Comparing the Fairness of Recursively Balanced Picking Sequences

Karen Frilya Celine
National University of Singapore
Singapore
karen.celine@u.nus.edu

Warut Suksompong
National University of Singapore
Singapore
warut@comp.nus.edu.sg

Sheung Man Yuen
National University of Singapore
Singapore
ysm@u.nus.edu

ABSTRACT

Picking sequences are well-established methods for allocating indivisible goods. Among the various picking sequences, *recursively balanced picking sequences*—whereby each agent picks one good in every round—are notable for guaranteeing allocations that satisfy envy-freeness up to one good. In this paper, we compare the fairness of different recursively balanced picking sequences using two key measures. Firstly, we demonstrate that all such sequences have the same price in terms of egalitarian welfare relative to other picking sequences. Secondly, we characterize the approximate maximin share (MMS) guarantees of these sequences. In particular, we show that compensating the agent who picks last in the first round by letting her pick first in every subsequent round yields the best MMS guarantee.

KEYWORDS

Picking sequence; Egalitarian welfare; Maximin share; Fair division

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1 INTRODUCTION

As humans living together and sharing limited resources, we are bound to face the problem of how to fairly divide the resources, also known as *fair division* [31]. An important application of fair division is the allocation of *indivisible goods*, such as allocating course slots to students, assigning sports players to teams, or distributing research equipment among labs [2, 14, 22].

While numerous methods have been proposed for fairly allocating indivisible goods, one of the most fundamental class of methods is that of *picking sequences* [13, 16, 27]. Picking sequences allow agents to select goods according to a prespecified agent order—at each turn, the designated agent chooses her favorite good from the remaining goods. Not only are picking sequences intuitive and easy to implement, but they also help preserve the privacy of the participating agents, as the agents only need to reveal their picks rather than their entire valuations for the goods. A picking sequence is called *recursively balanced* if at every point in the sequence, the difference between the number of turns taken by each pair of agents is at most one. Note that a recursively balanced picking sequence can be divided into *rounds*, where each agent picks exactly once in every round (except possibly the last round, in which some agents may not receive a pick). Recursively balanced picking sequences are notable because they always produce allocations satisfying the fairness notion of *envy-freeness up to one good (EF1)*, which means that if an agent envies another agent, then the envy can be eliminated by removing a good from the latter agent's bundle [2, p. 5].[1]

Among recursively balanced picking sequences, the most widely studied one is *round-robin*, which lets the agents pick in the order $(1, 2, \ldots, n \mid 1, 2, \ldots, n \mid 1, 2, \ldots)$, where n denotes the number of agents. However, round-robin is far from the only recursively balanced picking sequence. Another natural recursively balanced sequence is *balanced alternation*, which reverses the ordering in every alternate round: $(1, 2, \ldots, n \mid n, n-1, \ldots, 1 \mid 1, 2, \ldots)$.[2] Intuitively, balanced alternation appears fairer than round-robin, as it gives the agents more equal opportunities to choose their preferred goods. The applicability of balanced alternation is demonstrated by the fact that it is used to allocate courses to students at Harvard Business School [17]. Nevertheless, given that all recursively balanced picking sequences ensure EF1, it is unclear which sequence should be considered the "fairest". Are there recursively balanced picking sequences that fare better with respect to certain fairness criteria than these well-known sequences?

In this paper, we compare the fairness of recursively balanced picking sequences using two established measures: egalitarian welfare and maximin share (MMS).

1.1 Our Results

Assume that there are n agents with additive utilities over m indivisible goods. If $m < n$, all recursively balanced picking sequences are effectively equivalent, so we assume that $m \geq n$. Without loss of generality, we consider picking sequences starting with the prefix $(1, 2, \ldots, n)$. Our model is described formally in Section 2.

In Section 3, we compare recursively balanced picking sequences based on their worst-case egalitarian welfare relative to other picking sequences. Specifically, we define the *egalitarian price* of a picking sequence as the supremum, taken across all possible instances, of the ratio between the "optimal egalitarian welfare" and the egalitarian welfare of the given picking sequence. We show that this ratio is infinite for any recursively balanced picking sequence if the

[1]This guarantee relies on the assumption that agents have additive utilities over the goods. This assumption is common in the fair division literature, and we will make it throughout our paper.
[2]The term *balanced alternation* has also been used to refer to other picking sequences [15, 16].

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optimal egalitarian welfare is taken among *all allocations* or *all picking sequences* within the instance, which renders the comparison impractical. Hence, in Section 3.1, we take the optimal egalitarian welfare among allocations obtained via picking sequences with the same prefix $(1, 2, \ldots, n)$. In Section 3.2, we further restrict the optimal welfare to be across allocations produced by *recursively balanced* picking sequences starting with $(1, 2, \ldots, n)$. Perhaps surprisingly, for both variants, we find that all recursively balanced picking sequences have the same egalitarian price: $\min\{m-n+1, n\}$ and $\min\{\lceil m/n \rceil, \lfloor \log_2 n \rfloor + 1\}$ respectively. For the latter variant, our proof involves analyzing paths along a directed graph constructed from the agents' picks.

Next, in Section 4, we compare recursively balanced picking sequences using their approximate MMS guarantees. To state these guarantees, we define conditions for a picking sequence to be *regular*. Most picking sequences—including all sequences with $m \geq 2n$—are regular, and we characterize their MMS guarantees in Theorem 4.2. For such sequences, the MMS guarantee depends only on the picks of agent n (who picks last in the first round). However, there are a small number of *irregular* picking sequences, for which the agent with the worst MMS guarantee is agent $n-1$ instead of agent n. The MMS guarantees of these sequences are characterized in Theorem 4.5. Our characterizations allow us to determine the best picking sequences with respect to MMS in Theorem 4.6; these include the sequences that compensate agent n by letting her pick first in every round after the first. On the other end of the spectrum, we also identify the picking sequences with the worst MMS guarantee in Theorem 4.7. While these include the round-robin sequence as one may expect, more interestingly, they also include the balanced alternation sequence whenever $m \geq 3n - 1$.

1.2 Further Related Work

As mentioned earlier, picking sequences have long been studied in fair division [12, 16, 27]. Since it can be beneficial for an agent to avoid picking her favorite good if she is aware of other agents' preferences, several authors have investigated picking sequences from a strategic perspective [6, 13, 26, 33]. Among picking sequences, round-robin has received particular attention due to its simplicity and EF1 fairness guarantee [3, 30]. Bouveret et al. [11] focused on *constrained serial dictatorships*, where the turns of each agent occur consecutively, e.g., $(1, 2, 2, 3, 3, 3, 3)$. While such picking sequences are strategyproof, they are generally far from guaranteeing EF1. Chakraborty et al. [21] showed that picking sequences provide meaningful fairness guarantees when agents have different entitlements, while Gourvès et al. [24] defined fairness criteria based on the picking sequences themselves. Aziz et al. [7] attained "best-of-both-worlds" fairness via a lottery over picking sequences.

