incompatibility with STAY and EA (as shown by Ge et al. [22]), *SF* is conceptually misaligned with the sequential nature of online value sharing.

2.3 New Incentive Axioms

In the previous section, we revisited two participation incentive axioms, Stay and EA, originally introduced by Ge et al. [22]. We now turn to additional natural axioms that further refine our understanding of participation incentives in online cooperative games. To formulate these axioms, we first introduce an auxiliary notion for players whose arrival generates a strictly positive marginal contribution, which we refer to as *contributing players.*

Definition 2.5 (Contributing Player). Given an OCG $G = (N, v, \pi)$, for each player i in arriving order π, if $v(N_{\pi|i}) > v(N_{\pi|i} \setminus \{i\})$, then player i is called a *contributing* player under permutation π in G.

Building on this notion, we propose new participation incentive axioms. The first, termed *Strong Incentive to Stay* (S-Stay), strengthens the Stay axiom by imposing a more stringent requirement.

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[1] In the original paper [22], this property is referred to as Online Individual Rationality (OIR). However, it differs from the classic notion of individual rationality. As discussed in Section 2.3, we revisit the axiom of individual rationality (IR) that aligns with the classic notion.

Definition 2.6 (Strong Incentive to Stay (S-Stay)). An online value-sharing rule $\phi(G)$ satisfies *Strong Incentive to Stay* (S-Stay) if, for any OCG $G = (N, v, \pi)$, it satisfies the STAY axiom, and, for every contributing player i, and every player $j \prec_\pi i$ such that $v(N_{\pi|i}) > v(N_{\pi|i} \setminus \{j\})$, it holds that for player j, $\phi(G^i, j) > \phi(G^j, j)$.

Intuitively, S-Stay requires not only that each player's cumulative allocation be non-decreasing, but also that if an already-arrived player j is *essential* for creating the positive marginal value generated by a newly-arrived contributing player i, then j must receive a share of this newly created value. In this way, S-Stay provides stronger incentives for players to remain in the grand coalition, as it ensures that they can benefit from potential future cooperation with later arrivals. We refer to the second axiom as *Incentive for Participation* (Part), which ensures that each arriving player receives an immediate share of value if they contribute to the coalition. We consider this a minimal and natural requirement in the online setting.

Definition 2.7 (Incentive for Participation (Part)). An online value-sharing rule $\phi(G)$ satisfies *Incentive for Participation* (Part) if, for any OCG $G = (N, v, \pi)$ and every player i, whenever i is a contributing player, it holds that $\phi(G^i, i) > 0$.

Intuitively, PART requires that in any OCG G, every contributing player i receives a strictly positive share immediately upon joining the coalition. S-Stay and Part are two natural axioms which give players strong incentive to participate to join the coalition. To give a more intuitive feeling, consider a simple example, an SOCG with two players, player 1 joins the coalition first. When player 2 subsequently joins, the coalition value increases to 1, whereas neither player alone can generate this value. In this case, S-Stay provides a strong guarantee for player 1 to join and remain in the coalition as she will benefit once player 2 arrives. Part incentivizes player 2 joining since she is a contributing player and she will receive immediate reward. However, in the next section, we will show that prior existing sharing rules fail these simple and natural axioms. Apart from these two new axioms, we also consider the classical notion of *Individual Rationality* (IR), which aligns with its definition in standard cooperative games and can likewise be viewed as a participation incentive axiom.

Definition 2.8 (Individual Rationality (IR)). Given an OCG $G = (N, v, \pi)$, an online value-sharing rule ϕ is individual rational (IR) if for each player $i \in N$, $\phi(G, i) \geq v(\{i\})$.

Although we relax the Shapley-Fairness axiom in this paper, we consider an alternative fairness criterion by introducing the concept of *Online-Dummy* (OD) to evaluate value-sharing rules. The Online-Dummy concept is defined within each prefix subgame G^i, requiring that any *dummy player* in G^i receives no share of the marginal value $\mathrm{MC}(G^i, N_{\pi|i} \setminus i, i)$. Consequently, the Online-Dummy property generalizes the Dummy axiom from classical cooperative games. We begin by formally defining a *dummy player.*

Definition 2.9 (Dummy Player). Given an OCG $G = (N, v, \pi)$, for player i, if $v(S \cup \{i\}) = v(S)$ for any $S \subseteq N$, then player i is a dummy player in G.