Given the variety of picking sequences, a natural direction is to compare them with respect to particular criteria. Bouveret and Lang [12] studied the *expected* utilitarian and egalitarian welfare of picking sequences, and computed the optimal picking sequences for small numbers of agents and goods (see their Table 1). Subsequently, Kalinowski et al. [25] proved that round-robin yields the optimal expected utilitarian welfare when there are two agents, under certain distributions of the agents' utilities (see their Theorem 1). Unlike these average-case analyses, our results are worst-case and do not rely on any distributional assumptions. Aziz et al. [8] examined the complexity of possible and necessary allocation problems for different classes of picking sequences.

Our egalitarian price notion is inspired by the well-established *price of fairness* concept, which captures the ratio between the optimal welfare overall and the optimal welfare subject to a given fairness requirement such as EF1 [10, 18, 19, 29]. While both egalitarian and utilitarian welfare have been considered in this line of work, we focus on egalitarian welfare in our paper since it is widely regarded as a fairness measure (whereas utilitarian welfare is typically viewed as a measure of efficiency). Baumeister et al. [9, Sec. 6.2] also studied the price of picking sequences but restricted utilities to follow a specific scoring vector. Finally, approximate MMS has received significant attention in recent years [1, 23, 28].

2 PRELIMINARIES

Let $N = \{1, \ldots, n\}$ be a set of $n \geq 2$ agents, and M be a set of m goods; we typically denote the goods by $g_1, \ldots, g_m$. A *bundle* refers to a (possibly empty) set of goods in M. Each agent $i \in N$ has a utility function u_i such that $u_i(S)$ is agent i's utility for the bundle $S \subseteq M$; we write $u_i(g)$ instead of $u_i(\{g\})$ for a single good $g \in M$. Each utility function is *additive*, i.e., $u_i(S) = \sum_{g \in S} u_i(g)$ for any $S \subseteq M$. The utilities are *identical* if $u_i = u_j$ for all $i, j \in N$. Furthermore, each agent i has a *picking preference order* $\succ_i$, which is a total order on the set of goods M: for any $g, g' \in M$, $g \succ_i g'$ means that agent i would choose good g before good g', provided both goods are available. We assume that $u_i(g) > u_i(g')$ implies $g \succ_i g'$ for any $g, g' \in M$. We require $\succ_i$ to be a total order to facilitate tie-breaking.[3] An *instance* consists of N, M, $(u_i)_{i \in N}$, and $(\succ_i)_{i \in N}$. Denote by $\mathcal{I}_{n,m}$ the set of all instances with n agents and m goods.

Given an instance $\mathcal{I} \in \mathcal{I}_{n,m}$, let $\mathcal{P}$ be the set of all partitions of M into n bundles. The *maximin share (MMS) of agent i* is defined as

$$\text{MMS}_i \coloneqq \max_{\{P_1, \ldots, P_n\} \in \mathcal{P}} \min_{j \in \{1, \ldots, n\}} u_i(P_j).$$

An allocation $\mathcal{A} = (A_1, \ldots, A_n)$ is an ordered partition of M into n bundles $A_1, \ldots, A_n$ such that A_i is allocated to agent $i \in N$. An allocation $\mathcal{A}$ is called *envy-free up to one good (EF1)* if for any $i, j \in N$ with $A_j \neq \emptyset$, there exists $g \in A_j$ such that $u_i(A_i) \geq u_i(A_j \setminus \{g\})$. The *egalitarian welfare* of allocation $\mathcal{A}$ for instance $\mathcal{I}$ is defined as $\text{EW}(\mathcal{A}, \mathcal{I}) \coloneqq \min_{i \in N} u_i(A_i)$.

A *picking sequence* is a sequence $\pi = (a_1, \ldots, a_m)$, where $a_j \in N$ for each $j \in \{1, \ldots, m\}$. For each agent $i \in N$, *the picking sequence of agent i in* π is defined as $\pi_i = (t_1, \ldots, t_R)$, where $a_t = i$ if and only if $t \in \{t_1, \ldots, t_R\}$, and $t_1 < \cdots < t_R$. For each $r \in \{1, \ldots, R\}$, we call t_r *the index of agent i's r-th pick in* π.

A picking sequence π is *recursively balanced* if for every prefix of π and every pair of agents, the difference in the number of times that the two agents appear in the prefix is at most 1. Note that for each $j \in \{1, \ldots, \lfloor m/n \rfloor\}$, the prefix of length jn of a recursively balanced picking sequence π contains each agent exactly j times. Hence, the subsequence $(a_{(j-1)n+1}, \ldots, a_{jn})$ contains each agent exactly once; we call this subsequence *the j-th round* of π. When m is not divisible by n, *the* $\lceil m/n \rceil$*-th round* of π is the subsequence

[3] Alternatively, one could assume that every agent always breaks ties by picking a good with a lower index before one with a higher index if the agent values both goods equally. All specific instances we use in our proofs satisfy this property.

$(a_{\lfloor m/n \rfloor n+1}, \ldots, a_m)$, which contains each agent at most once. For clarity, we may use a vertical bar (|) instead of a comma (,) to separate the sequences of different rounds. That is, given a picking sequence π, we may write $\pi = (a_1, \ldots, a_n \mid a_{n+1}, \ldots, a_{2n} \mid \cdots \mid a_{\lfloor m/n \rfloor n+1}, \ldots, a_m)$. We sometimes use the term *round* to denote the same picks for picking sequences that are not recursively balanced—for such sequences, it is not necessary that every agent picks at most once in each round. The *round-robin sequence* is the recursively balanced picking sequence

$$\pi_{\text{RR}} \coloneqq (1, 2, \ldots, n \mid \cdots \mid 1, 2, \ldots, n \mid 1, 2, \ldots, a_m),$$

where $a_m = m - (\lceil m/n \rceil - 1)n$.

We denote by $\Pi_{n,m}$ the set of all picking sequences with n agents and m goods prefixed by $(1, 2, \ldots, n)$. Moreover, we denote by $\mathcal{R}_{n,m}$ the set of all *recursively balanced* picking sequences with n agents and m goods prefixed by $(1, 2, \ldots, n)$. Note that $\Pi_{n,m}$ (resp. $\mathcal{R}_{n,m}$) does not contain all picking sequences (resp. recursively balanced picking sequences). However, for any picking sequence (resp. recursively balanced picking sequence) π with all agents in the first n picks, there exists a picking sequence π' in $\Pi_{n,m}$ (resp. $\mathcal{R}_{n,m}$) that is equivalent to π up to some relabeling of the agents.

Given an instance and a picking sequence $\pi = (a_1, \ldots, a_m)$, the allocation $\mathcal{A}^\pi = (A_1^\pi, \ldots, A_n^\pi)$ obtained with π is given as follows: each agent begins with an empty bundle, and at each step $j \in \{1, \ldots, m\}$, agent a_j selects the $\succ_{a_j}$-maximum good available among the unallocated goods and adds this good to her bundle. The *round-robin allocation* is the allocation obtained with π_{RR}. Given a picking sequence π and an instance $\mathcal{I}$, we write $\text{EW}(\pi, \mathcal{I}) \coloneqq \text{EW}(\mathcal{A}^\pi, \mathcal{I})$ to mean the egalitarian welfare of the allocation obtained by π.

We make the following observation. The proof of this observation, along with all other omitted proofs, can be found in the full version of our paper [20].