In classical cooperative games, the *Dummy* axiom requires that every dummy player receives no positive share of the value. We now extend this axiom to the online setting by introducing the *Online Dummy* axiom.

Definition 2.10 (Online Dummy (OD)). An online value-sharing rule ϕ satisfies Online-Dummy if for each prefix subgame $G^i = (N_{|i}, v_{|i}, \pi_{|i})$, for any dummy player j in game G^i, $\phi(G^i, j) = 0$.

OD axiom requires that dummy players receive zero payoff not only in the final allocation but also at every subgames. In particular, any dummy player in a certain prefix subgame receives no share at that stage, even though they may later contribute positively through interactions with subsequently arriving players.

3 LIMITATION OF EXISTING RULES

In this section, we first revisit three existing online value-sharing rules: the Distribute Marginal Contribution (DMC) rule, the Shapley Value (SV) rule, and the extended Reward First Critical Player (eRFC) rule [22] and highlight their limitations with respect to the participation incentive axioms introduced above.

The DMC rule simply distributes the whole new marginal value to the newly arrived player, i.e., for each arriving player i, $\phi(G, i) = \mathrm{MC}(G^i, N_{\pi|i} \setminus \{i\}, i)$. Ge et al. [22] prove that it satisfies Stay and SF, but fails EA in general[2]. SV rule simply computes the Shapley value for each prefix subgame and distributes these values among the players. When the grand coalition is eventually formed, each player's allocation coincides with its Shapley value. Ge et al. [22] prove that it satisfies both EA and SF, but fails STAY. Furthermore, Ge et al. [22] proposed a novel rule, termed the **R**eward the **F**irst **C**ritical Player (RFC) rule, which was initially developed for SOCGs and later extended to general OCGs through game decomposition. Given any SOCG G and any prefix subgame G^i with arriving player i, the RFC rule first identifies the set of players S^i that are essential for generating the positive marginal contribution, namely, those players whose absence would cause the coalition value to drop from 1 to 0. The entire marginal contribution is then allocated to the first-arriving player in S^i according to the arrival order π.

Algorithm 1 Reward First Critical Player (RFC) rule

```
Input: OCG $G = (N, v, \pi)$.
for each arriving player $i$ in order $\pi$ do
    Initialize $\phi(G, i) \leftarrow 0$.
    $S^i \leftarrow \{j \mid j \in N_{\pi|i}, v(N_{\pi|i}) > v(N_{\pi|i} \setminus \{j\})\}$.
    $j^* \leftarrow$ first arrived player in $S^i$.
    $\phi(G, j^*) \leftarrow \phi(G, j^*) + \mathrm{MC}(G^i, N_{\pi|i} \setminus \{i\}, i)$
end for
Output: Allocation $\phi(G)$.
```

For general OCGs, Ge et al. [22] further proposed a greedy monotone (GM) decomposition method (see Appendix for algorithm details and examples) and extended the RFC rule to the eRFC rule by decomposing an OCG into multiple SOCGs and aggregating the results obtained from applying the RFC rule to each decomposed SOCG. The RFC rule is shown to satisfy Stay and SF, but not EA, even within SOCGs. Moreover, Ge et al. [22] demonstrated the inherent incompatibility among Stay, EA, and SF by proving the

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[2] It satisfies EA if and only if $v(\cdot)$ is submodular

existence of a class of unsolvable games, in which no value-sharing rule can simultaneously satisfy all three properties. The RFC rule thus satisfies all three properties except for these unsolvable cases.

In light of the aforementioned impossibility result and our focus on participation incentive axioms, we relax the SF axiom and investigate the extent to which all participation incentive axioms, including Stay, EA, S-Stay, and Part, can be simultaneously satisfied by value-sharing rules. Our first observation is that, unfortunately, the three existing value-sharing rules not only violate Stay and EA, but also fail to satisfy our newly introduced S-Stay and Part axioms. We summarize these shortcomings below.