Proposition 2.1. *A picking sequence always produces an EF1 allocation if and only if it is recursively balanced.*

3 EGALITARIAN PRICE

In this section, we compare recursively balanced picking sequences using the egalitarian welfare. We first observe that if the utilities are identical, then round-robin is always the worst among such sequences.

Proposition 3.1. *Let $\mathcal{I}$ be an instance with n agents with identical utilities and m goods, and let $\pi \in \mathcal{R}_{n,m}$. Then, the egalitarian welfare of the allocation obtained with π is at least the corresponding welfare with π_{RR}.*

Interestingly, Proposition 3.1 ceases to hold for non-identical utilities, even with identical orders of preference.

Example 3.2. Consider the following instance.

g	g_1	g_2	g_3	g_4
$u_1(g)$	8	7	5	0
$u_2(g)$	7	6	4	3

The round-robin allocation is $(\{g_1, g_3\}, \{g_2, g_4\})$, which has an egalitarian welfare of 9. However, the allocation obtained with $\pi = (1, 2, 2, 1)$ is $(\{g_1, g_4\}, \{g_2, g_3\})$, which has an egalitarian welfare of only 8.

In order to compare different sequences, a natural approach is to consider their egalitarian welfare in the worst case. However, this approach is not meaningful because for any sequence, this value can be 0 even when the optimal egalitarian welfare is positive.

Example 3.3. Given any $n \geq 2$, $m \geq n$, and $\pi \in \mathcal{R}_{n,m}$, consider an instance with the following utilities:

- $u_1(g_1) = 2$, $u_1(g_2) = 1$, and $u_2(g_1) = 3$.
- For each agent $i \in N \setminus \{1, 2\}$, we have $u_i(g_i) = 3$.
- $u_i(g_j) = 0$ for all other pairs (i, j).

Agents break ties in favor of lower-index goods.

Observe that according to π, the goods will be chosen in increasing order of index, i.e., each agent $i \in N$ picks good g_i in her first turn, and picks a good with value 0 in any subsequent turn. Hence, agent 1 receives utility 2, agent 2 receives utility 0, and each agent $i \in N \setminus \{1, 2\}$ receives utility 3. Therefore, the egalitarian welfare obtained with π is 0. On the other hand, if agents 1 and 2 swap g_1 and g_2, the egalitarian welfare becomes 1. Moreover, this allocation can be obtained with the picking sequence π' derived from π by switching the first two picks (between agents 1 and 2).

Example 3.3 implies that the conventional definition of the *egalitarian price* of $\pi \in \mathcal{R}_{n,m}$, i.e.,

$$\sup_{\mathcal{I} \in \mathcal{I}_{n,m}} \max_{\mathcal{A}} \frac{\text{EW}(\mathcal{A}, \mathcal{I})}{\text{EW}(\pi, \mathcal{I})},$$

where the maximum is taken across all allocations $\mathcal{A}$ in the instance $\mathcal{I}$ [5, 19, 32], is not useful as a metric for comparison.[4] Indeed, with this definition, the egalitarian price of any picking sequence $\pi \in \mathcal{R}_{n,m}$ would be ∞ for any $n \geq 2$ and $m \geq n$. Moreover, the picking sequence π' in Example 3.3 yields positive egalitarian welfare.[5] Hence, this also rules out the following definition of the egalitarian price of $\pi \in \mathcal{R}_{n,m}$ with respect to all other picking sequences:

$$\sup_{\mathcal{I} \in \mathcal{I}_{n,m}} \max_{\pi'} \frac{\text{EW}(\pi', \mathcal{I})}{\text{EW}(\pi, \mathcal{I})}.$$

In a bid to find a meaningful comparison metric, we observe that the picking sequence π' in Example 3.3 does not belong to $\Pi_{n,m}$ or $\mathcal{R}_{n,m}$, since it does not start with the prefix $(1, 2, \ldots, n)$. Recall from Section 2 that, when comparing different picking sequences, we can restrict our attention to picking sequences that start with $(1, 2, \ldots, n)$ without loss of generality, since any picking sequence with all agents in the first n picks is equivalent to such a sequence up to relabeling of the agents. With the first-round prefix fixed, $\mathcal{R}_{n,m}$ is precisely the set of picking sequences that guarantee EF1 (Proposition 2.1), and $\Pi_{n,m}$ is the set of picking sequences that we investigate. From this perspective, it is reasonable to use the allocations produced by picking sequences in $\Pi_{n,m}$ or $\mathcal{R}_{n,m}$ as comparison benchmarks.

As it turns out, defining the optimal welfare as the maximum welfare over all picking sequences $\pi' \in \Pi_{n,m}$ or $\pi' \in \mathcal{R}_{n,m}$ also

[4] For such fractions, we interpret $\frac{0}{0}$ to be equal to 1.

[5] In fact, Celine et al. [19] showed that for every instance, there always exists a recursively balanced picking sequence with the same agent ordering in every round, such that the resulting allocation has an egalitarian welfare of at least $1/(2n-1)$ times the optimum. However, the ordering of agents in each round of such a picking sequence is not fixed for all instances, but may differ depending on the instance.

allows us to circumvent the issue that the welfare provided by a given picking sequence π can be 0 even when the optimal welfare is positive. In Section 3.1, we consider the egalitarian price of $\pi \in \mathcal{R}_{n,m}$ with respect to all picking sequences $\pi' \in \Pi_{n,m}$. In Section 3.2, we restrict the search of the optimal egalitarian welfare to recursively balanced picking sequences, and consider the egalitarian price of $\pi \in \mathcal{R}_{n,m}$ with respect to all recursively balanced picking sequences $\pi' \in \mathcal{R}_{n,m}$. Interestingly, we show that for each metric, the egalitarian price of every recursively balanced picking sequence π is the same. This means that all recursively balanced picking sequences are "equally fair" with respect to these metrics based on egalitarian welfare.

Before proceeding further, we remark that while the price of fairness literature commonly normalizes the agents' utilities so that each agent has utility 1 for the entire set of goods [10, 18, 19], this is not necessary for our results. Indeed, our upper bounds hold even without normalization, whereas all of the instances used for our lower bound proofs are normalized.

3.1 Price Relative to All Picking Sequences

First, we derive a tight bound for the egalitarian price relative to other picking sequences starting with the same n picks, as stated in Theorem 3.4 below. In particular, we find that the egalitarian price depends only on n and m, and not on the specific picking sequence π. Therefore, every picking sequence is equally fair with respect to this version of the egalitarian price.

Theorem 3.4. *For any $n \geq 2$, $m \geq n$, and $\pi \in \mathcal{R}_{n,m}$,*

$$\sup_{\mathcal{I} \in \mathcal{I}_{n,m}} \max_{\pi' \in \Pi_{n,m}} \frac{\mathrm{EW}(\pi', \mathcal{I})}{\mathrm{EW}(\pi, \mathcal{I})} = \min\{m - n + 1, n\}.$$

Proof. We begin by establishing the upper bound. Let $\pi \in \mathcal{R}_{n,m}$, $\mathcal{I} \in \mathcal{I}_{n,m}$, and $\pi' \in \Pi_{n,m}$.