Proposition 3.1. *SV and DMC satisfy Part while eRFC does not.*

Proof. Consider an SOCG $G = (N, v, \pi)$, where $N = \{1, 2\}, \pi = (1, 2)$ and $v(\{1\}) = v(\{2\}) = 0, v(\{1, 2\}) = 1$. eRFC rule allocates the entire value 1 to player 1, who is the first arrived player and contributes to the grand coalition. However, player 2, as a contributional player, receives zero shared value, thereby violating the PART axiom. □

Proposition 3.2. *DMC, SV, and eRFC rules do not satisfy S-Stay.*

Proof. Since SV rule fails to satisfy STAY, it does not satisfy S-Stay. For DMC and eRFC rules, consider an SOCG $G = (N, v, \pi)$ where $N = \{1, 2, 3\}$, $\pi = (1, 2, 3)$, and $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = 0, v(\{1, 2, 3\}) = 1$. DMC allocates the value 1 to player 3 because 3's arrival creates the new marginal value 1, i.e., $\phi(G, 1) = \phi(G, 2) = 0, \phi(G, 3) = 1$. It does not satisfy S-Stay for player 1 and 2 as they both contribute to the grand coalition after player 3's arrival, in which S-Stay requires player 1 and 2 should receive some positive value. With regard to eRFC rule, value 1 will be wholly allocated to player 1, i.e., $\phi(G, 1) = 1, \phi(G, 2) = \phi(G, 3) = 0$. Note that player 2, who is essential for creating the value 1, however, gets 0 in G. This violates S-Stay which requires that $\phi(G, 2) > 0$. □

Remark 1. *The DMC, and eRFC rules fail not only the EA axiom but also our newly introduced incentive axioms, S-Stay and Part, in general. Although the SV rule satisfies EA, this is rather trivial, as its allocation is entirely independent of the arrival order. These observations motivate us to investigate whether there exist sharing rules satisfying all three main incentive axioms simultaneously. In the next section, we address this question by introducing a family of sharing rules, termed the "Equal Sharing" rules.*

4 NEW DESIRABLE RULES

In the previous section, we highlighted the failure of existing rules to satisfy the existing and newly proposed incentive axioms, including EA, S-Stay, Part, etc. Motivated by these concerns, we study a family of value-sharing rules, termed **E**qual **S**haring rules (Algorithm 2), which is based on the simple concept that whenever a new contributing player joins, the marginal value is shared equally among a specific subset of players. The composition of this subset varies across the different rules we propose.

Notably, Algorithm 2 is quite general, and both the DMC and RFC rules can be interpreted as special cases within the Equal Sharing Scheme framework. In particular, each employs a singleton sharing set: under DMC, the sharing set consists solely of the arriving player i, whereas under RFC, it consists solely of the critical player j^* (see Algorithm 1). However, neither DMC nor RFC simultaneously satisfies the axioms S-Stay, Part, and EA. This naturally raises the question of whether there exists an allocation rule within the Equal Sharing Scheme that fulfills all participation incentives. We answer this question affirmatively by showing that there exists a class of Equal Sharing rules that satisfies these desirable axioms. Observe that the axioms S-Stay and Part are relatively easier to satisfy, whereas EA is more demanding. We identify two structural properties for sharing rules, one is *sharing consistency* and the other is *order independence*,under which the satisfaction of S-Stay and Part implies EA.

Algorithm 2 Equal Sharing Scheme

```
Input: OCG $G = (N, v, \pi)$.
for each arriving player $i$ in order $\pi$ do
    Initialize $\phi(G, i) \leftarrow 0$.
    Determine a sharing set $S^i \subseteq N_{\pi|i}$.
    for player $j$ in $S^i$ do
        $\phi(G, j) \leftarrow \phi(G, j) + \frac{1}{|S^i|} \mathrm{MC}(G^i, N_{\pi|i} \setminus \{i\}, i)$.
    end for
end for
Output: Allocation $\phi(G)$.
```

Definition 4.1 (Sharing Consistency). Given any OCG $G = (N, v, \pi)$, for any Equal Sharing rule ϕ, we say that ϕ is *sharing consistent* if, for any prefix subgames G^i and G^j with arriving players i and j, where $i \prec_\pi j$, it holds that $S^i \subseteq S^j$.

Recall that for any prefix subgame G^i with arriving player i, S^i denotes the sharing set of players in G^i under the Equal Sharing scheme. The property of *sharing consistency* requires that once an player $k \in S^i$, this player remains in the sharing set for all subsequent prefix subgames G^j where $j \succ_\pi i$.

Definition 4.2 (Order Independence). Given two OCGs $G_1 = (N, v, \pi_1)$ and $G_2 = (N, v, \pi_2)$, for any Equal Sharing rule ϕ, we say that ϕ is *order independent* if, for any $i, j \in N$ such that $N_{\pi_1|i} = N_{\pi_2|j}$, it holds that $S^i_1 = S^j_2$.