First, we show that when $n \leq m \leq 2n - 1$, an upper bound is $\min\{m - n + 1, n\} = m - n + 1$. Fix $k \in N$, and let g be the good picked by agent k in the first round of π. Note that g is also picked by agent k in the first round of π', since both picking sequences have the same prefix for the first round. Agent k receives a utility of at least $u_k(g)$ in the allocation obtained by π. In π', agent k receives at most $1 + (m - n)$ goods, and the utility of each good is at most $u_k(g)$, so agent k receives a utility of at most $(m - n + 1) \cdot u_k(g)$. Therefore, the ratio of agent k's utility in the allocation obtained by π' to the corresponding utility for π is at most $m - n + 1$. Since $k \in N$ was arbitrarily chosen, we have

$$\frac{\mathrm{EW}(\pi', \mathcal{I})}{\mathrm{EW}(\pi, \mathcal{I})} \leq m - n + 1,$$

proving the upper bound when $n \leq m \leq 2n - 1$.

Next, we show that when $m \geq 2n$, an upper bound is $\min\{m - n + 1, n\} = n$. Fix $k \in N$. Again, the goods picked by the agents in the first round of π are identical to those in the first round of π'. Without loss of generality, relabel the goods as follows: let g_i be the good picked by agent $i \in N$ in the first round of π and π', and let the goods in $M' = \{g_{n+1}, \ldots, g_m\}$ be arranged in descending order of agent k's picking preference, i.e., for all $n + 1 \leq j_1, j_2 \leq m$, we have $j_1 < j_2$ if and only if $g_{j_1} \succ_k g_{j_2}$.

Consider the allocation obtained by π. For every $r \in \{2, \ldots, \lfloor m/n \rfloor\}$, agent k is able to select some good from $\{g_{n+1}, \ldots, g_{rn}\}$ in round r. This is because π is recursively balanced, so when it is agent k's turn in round r, at most $rn - 1$ goods (including those in $M \setminus M'$) have been taken. This means that the most preferred good for agent k in round r is no worse than g_{rn}. Agent k's utility of her bundle from π is therefore at least $u_k(g_k) + \sum_{r=2}^{\lfloor m/n \rfloor} u_k(g_{rn})$.

Now, consider the allocation obtained by π'. Agent k receives a subset of $\{g_k\} \cup M'$, which has utility at most

$$\begin{aligned}
&u_k(g_k) + u_k(M') \\
&= u_k(g_k) + \sum_{j=n+1}^{2n-1} u_k(g_j) + \sum_{r=2}^{\lfloor m/n \rfloor - 1} \sum_{j=0}^{n-1} u_k(g_{rn+j}) \\
&\qquad + \sum_{j=\lfloor m/n \rfloor n}^{m} u_k(g_j) \\
&\leq u_k(g_k) + \sum_{j=n+1}^{2n-1} u_k(g_k) + \sum_{r=2}^{\lfloor m/n \rfloor - 1} \sum_{j=0}^{n-1} u_k(g_{rn}) \\
&\qquad + \sum_{j=\lfloor m/n \rfloor n}^{m} u_k(g_{\lfloor m/n \rfloor n}) \\
&\leq n \cdot u_k(g_k) + \sum_{r=2}^{\lfloor m/n \rfloor - 1} n \cdot u_k(g_{rn}) + n \cdot u_k(g_{\lfloor m/n \rfloor n}) \\
&= n \cdot \left(u_k(g_k) + \sum_{r=2}^{\lfloor m/n \rfloor} u_k(g_{rn}) \right),
\end{aligned}$$

which is at most n multiplied by agent k's utility of her bundle from π. Since $k \in N$ was arbitrarily chosen, we have

$$\frac{\mathrm{EW}(\pi', \mathcal{I})}{\mathrm{EW}(\pi, \mathcal{I})} \leq n,$$

proving the upper bound when $m \geq 2n$.

We next turn to the lower bound. First, we show that when $n \leq m \leq 2n - 1$, a lower bound is $\min\{m - n + 1, n\} = m - n + 1$. Let $\pi \in \mathcal{R}_{n,m}$, and let $k \in N$ be an agent who does not get to pick in the second round—such an agent must exist since $m \leq 2n - 1$ and so the second round, if it exists, is incomplete. Consider an instance $\mathcal{I} \in \mathcal{I}_{n,m}$ with the following utilities:

- $u_k(g_j) = 1/(m - n + 1)$ for $j \in \{k\} \cup \{n + 1, \ldots, m\}$.
- For each agent $i \in N \setminus \{k\}$, we have $u_i(g_i) = 1$.
- $u_i(g_j) = 0$ for all other pairs (i, j).

Agents break ties in favor of lower-index goods.

Consider the allocation obtained by π. In the first round, every agent $i \in N$ selects g_i. In the second round, agent k does not select any good. Agent k receives a utility of $1/(m - n + 1)$ from g_k, and every other agent i receives a utility of 1 from g_i, so the egalitarian welfare is $1/(m - n + 1)$.

Now, consider the allocation obtained by $\pi' = (1, 2, \ldots, n, k, k, \ldots, k) \in \Pi_{n,m}$. In the first round, every agent $i \in N$ selects g_i. In the second round, agent k selects all of $g_{n+1}, \ldots, g_m$. Agent k receives a utility of 1 from picking all $m - n + 1$ goods valuable to her, and every other agent i receives a utility of 1 from g_i, so the egalitarian welfare is 1. This shows that

$$\frac{\mathrm{EW}(\pi', \mathcal{I})}{\mathrm{EW}(\pi, \mathcal{I})} = \frac{1}{1/(m-n+1)} = m - n + 1,$$

proving the lower bound when $n \leq m \leq 2n - 1$.

Next, we show that when $m \geq 2n$, a lower bound is $\min\{m-n+1, n\} = n$. Let $\pi \in \mathcal{R}_{n,m}$, and let $k = a_{2n}$, i.e., the agent who gets the $(2n)$-th pick in π. Consider an instance $\mathcal{I} \in \mathcal{I}_{n,m}$ with the following utilities:

- $u_k(g_j) = 1/n$ for $j \in \{k\} \cup \{n+1, \ldots, 2n-1\}$.
- For each agent $i \in N \setminus \{k\}$, we have $u_i(g_i) = 1$.
- $u_i(g_j) = 0$ for all other pairs (i, j).

Agents break ties in favor of lower-index goods.

Consider the allocation obtained by π. In the first round, every agent $i \in N$ selects g_i. In every subsequent round, agent k does not get to select any good valuable to her. Agent k receives a utility of $1/n$ from g_k, and every other agent i receives a utility of 1 from g_i, so the egalitarian welfare is $1/n$.

Now, consider the allocation obtained by $\pi' = (1, 2, \ldots, n, k, k, \ldots, k) \in \Pi_{n,m}$. In the first round, every agent $i \in N$ selects g_i. In the subsequent rounds, agent k selects all of $g_{n+1}, g_{n+2}, \ldots, g_m$. Agent k receives a utility of 1 from picking all n goods valuable to her, and every other agent i receives a utility of 1 from g_i, so the egalitarian welfare is 1. This shows that

$$\frac{\mathrm{EW}(\pi', \mathcal{I})}{\mathrm{EW}(\pi, \mathcal{I})} = \frac{1}{1/n} = n,$$

proving the lower bound when $m \geq 2n$. □

3.2 Price Relative to Recursively Balanced Picking Sequences

Next, we consider the egalitarian price with respect to all *recursively balanced* picking sequences with the same first-round prefix. Again, we find that the egalitarian price is the same regardless of the picking sequence; however, the proof for this is more involved.

Theorem 3.5. *For any $n \geq 2$, $m \geq n$, and $\pi \in \mathcal{R}_{n,m}$,*

$$\sup_{\mathcal{I} \in \mathcal{I}_{n,m}} \max_{\pi' \in \mathcal{R}_{n,m}} \frac{\mathrm{EW}(\pi', \mathcal{I})}{\mathrm{EW}(\pi, \mathcal{I})} = \min\{\lceil m/n \rceil, \lfloor \log_2 n \rfloor + 1\}.$$

Proof. We prove the upper bound here and defer the proof of the lower bound to the full version of our paper [20]. Let $\pi, \pi' \in \mathcal{R}_{n,m}$ and $\mathcal{I} \in \mathcal{I}_{n,m}$.

First, we show that when $n \leq m \leq n(\lfloor \log_2 n \rfloor + 1)$, an upper bound is $\min\{\lceil m/n \rceil, \lfloor \log_2 n \rfloor + 1\} = \lceil m/n \rceil$. Fix $k \in N$, and let g be the good picked by agent k in the first round of π. Note that this good is also picked by agent k in the first round of π', since both picking sequences have the same prefix for the first round. Agent k receives a utility of at least $u_k(g)$ in the allocation obtained by π. In π', agent k receives at most $\lceil m/n \rceil$ goods since π' is recursively balanced, and the utility of each good is at most $u_k(g)$, so agent k receives a utility of at most $\lceil m/n \rceil \cdot u_k(g)$. Therefore, the ratio of agent k's utility in the allocation obtained by π' to that in the allocation obtained by π is at most $\lceil m/n \rceil$. Since $k \in N$ was arbitrarily chosen, we have

$$\frac{\mathrm{EW}(\pi', \mathcal{I})}{\mathrm{EW}(\pi, \mathcal{I})} \leq \lceil m/n \rceil,$$

proving the upper bound when $n \leq m \leq n(\lfloor \log_2 n \rfloor + 1)$.

Next, we show that when $m > n(\lfloor \log_2 n \rfloor + 1)$, an upper bound is $\min\{\lceil m/n \rceil, \lfloor \log_2 n \rfloor + 1\} = \lfloor \log_2 n \rfloor + 1$. For each $i \in N$ and positive integer r, let $g_r^{i,\pi}$ and $g_r^{i,\pi'}$ be the goods selected by agent i in round r of π and π' respectively (for notational simplicity, if agent i does not select a good in round r, then we define the good $g_r^{i,\pi}$ or $g_r^{i,\pi'}$ to be a dummy good of zero utility). Let $L = \lfloor \log_2 n \rfloor + 1$. We first show the relationships between the goods via the following lemma.

Lemma 3.6. *For each $i \in N$ and positive integer s, we have*

$$g_s^{i,\pi} \succeq_i g_{(s-1)L+1}^{i,\pi'}.$$

Proof Sketch. The proof proceeds by induction on s. For the inductive step, we assume for contradiction that $g_s^{k,\pi} \prec_k g_{s_1}^{k,\pi'}$ for $s_1 = (s-1)L + 1$ and some $k \in N$. We consider the set of goods $M' = \{g_t^{j,\pi'} \mid j \in N, s_0 \leq t \leq s_1\}$, where $s_0 = (s-2)L + 2$, and show that every good in M' is available at the start of round s of π. We then define a directed graph where the vertices correspond to the goods in M' and the edges represent the relations between the bundles obtained with π' and π. We show that the subgraph of this graph induced by $g_{s_1}^{k,\pi'}$ along with all vertices reachable from this vertex is a perfect binary tree. Furthermore, every vertex in this tree is selected in the same round s of π by some agent $j \neq k$. Finally, a counting argument on the number of vertices in this tree yields the desired contradiction and completes the induction. ◁

We continue the proof of (the upper bound of) Theorem 3.5. Fix $k \in N$. We have

$$\begin{aligned}
u_k\left(A_k^{\pi'}\right) &= \sum_{r=1}^{\lceil m/n \rceil} u_k\left(g_r^{k,\pi'}\right) \\
&= \sum_{r=1}^{L\lceil m/n \rceil} u_k\left(g_r^{k,\pi'}\right) \qquad \left(u_k\left(g_r^{k,\pi'}\right) = 0 \text{ for } r > \lceil m/n \rceil\right) \\
&= \sum_{s=1}^{\lceil m/n \rceil} \sum_{r=(s-1)L+1}^{sL} u_k\left(g_r^{k,\pi'}\right) \\
&\leq \sum_{s=1}^{\lceil m/n \rceil} \sum_{r=(s-1)L+1}^{sL} u_k\left(g_{(s-1)L+1}^{k,\pi'}\right) \\
&= \sum_{s=1}^{\lceil m/n \rceil} L \cdot u_k\left(g_{(s-1)L+1}^{k,\pi'}\right) \\
&\leq \sum_{s=1}^{\lceil m/n \rceil} L \cdot u_k\left(g_s^{k,\pi}\right) \qquad \text{(by Lemma 3.6)} \\
&= L \cdot u_k\left(A_k^{\pi}\right).
\end{aligned}$$

This shows that in the allocation obtained with π', agent k receives a utility of at most L times her utility in the allocation obtained with π. Since $k \in N$ was arbitrarily chosen, we have

$$\frac{\mathrm{EW}(\pi', \mathcal{I})}{\mathrm{EW}(\pi, \mathcal{I})} \leq L = \lfloor \log_2 n \rfloor + 1,$$

proving the upper bound when $m > n(\lfloor \log_2 n \rfloor + 1)$. □

4 MAXIMIN SHARE

In this section, we compare recursively balanced picking sequences using MMS approximation. Interestingly, for most sequences, the MMS guarantee depends only on the picking sequence of agent n (who picks last in the first round), regardless of the picking sequences of other agents. Intuitively speaking, the disadvantage incurred by agent n in the first round is so significant that letting her pick first in every subsequent round does not sufficiently compensate for it—she will still have the lowest MMS guarantee among all agents. However, there are some exceptions where agent $n-1$ has a lower MMS guarantee. Hence, before we state the MMS guarantees, we distinguish between the general case and the exception by defining "irregular" picking sequences.

Definition 4.1. Let $n \geq 2$ and $m \geq n$. A picking sequence $\pi \in \mathcal{R}_{n,m}$ is called *irregular* if it has a second round that satisfies the following conditions:

(1) The second round has a positive even number of turns.
(2) Agent $n-1$ does not get to pick in the second round.
(3) Agent n picks in the first $(m-n)/2$ turns of the second round.

Otherwise, we say that π is *regular*.

Observe that a picking sequence $\pi \in \mathcal{R}_{n,m}$ can be irregular only if $n+2 \leq m \leq 2n-1$. Indeed, if $m \leq n+1$, then the second round either does not exist or has an odd number of turns. If $m \geq 2n$, agent $n-1$ has to pick in the second round as π is recursively balanced.

We begin by characterizing the MMS guarantees for regular picking sequences. Note that these guarantees depend only on the picking sequence π_n of agent n.