Intuitively, *order independence* means that for any two prefix subgames G^i_1 and G^j_2 with the arriving players i and j, if the set of existing arrived players (including i and j) is identical, i.e., $N_{\pi_1|i} = N_{\pi_2|j}$, then the resulting sharing sets S^i_1 and S^j_2 are the same. In other words, the sharing sets of subgame G^i_1 and G^j_2 are independent of the arrival order among the players in $N_{\pi_1|i}$ and $N_{\pi_2|j}$.

Theorem 4.3. *For any $G = (N, v, \pi)$ and any Equal Sharing rule ϕ satisfying S-Stay and Part, if ϕ is Sharing Consistent and Order Independent, then ϕ satisfies EA.*

Proof. Let ϕ denote an Equal Sharing rule that satisfies S-Stay and Part. Assume further that ϕ satisfies *Sharing Consistency* and *Order Independence.* Consider two OCGs $G_1 = (N, v, \pi_1)$ and $G_2 = (N, v, \pi_2)$ that differ only in the arrival order of some player $i \in N$, where i delays her arrival in π_2 relative to π_1:

$$\pi_1 : (1, 2, \ldots, i-1, \mathbf{i}, i+1, \ldots, j-1, j, j+1, \ldots, n);$$

$$\pi_2 : (1, 2, \ldots, i-1, i+1, \ldots, j-1, j, \mathbf{i}, j+1, \ldots, n).$$

We prove that ϕ satisfies EA axiom by discussing two cases based on whether i is a contributing player in G_1. For any player i, let S_1^i (resp. S_2^i) denote the sharing set in subgame G_1^i (resp. G_2^i).

Case 1. i is a contributing player in G_1. Since ϕ satisfies Part, we have $i \in S_1^i$. By sharing consistency, the total reward received by i in G_1 can be expressed as

$$\phi(G_1, i) = \sum_{k=i}^{n} \frac{1}{|S_1^k|} \mathrm{MC}(G_1, N_{\pi_1|k} \setminus \{k\}, k).$$

On the other hand, in G_2, player i's total share can be upper-bounded by

$$\phi(G_2, i) \leq \frac{1}{|S_2^i|} \mathrm{MC}(G_2^i, N_{\pi_2|i} \setminus \{i\}, i) + \sum_{k=j+1}^{n} \frac{1}{|S_2^k|} \mathrm{MC}(G_2, N_{\pi_2|k} \setminus \{k\}, k).$$

Since ϕ satisfies order independence, the corresponding sharing sets in the two games satisfy $S_1^j = S_2^i$, and for all $k \in \{j+1, \ldots, n\}$, $S_1^k = S_2^k$. Thus, the value shared with i from players arriving after j is identical in both games and cancels out in the comparison. We therefore focus on the difference of shared values for player j and derive that

$$\begin{aligned}
&\phi(G_1, i) - \phi(G_2, i) \\
\geq& \sum_{k=i}^{j} \frac{1}{|S_1^k|} \mathrm{MC}(G_1, N_{\pi_1|k} \setminus \{k\}, k) - \frac{1}{|S_2^i|} \mathrm{MC}(G_2^i, N_{\pi_2|i} \setminus \{i\}, i) \\
\geq& \frac{1}{|S_1^j|} \sum_{k=i}^{j} \mathrm{MC}(G_1, N_{\pi_1|k} \setminus \{k\}, k) - \frac{1}{|S_2^i|} \mathrm{MC}(G_2^i, N_{\pi_2|i} \setminus \{i\}, i) \\
& \hfill \text{(Sharing consistency)} \\
\geq& \frac{1}{|S_1^j|} \big(v(N_{\pi_1|j}) - v(N_{\pi_1|i} \setminus \{i\})\big) - \frac{1}{|S_2^i|} \big(v(N_{\pi_2|i}) - v(N_{\pi_2|j})\big) \\
& \hfill \text{(MC definition)} \\
\geq& \frac{1}{|S_1^j|} \big(v(N_{\pi_2|j} - v(N_{\pi_1|i} \setminus \{i\}))\big) \qquad (S_1^j = S_2^i, N_{\pi_1|j} = N_{\pi_2|i}) \\
\geq& 0. \qquad \text{(Valuation monotonicity)}
\end{aligned}$$

This implies that when player i is a contributing player, i has no incentive to delay her arrival.

Case 2. If i is not a contributing player in G_1. The first sub-case is that i is also not a contributing player in G_2, then it follows that $\mathrm{MC}(G_2^i, N_{\pi_2|i} \setminus \{i\}, i) = 0$. Since ϕ satisfies order consistency, for every $k \in \{j+1, \ldots, n\}$ we have $S_1^k = S_2^k$, implying that after player $j+1$'s arrival, the value shared with i is identical in G_1 and G_2. However, because ϕ satisfies S-Stay, player i may obtain extra benefit in G_1 if she is essential in generating the marginal contribution of some player q who arrives between i and j in π_1. Thus, i again has no incentive to delay her arrival. Now consider the remaining sub-case where i is not a contributing player in G_1 but becomes one in G_2 after delaying. This implies that i is essential for generating the marginal contribution of some player arriving between i and j in π_1 (Otherwise i cannot be a contributing player after delay the arrival). Let q be the **first** such player, i.e., $v(N_{\pi_1|q}) > v(N_{\pi_1|q} \setminus i)$ with $i \prec_{\pi_1} q \prec_{\pi_1} j$ (or $q = j$). Recall that ϕ satisfies S-Stay and sharing consistency. i will be included in the sharing set since q's arrival. Hence, i's total share in G_1 is

$$\phi(G_1, i) = \sum_{k=q}^{n} \frac{1}{|S_1^k|} \mathrm{MC}(G_1, N_{\pi_1|k} \setminus \{k\}, k).$$

For G_2, by Part and sharing consistency, we have

$$\phi(G_2, i) = \frac{1}{|S_2^i|} \mathrm{MC}(G_2^i, N_{\pi_2|i} \setminus \{i\}, i) + \sum_{k=j+1}^{n} \frac{1}{|S_2^k|} \mathrm{MC}(G_2, N_{\pi_2|k} \setminus \{k\}, k).$$

Applying the same reasoning as in Case 1, we obtain the difference between $\phi(G_1, i)$ and $\phi(G_2, i)$

$$\begin{aligned}
\phi(G_1, i) - \phi(G_2, i) &= \frac{1}{|S_1^j|} \big(v(N_{\pi_2|j}) - v(N_{\pi_1|(q-1)})\big) \\
&= \frac{1}{|S_1^j|} \big(v(N_{\pi_1|j} \setminus \{i\}) - v(N_{\pi_1|(q-1)})\big).
\end{aligned}$$

We claim that $v(N_{\pi_1|(q-1)}) = v(N_{\pi_1|(q-1)} \setminus \{i\})$. To see this, suppose not, by monotonicity, it must be the case that $v(N_{\pi_1|(q-1)}) > v(N_{\pi_1|(q-1)} \setminus \{i\})$, implying that player i is essential to create the marginal value when player $q-1$ joins the coalition, contradicting our assumption that player q is the first player such that i is essential to create the marginal contribution. Therefore, we have $v(N_{\pi_1|(q-1)}) = v(N_{\pi_1|(q-1)} \setminus \{i\})$. Consequently,

$$\begin{aligned}
\phi(G_1, i) - \phi(G_2, i) &= \frac{1}{|S_1^j|} \big(v(N_{\pi_1|j} \setminus \{i\}) - v(N_{\pi_1|(q-1)})\big) \\
&= \frac{1}{|S_1^j|} \big(v(N_{\pi_1|j} \setminus \{i\}) - v(N_{\pi_1|(q-1)} \setminus \{i\})\big) \geq 0. \\
& \hfill \text{(Valuation monotonicity)}
\end{aligned}$$

Combining both cases, we conclude that player i never benefits from delaying her arrival. Therefore, any Equal Sharing rule ϕ that satisfies S-Stay, Part, sharing consistency, and order independence also satisfies EA. □