Theorem 4.2. *Let $n \geq 2$, $m \geq n$, and $\pi \in \mathcal{R}_{n,m}$. Suppose that π is regular. Let agent n's picking sequence in π be $\pi_n = (t_1, \dots, t_R)$ where $t_1 = n$, and define $t_{R+1} = m+1$. Let*

$$\alpha = \min_{r \in \{2,\dots,R+1\}} \frac{r-1}{t_r - n}.$$

Then, the following hold for the allocation obtained with π:

(a) Every agent is always guaranteed to get at least α times her MMS.
(b) There exists an instance $\mathcal{I} \in \mathcal{I}_{n,m}$ where agent n has a positive MMS and gets no more than α times her MMS.

The proof of Theorem 4.2(a) can be found in the full version of our paper [20]. For Theorem 4.2(b), we prove a stronger result in the form of the following lemma. This lemma will also be used in the proof of Theorem 4.6.

Lemma 4.3. *Let $n \geq 2$, $m \geq n$, $\pi \in \mathcal{R}_{n,m}$, and $i \in N$. Let agent i's picking sequence be $\pi_i = (t_1, \dots, t_R)$ and define $t_{R+1} = m+1$. Then, there exists an instance such that under the picking sequence π, agent i has a positive MMS and gets no more than*

$$\min_{r \in \{2,\dots,R+1\}} \frac{r-1}{\lfloor (t_r - i)/(n+1-i) \rfloor}$$

times her MMS.

Proof Sketch. Choose $s \in \{2, \dots, R+1\}$ minimizing $(r-1)/\lfloor (t_r - i)/(n+1-i) \rfloor$. Consider the following instance:

- $u_i(g_j) = m$ for each $j \in \{1, \dots, i-1\}$.
- $u_i(g_j) = 1$ for each $j \in \{i, \dots, t_s - 1\}$. Note that this set is non-empty, since $s \geq 2$ and so $t_s > n \geq i$.
- For each agent $k \in N \setminus \{i\}$, we have $u_k(g_k) = 1$.
- $u_\ell(g_j) = 0$ for all other pairs (ℓ, j).

Assuming that agents break ties in favor of lower-index goods, one can check that in this instance, agent i's utility under the picking sequence π is exactly $(s-1)/\lfloor (t_s - i)/(n+1-i) \rfloor$ times her MMS. □

Then, Theorem 4.2(b) follows from Lemma 4.3 with $i = n$.

We next show that the MMS guarantee in Theorem 4.2(a) does not necessarily apply to irregular picking sequences.

Example 4.4. Let $n = 3$, $m = 5$, and $\pi = (1, 2, 3 \mid 3, 1)$. Note that π is irregular. Consider the following instance:

g	g_1	g_2	g_3	g_4	g_5
$u_1(g)$	6	0	0	0	0
$u_2(g)$	2	1	1	1	1
$u_3(g)$	6	0	0	0	0

Agents break ties in favor of lower-index goods.

Note that agent 2 receives exactly one good. Furthermore, this good cannot be g_1 since g_1 is picked by agent 1 in her first pick. Hence, agent 2 gets a utility of 1. On the other hand, agent 2 has an MMS of 2 by the partition $\{\{g_1\}, \{g_2, g_3\}, \{g_4, g_5\}\}$. Thus, agent 2 gets exactly $1/2$ of her MMS.

We compute α as defined in Theorem 4.2. The picking sequence of agent 3 in π is $\pi_3 = (3, 4)$. In particular, we have $R = 2$, $t_2 = 4$, and $t_3 = 6$, which gives $\alpha = \min_{r \in \{2,3\}} (r-1)/(t_r - n) = \min\{1, 2/3\} = 2/3$. Since agent 2 only gets $1/2 < \alpha$ times her MMS, this shows that the guarantee in Theorem 4.2(a) does not necessarily apply to irregular picking sequences.

In light of Example 4.4, we separately determine the MMS guarantees for irregular picking sequences.

Theorem 4.5. *Let $n \geq 2$, $m \geq n$, and $\pi \in \mathcal{R}_{n,m}$. Suppose that π is irregular. Then, the following hold for the allocation obtained with π:*

(a) Every agent is always guaranteed to get at least $2/(m-n+2)$ times her MMS.
(b) There exists an instance $\mathcal{I} \in \mathcal{I}_{n,m}$ where agent $n-1$ has a positive MMS and gets no more than $2/(m-n+2)$ times her MMS.

Theorems 4.2 and 4.5 allow us to determine the best recursively balanced picking sequences with respect to MMS.

Theorem 4.6 (Best Picking Sequences). *Let $n \geq 2$ and $m \geq n$. Denote*

$$\alpha_{\max} = \min \left\{ \frac{\lfloor m/n \rfloor}{\lfloor m/n \rfloor n - n + 1}, \frac{\lceil m/n \rceil}{m - n + 1} \right\}.$$

Then, the following statements hold:

(a) For every picking sequence $\pi \in \mathcal{R}_{n,m}$, there exists an instance $\mathcal{I} \in \mathcal{I}_{n,m}$ such that some agent with a positive MMS gets no more than $\alpha_{\max}$ times her MMS in the allocation obtained with π.
(b) There exists a picking sequence $\pi \in \mathcal{R}_{n,m}$ such that every agent is always guaranteed to get at least $\alpha_{\max}$ times her MMS in the allocation obtained with π.
(c) For any picking sequence $\pi \in \mathcal{R}_{n,m}$, every agent is always guaranteed to get at least $\alpha_{\max}$ times her MMS in the allocation obtained with π if and only if π satisfies the following:
  (i) π is regular, and
  (ii) for each $r \in \{2, \dots, \lceil m/n \rceil\}$, agent n gets to pick in round r and the index of her r-th pick is at most $n + (r-1)/\alpha_{\max}$.

Proof. We first prove statement (a). Consider an instance $\mathcal{I}_1 \in \mathcal{I}_{n,m}$ with the following utilities:

- For $i \in N \setminus \{n\}$, $u_i(g_i) = m$ and $u_i(g_j) = 0$ if $j \neq i$.
- $u_n(g_j) = m$ if $j < n$.
- $u_n(g_j) = 1$ if $n \leq j \leq \lfloor m/n \rfloor n$.
- $u_n(g_j) = 0$ if $j > \lfloor m/n \rfloor n$.

Agents break ties in favor of lower-index goods.

We observe that agent n's MMS is exactly $u_n(\{g_n, \dots, g_m\}) = \lfloor m/n \rfloor n - n + 1 \geq 1$, as obtained by the partition $\{\{g_1\}, \{g_2\}, \dots, \{g_{n-1}\}, \{g_n, g_{n+1}, \dots, g_m\}\}$. In each round $r \in \{1, \dots, \lfloor m/n \rfloor\}$, agent n gets a good with utility 1. If there is an incomplete last round $r = \lfloor m/n \rfloor + 1$, either agent n does not get to pick at all, or she picks a good with utility 0. In both cases, agent n's total utility is $\lfloor m/n \rfloor$. Hence, agent n gets exactly $\lfloor m/n \rfloor / (\lfloor m/n \rfloor n - n + 1)$ times her MMS.