4.1 MES and NDMES

Based on Theorem 4.3, we introduce two simple yet representative Equal Sharing rules, one is the **Marginal Equal Share** (MES) rule, and the other is the **Non-Dummy Marginal Equal Share** (NDMES) rule. The MES rule is based on the simple principle of distributing each player's marginal contribution equally among all existing players. Specifically, for each prefix subgame G^i with arriving player i, the sharing set is defined as $S^i = N_{\pi|i}$. Interestingly, MES satisfies not only Part and S-Stay, but also EA. A potential drawback of the MES rule is that its value sharing may be perceived as unfair, since some players may act as free riders: dummy players who never contribute to the grand coalition can still receive positive rewards. In other words, MES violates the OD axiom. It naturally raises the question of whether such dummy players can be excluded while preserving the desirable incentive properties. This leads to our second value-sharing rule, the Non-Dummy Marginal Equal Share (NDMES) rule, which allocates the marginal contribution only among existing non-dummy players. Specifically, for each prefix subgame G^i, let D_i denote the dummy player set under G^i. Define $S^i = N_{\pi|i} \setminus D_i$ as the sharing player set. For each player $j \in S^i$, j receives an equal share of $\frac{1}{|S^i|} \mathrm{MC}(G^i, N_{\pi|i} \setminus \{i\}, i)$. By construction, the NDMES rule satisfies OD, since dummy players are never included in the sharing set in any prefix subgame. Surprisingly, NDMES not only satisfies OD, but also maintains the incentive properties S-Stay, EA, and Part.

Proposition 4.4. *MES and NDMES satisfy S-Stay, Part, and EA.*

Proof. We first show that both the MES and NDMES rules satisfy S-Stay and Part. For the MES and NDMES rules, the Stay axiom is immediately satisfied since every player's shared value is non-decreasing throughout the process. Now consider any online cooperative game $G = (N, v, \pi)$ and any prefix subgame G^i corresponding to the arrival of player i. Take any player j such that $j \prec_\pi i$ and $v(N_{\pi|i}) > v(N_{\pi|i} \setminus j)$. For the MES rule, the sharing set in G^i is $N_{\pi|i}$, which includes j. Therefore, player j receives an additional positive share in G^i, given by $\frac{1}{|N_{\pi|i}|} \mathrm{MC}(G^i, N_{\pi|i} \setminus \{i\}, i) > 0$, which implies that $\phi(G^i, j) > \phi(G^j, j)$. The same reasoning applies to the NDMES rule: since every player j with $v(N_{\pi|i}) > v(N_{\pi|i} \setminus j)$ is a non-dummy player, j is included in the sharing set S^i of G^i, and thus also satisfies S-Stay. Next, we verify Part. Under the MES rule, any newly arriving player who makes a positive marginal contribution immediately receives a positive share, $\frac{1}{|N_{\pi|i}|} \mathrm{MC}(G^i, N_{\pi|i} \setminus \{i\}, i) > 0$, which establishes Part. For the NDMES rule, any such contributing player is non-dummy and thus included in the sharing set S^i, again satisfying Part. Having established that both the MES and NDMES rules satisfy S-Stay and Part, by Theorem 4.3, it remains to show that they satisfy sharing consistency and order independence.

The MES rule trivially satisfies both properties, as its sharing set in each subgame is simply the set of all existing players. For the NDMES rule, observe that for any two subgames G^i and G^j with $i \prec_\pi j$, every non-dummy player in G^i remains non-dummy in G^j. Moreover, the set of non-dummy players in each subgame depends solely on the player set rather than the arrival order. Hence, NDMES also satisfies sharing consistency and order independence. Applying Theorem 4.3, we conclude that both MES and NDMES satisfy the EA axiom. □

4.2 Alternative Rules: ULMES and eULMES

In the previous subsection, we characterized a class of Equal Sharing rules that satisfy three key participation incentive axioms, S-Stay, Part, and EA, and introduced two representative instances: MES and NDMES. However, the MES rule fails to satisfy the fairness notion of Online Dummy, while the NDMES rule faces computational challenges, as identifying all dummy players in each subgame requires exponential time due to the need to verify all coalitions[3].

We next move beyond the scope of the previous characterization and focus on designing a polynomial-time computable Equal Sharing rule. We propose a novel rule termed **Upward Lexicographic Marginal Equal Share** (ULMES) rule.

The ULMES rule follows a greedy procedure. For any prefix subgame G^i, it first initializes the sharing set as $S^i = N_{\pi|i}$ and then determines which players to retain in S^i in an upward lexicographic order. Specifically, for each player ℓ (starting from i), the rule checks whether removing ℓ decreases the grand coalition value in G^i. If the coalition $S^i \setminus \ell$ yields the same value as $v(N_{\pi|i})$, then player ℓ is considered dispensable for generating the marginal contribution and is removed from S^i. This procedure proceeds iteratively with $\ell \leftarrow \ell - 1$ until all players have been examined. The remaining players in S^i equally share the marginal contribution.