Now, consider a different instance $\mathcal{I}_2 \in \mathcal{I}_{n,m}$ with the following utilities:

- For $i \in N \setminus \{n\}$, $u_i(g_i) = m$ and $u_i(g_j) = 0$ if $j \neq i$.
- $u_n(g_j) = m$ if $j < n$.
- $u_n(g_j) = 1$ if $n \leq j \leq m$.

Again, agents break ties in favor of lower-index goods.

In this instance, agent n's MMS is exactly $u_n(\{g_n, \dots, g_m\}) = m - n + 1 \geq 1$, as obtained by the partition $\{\{g_1\}, \{g_2\}, \dots, \{g_{n-1}\}, \{g_n, g_{n+1}, \dots, g_m\}\}$. Moreover, in each round, agent n gets a good with utility at most 1, giving a total utility of at most $\lceil m/n \rceil$. It follows that agent n gets at most $\lceil m/n \rceil / (m - n + 1)$ times her MMS.

Therefore, for one of the instances $\mathcal{I}_1$ and $\mathcal{I}_2$, agent n gets no more than

$$\alpha_{\max} = \min \left\{ \frac{\lfloor m/n \rfloor}{\lfloor m/n \rfloor n - n + 1}, \frac{\lceil m/n \rceil}{m - n + 1} \right\}$$

times her MMS.

Next, we prove statement (b) using statement (c), which will be proven later. Consider the picking sequence π where the sequence in the second round onwards is the reverse of the first round—that is, $\pi = (1, 2, \dots, n \mid n, n-1, \dots, 1 \mid \cdots \mid n, n-1, \dots, 1 \mid n, n-1, \dots, a_m)$ where $a_m = \lceil m/n \rceil n - m + 1$. Observe that π is regular. Indeed, otherwise the second round exists and has an even number of turns, so agent $n-1$ gets to pick in the second round, which means that π cannot be irregular. We now prove that π satisfies (ii). Take any $r \in \{2, \dots, \lceil m/n \rceil\}$, and let t_r be the index of agent n's r-th pick. Note that $t_r = (r-1)n + 1$ and $r \leq \lceil m/n \rceil \leq \lfloor m/n \rfloor + 1$. Then,

$$\begin{aligned}
\frac{r-1}{t_r - n} &= \frac{r-1}{(r-1)n + 1 - n} \\
&= \frac{1}{n - \frac{n-1}{r-1}} \\
&\geq \frac{1}{n - \frac{n-1}{\lfloor m/n \rfloor}} && \text{(since } r - 1 \leq \lfloor m/n \rfloor\text{)} \\
&= \frac{\lfloor m/n \rfloor}{\lfloor m/n \rfloor n - n + 1} \geq \alpha_{\max}.
\end{aligned}$$

This means that $t_r \leq n + (r-1)/\alpha_{\max}$. Hence, π also satisfies condition (ii). By statement (c), under picking sequence π, every agent is guaranteed to get at least $\alpha_{\max}$ times her MMS.

We now prove the backward direction of statement (c). Take any picking sequence $\pi \in \mathcal{R}_{n,m}$ that satisfies both conditions (i) and (ii). In particular, π is regular. We shall use Theorem 4.2(a) to prove the statement. Let $\pi_n = (t_1, \dots, t_R)$ be agent n's picking sequence under π and $t_{R+1} = m + 1$. By condition (ii), we have $R = \lceil m/n \rceil$ and $t_r \leq n + (r-1)/\alpha_{\max}$ for each $r \in \{2, \dots, R\}$. It follows that

$$\begin{aligned}
&\min_{r \in \{2,\dots,R+1\}} \frac{r-1}{t_r - n} \\
&= \min \left\{ \min_{r \in \{2,\dots,R\}} \frac{r-1}{t_r - n}, \frac{R}{t_{R+1} - n} \right\} \\
&\geq \min \left\{ \min_{r \in \{2,\dots,R\}} \frac{r-1}{(r-1)/\alpha_{\max}}, \frac{R}{t_{R+1} - n} \right\} \\
&\qquad \text{(by the assumption on } t_r\text{)} \\
&= \min \left\{ \alpha_{\max}, \frac{\lceil m/n \rceil}{m - n + 1} \right\} \quad \text{(since } R = \lceil m/n \rceil \text{ and } t_{R+1} = m + 1\text{)} \\
&= \alpha_{\max}. \qquad \text{(by definition of } \alpha_{\max}\text{)}
\end{aligned}$$

Hence, by Theorem 4.2(a), every agent is guaranteed to get at least $\alpha_{\max}$ times her MMS.

It remains to prove the forward direction of statement (c). Let $\pi \in \mathcal{R}_{n,m}$ be such that every agent is guaranteed to get at least $\alpha_{\max}$ times her MMS. Let $\pi_n = (t_1, \dots, t_R)$ be agent n's picking sequence under π and $t_{R+1} = m + 1$.

Assume for the sake of contradiction that π is irregular. Then, the second round exists and is incomplete. This means that $\lfloor m/n \rfloor = 1$ and $\lceil m/n \rceil = 2$. Hence, $\alpha_{\max} = \min\{1, 2/(m-n+1)\} = 2/(m-n+1)$. On the other hand, by Theorem 4.5(b), there exists an instance where some agent with a positive MMS gets no more than $2/(m - n + 2)$ times her MMS. Since $2/(m-n+2) < 2/(m-n+1) = \alpha_{\max}$, this contradicts the assumption that every agent gets at least $\alpha_{\max}$ times her MMS. It follows that π must be regular.

We can now prove, by contradiction, that agent n must pick in the last round $r = \lceil m/n \rceil$. Suppose otherwise. Since π is recursively balanced, the last round where agent n gets to pick is $R = \lfloor m/n \rfloor \leq \lceil m/n \rceil - 1$; that is, m is not divisible by n. Then, we have

$$\begin{aligned}
&\min_{r \in \{2,\dots,R+1\}} \frac{r-1}{t_r - n} \\
&\leq \frac{R}{t_{R+1} - n} \\
&= \frac{\lfloor m/n \rfloor}{m - n + 1} \qquad \text{(since } R = \lfloor m/n \rfloor \text{ and } t_{R+1} = m + 1\text{)} \\
&< \min \left\{ \frac{\lfloor m/n \rfloor}{\lfloor m/n \rfloor n - n + 1}, \frac{\lceil m/n \rceil}{m - n + 1} \right\} \\
&\qquad \text{(since } m > \lfloor m/n \rfloor n \text{ and } \lfloor m/n \rfloor < \lceil m/n \rceil\text{)} \\
&= \alpha_{\max}.
\end{aligned}$$

By Lemma 4.3, this implies that agent n is not always guaranteed to get at least $\alpha_{\max}$ times her MMS, a contradiction. Hence, agent n must pick in every round.