Algorithm 3 ULMES rule

```
Input: OCG $G = (N, v, \pi)$.
for each arriving player $i$ in order $\pi$ do
    Initialize $\phi(G, i) \leftarrow 0$ and $S^i \leftarrow N_{\pi|i}, \ell \leftarrow i$.
    while $\ell > 0$ do
        if $v(S^i \setminus \{\ell\}) = v(N_{\pi|i})$ then
            Update $S^i \leftarrow S^i \setminus \{\ell\}$; $\ell \leftarrow \ell - 1$.
        end if
    end while
    for player $j$ in $S^i$ do
        $\phi(G, j) \leftarrow \phi(G, j) + \frac{1}{|S^i|} \mathrm{MC}(G^i, N_{\pi|i} \setminus \{i\}, i)$.
    end for
end for
Output: Allocation $\phi(G)$.
```

Intuitively, when a new player i arrives, there may exist multiple coalitions capable of generating the same marginal contribution. Among all such coalitions, ULMES selects the player set S^i for which the last arriving player is the earliest, that is, it favors the coalition that is **lexicographically minimal in terms of arrival order**. The ULMES rule runs in $O(n^2)$ time as computing the sharing set S^i requires $O(n)$ time for verifying whether each existing player should be removed from S^i.

We show that ULMES satisfies the S-Stay, Part, and OD axioms. However, it unfortunately fails to satisfy the EA axiom. Due to space limitations, The formal proofs of these properties are deferred to the appendix. We further provide here a counterexample illustrating why ULMES fails to meet the EA condition.

Theorem 4.5. *ULMES satisfies S-Stay, Part, and OD, but fails EA.*

Example 4.6. Consider two OCGs $G_1 = (N, v, \pi_1)$, $N = \{1, 2, 3, 4\}$, $\pi_1 = (1, 2, 3, 4)$ and $G_2 = (N, v, \pi_2)$ where $\pi_2 = (1, 2, 4, 3)$, that is, the instance where player 3 delays her arrival in G_1, which leads to G_2. For the valuation function, we have

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{4\}) = 0;$$
$$v(\{1,2\}) = v(\{1,4\}) = v(\{2,4\}) = v(\{1,2,4\}) = 0;$$
$$0 \le v(\{1,3\}) \le v(\{2,3\}) < v(\{1,2,3\}) = x \ (x > 0);$$
$$v(\{3,4\}) = v(\{1,3,4\}) = v(\{2,3,4\}) = v(\{1,2,3,4\}) = y > x.$$

We now focus on player 3. For G_1, when player 3 arrives, the value jumps from 0 to x and ULMES selects the sharing set $S^3 = \{1, 2, 3\}$, so we have $\phi(G_1^3, 3) = \frac{x}{3}$, when player 4 comes, only $\{3, 4\}$ survive in ULMES and the marginal contribution $(y - x)$ is equally shared by 3 and 4. Hence we have $\phi(G_1, 3) = \frac{x}{3} + \frac{y-x}{2}$. In contrast, when 3 delays her arrival, i.e., in G_2, 1, 2, 4 share no value as $v(\{1, 2, 4\}) = 0$. When 3 arrives in π_2, the marginal contribution will be y and it is shared by $S_2^3 = \{3, 4\}$ according to ULMES rule. Then we have $\phi(G_2, 3) = \frac{y}{2}$, which is greater than $\phi(G_1, 3) = \frac{x}{3} + \frac{y-x}{2}$. This implies that ULMES fails EA axiom.

Intuitively, ULMES fails to satisfy EA in general because a player might choose to delay their arrival to become a contributing player

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[3] We show that NDMES runs in $O(n)$ time when the valuation function is subadditive. See appendix for more details.

who shares a larger marginal value with fewer players. Although this player might forfeit some shared value from previous timesteps, the potential gain from the new marginal value can outweigh the losses, thereby undermining the EA property. Although ULMES fails EA in general, we observe that it adheres to the EA axiom for every simple online cooperative game (SOCG). The proof proceeds via a case analysis, considering whether a player i is pivotal and how her delayed arrival position affects the sharing set. Due to space constraints, the detailed proof is deferred to the appendix.

Lemma 4.7. *For any SOCG, ULMES satisfies EA.*

Recall the Greedy Monotone (GM) decomposition algorithm proposed by Ge et al. [22], which generalizes the RFC rule for SOCGs to the eRFC rule for general OCGs without compromising any axiomatic guarantees. Following a similar idea, we extend our ULMES rule to the *extended Upward Lexicographic Marginal Equal Share* (eULMES) rule by leveraging the GM decomposition framework. We briefly outline the main steps of the GM decomposition and provide detailed explanations and examples in the appendix.