Lastly, we prove, again by contradiction, that $t_r \leq n + (r-1)/\alpha_{\max}$ for all $r \in \{2, \dots, \lceil m/n \rceil\}$. Suppose that there is some $r \in \{2, \dots, \lceil m/n \rceil\}$ such that $t_r > n + (r-1)/\alpha_{\max}$. This means that $\alpha_{\max} > (r-1)/(t_r - n)$. Like in the previous paragraph, Lemma 4.3 implies that agent n is not guaranteed to get $\alpha_{\max}$ times her MMS, contradicting our earlier assumption. Hence, it must hold that $t_r \leq n + (r-1)/\alpha_{\max}$ for all $r \in \{2, \dots, \lceil m/n \rceil\}$. Together with the fact that agent n picks in round $\lceil m/n \rceil$, this implies that condition (ii) is satisfied. □

At the other extreme, we characterize the recursively balanced picking sequences with the worst MMS guarantee.

Theorem 4.7 (Worst Picking Sequences). *Let $n \geq 2$ and $m \geq n$. Denote*

$$\alpha_{\min} = \max\left\{\frac{1}{n}, \frac{1}{m-n+1}\right\}.$$

Then, the following statements hold:

(a) For every picking sequence $\pi \in \mathcal{R}_{n,m}$, every agent is always guaranteed to get at least $\alpha_{\min}$ times her MMS in the allocation obtained with π.

(b) There exists a picking sequence $\pi \in \mathcal{R}_{n,m}$ such that in some instance $\mathcal{I} \in \mathcal{I}_{n,m}$, some agent with a positive MMS gets no more than $\alpha_{\min}$ times her MMS in the allocation obtained with π.

(c) For any picking sequence $\pi \in \mathcal{R}_{n,m}$, some agent with a positive MMS gets no more than $\alpha_{\min}$ times her MMS in the allocation obtained with π in some instance $\mathcal{I} \in \mathcal{I}_{n,m}$ if and only if π satisfies the following:

(i) $m \leq 2n-1$ and agent n only picks once, or

(ii) $m \geq 2n$ and there is some round $r \in \{2, \ldots, \lceil m/n \rceil\}$ consisting of at least $n-1$ turns where agent n does not pick in the first $n-1$ turns.

When $m \geq 2n-1$, we have $\alpha_{\min} = 1/n$. In this case, Theorem 4.7(a) follows from Proposition 3.6 of Amanatidis et al. [4], which shows that any EF1 allocation gives each agent at least $1/n$ times her MMS.

Note that round-robin always satisfies condition (i) or (ii) of Theorem 4.7(c), and therefore has the worst MMS guarantee. More interestingly, balanced alternation also satisfies condition (ii) when $m \geq 3n-1$, since agent n does not pick in the first $n-1$ turns of the third round.

Theorems 4.6 and 4.7 yield a simple classification of the MMS guarantees of (recursively balanced) picking sequences for two agents. Specifically, for each $m \geq 3$, there is a unique picking sequence with the best MMS guarantee, which lets agent 2 pick first in every round except the first round; all other picking sequences have the worst MMS guarantee. This is elaborated in Corollary 4.8.

Corollary 4.8. *Let $n = 2$, $m \geq 3$, and $\pi \in \mathcal{R}_{n,m}$. Then, the following statements hold:*

(a) If agent 2 picks first in every round from the second round onwards in π, then each agent is always guaranteed to get at least $1/(2 - 1/\lfloor m/2 \rfloor)$ times her MMS. Moreover, this bound is tight.

(b) Otherwise, each agent is always guaranteed to get at least $1/2$ times her MMS. Moreover, this bound is tight.

When there are more than two agents, however, the picture is not as simple. In particular, there is more than one picking sequence with the best MMS guarantee, and there are picking sequences with neither the best nor the worst MMS guarantee. We illustrate these points with the following example.

Example 4.9. Let $n = 3$ and $m = 7$.

First, we find all picking sequences in $\mathcal{R}_{3,7}$ with the best MMS guarantee. By Theorem 4.6, $\alpha_{\max} = \min\{2/4, 3/5\} = 1/2$. In order to guarantee that each agent always gets at least $\alpha_{\max} = 1/2$ times her MMS, a picking sequence must satisfy conditions (i) and (ii) of Theorem 4.6(c). Condition (i) is always satisfied since the second round is complete. Condition (ii) is satisfied if and only if agent 3 gets to pick three times and the indices of agent 3's second and third picks are at most $3 + 1/\alpha_{\max} = 5$ and $3 + 2/\alpha_{\max} = 7$ respectively. We can therefore list all satisfying picking sequences as follows:

$$(1,2,3 \mid \mathbf{3},1,2 \mid \mathbf{3}); \quad (1,2,3 \mid 1,\mathbf{3},2 \mid \mathbf{3});$$
$$(1,2,3 \mid \mathbf{3},2,1 \mid \mathbf{3}); \quad (1,2,3 \mid 2,\mathbf{3},1 \mid \mathbf{3}).$$

Next, we find all picking sequences in $\mathcal{R}_{3,7}$ with the worst MMS guarantee. By Theorem 4.7, $\alpha_{\min} = \max\{1/3, 1/5\} = 1/3$. Since $m \geq 2n$, these picking sequences must satisfy condition (ii) of Theorem 4.7(c), that is, agent 3 must pick last in the second round. All picking sequences satisfying this condition are as follows:

$$(1,2,3 \mid 1,2,\mathbf{3} \mid 1); \quad (1,2,3 \mid 2,1,\mathbf{3} \mid 1); \quad (1,2,3 \mid 1,2,\mathbf{3} \mid 2);$$
$$(1,2,3 \mid 2,1,\mathbf{3} \mid 2); \quad (1,2,3 \mid 1,2,\mathbf{3} \mid 3); \quad (1,2,3 \mid 2,1,\mathbf{3} \mid 3).$$

In total, there are four picking sequences with the best MMS guarantee of $\alpha_{\max} = 1/2$, and six sequences with the worst MMS guarantee of $\alpha_{\min} = 1/3$. Since the number of picking sequences in $\mathcal{R}_{3,7}$ is $3! \cdot 3 = 18$, there are eight picking sequences with neither the best nor the worst MMS guarantee, e.g., $(1,2,3 \mid 1,3,2 \mid 1)$.

5 CONCLUSION

In this paper, we have compared the fairness of recursively balanced picking sequences, all of which are known to guarantee envy-freeness up to one good (EF1). We used two important measures, egalitarian welfare and approximate maximin share (MMS), and showed that they yield highly different results. On the one hand, all recursively balanced picking sequences fare equally well when evaluated using worst-case egalitarian welfare relative to other picking sequences. On the other hand, various recursively balanced picking sequences offer differing MMS guarantees, with the round-robin and balanced alternation sequences being among the worst. Interestingly, the sequences with the best MMS guarantee include those in which the agent who picks last in the first round always picks first in every subsequent round. In light of their theoretical guarantees, we believe that these sequences merit consideration for practical adoption.

We conclude by proposing two directions for future research. Firstly, it would be interesting to see whether our worst-case results continue to hold in the average case. In this vein, Bouveret and Lang [12] computed the optimal picking sequences with respect to egalitarian (and utilitarian) welfare; however, their results are empirical and limited to small numbers of agents and goods.[6] Note that average-case results rely on assumptions on the distributions of agents' utilities, which may vary across different applications. Secondly, one could extend our analyses to picking sequences that are not recursively balanced. While such picking sequences do not provide the EF1 guarantee, they may nevertheless perform well according to other measures.

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[6] Moreover, our measure is somewhat different, as we consider the egalitarian *price* (see Section 3).

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