In general, the GM decomposition algorithm takes any OCG G as input and outputs a linear combination of component games, denoted as $D(G)$. Each component is represented as a pair $(c, \bar{G})$, where c is the coefficient and $\bar{G}$ is an SOCG. Given any OCG $G = (N, v, \pi)$, the eULMES rule proceeds as follows:

1. apply the GM decomposition to decompose G into multiple component games;
2. execute the ULMES rule within each component game;
3. aggregate the weighted results across all components to obtain each player's total share.

Theorem 4.8. *eULMES satisfies S-Stay, EA, Part, and OD.*

Due to space limitations, we defer the formal proof of theorem 4.8 to the appendix. To further illustrate the behavior of the eULMES rule and its distinction from ULMES, we provide a specific example in the appendix, in which ULMES fails the EA axiom, while eULMES successfully satisfies it.

5 INDIVIDUAL RATIONALITY UNDER SUPERADDITIVITY VALUATION

In the previous sections, we explored participation incentives through the lens of the EA, S-Stay, and Part axioms. We now shift our focus to the individual rationality (IR) axiom, which reflects a fundamental participation incentive, requiring that a player chooses to join the grand coalition only if the value they receive from it is at least as great as their standalone (singleton) valuation.

We begin by presenting an impossibility result regarding the satisfaction of the IR axiom in online cooperative games.

Proposition 5.1 (Impossibility). *There is no value sharing rule satisfying IR for OCG under general valuation functions.*

In light of the impossibility result, we restrict our attention to the *superadditive* valuation. Specifically, for an OCG $G = (N, v, \pi)$, we say that G is a *superadditive OCG* if its valuation function v is superadditive. In what follows, we first verify that both the DMC and SV rules satisfy the IR axiom, whereas the eRFC rule does not. We then introduce an IR-refinement paradigm that modifies all three rules to ensure IR satisfaction while preserving their other desirable axiomatic properties.

Proposition 5.2. *In OCGs with superadditive valuations, DMC and SV satisfy IR while eRFC does not.*

Unfortunately, MES, NDMES, and eULMES also fail IR axiom. To address this, we propose a simple IR refinement to modify these rules such that IR axiom is satisfied. The refinement follows a straightforward approach. For each subgame G^i, we first allocate value $v(\{i\})$ to player i to ensure satisfaction of the IR axiom. After that, we apply the Equal Sharing rule to determine the sharing set S^i, and then equally distribute the remaining marginal contribution, $\mathrm{MC}(G^i, N_{\pi|i} \setminus \{i\}, i) - v(\{i\})$, among S^i.

Algorithm 4 IR Refinement Framework

```
Input: OCG $G = (N, v, \pi)$.
for each arriving player $i$ in order $\pi$ do
    Initialize $\phi(G, i) \leftarrow 0$.
    $\hat{\phi}(G, i) \leftarrow v(\{i\})$.
    Compute sharing set $S^i$ by the rule $\phi$.
    for player $j$ in $S^i$ do
        $\hat{v}_i \leftarrow \mathrm{MC}(G^i, N_{\pi|i} \setminus \{i\}, i) - v(\{i\})$.
        $\hat{\phi}(G, i) \leftarrow \hat{\phi}(G, i) + \frac{1}{|S^i|}\hat{v}_i$.
    end for
end for
Output: Allocation $\phi(G)$.
```

For each Equal Sharing rule, we denote its refined counterpart by adding the "IR-" prefix, for example, eULMES becomes the IR-eULMES rule. This refinement guarantees compliance with the IR axiom while preserving all other desired properties.

Theorem 5.3. *For any superadditive OCG, the IR-MES, IR-NDMES, and IR-eULMES rules satisfy IR while preserving all other axioms that were satisfied prior to the refinement.*

6 DISCUSSION

In this paper, we study participation incentive axioms in online cooperative games with strategic arrivals, including the existing Stay and EA axioms, and our newly studied S-Stay, Part, and IR. We first identify that existing sharing rules fail the axioms. To address this, we design a class of "Equal Sharing" rules and provide sufficient conditions for satisfying S-Stay, Part, and EA. An intriguing open question that remains is: "What are the necessary and sufficient conditions for an Equal Sharing rule to satisfy these axioms?" Moreover, studying online hedonic games with strategic arrivals presents a promising direction for future research.

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