Maximin Shares with Lower Quotas*

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ABSTRACT

We study the fair division of indivisible items among n agents with heterogeneous additive valuations, subject to *lower* and *upper quotas* on the number of items allocated to each agent. Such constraints are crucial in various applications, ranging from personnel assignments to computing resource distribution. This paper focuses on the fairness criterion known as *maximin shares (MMS)* and its approximations. Under arbitrary lower and upper quotas, we show that a $\left(\frac{2n}{3n-1}\right)$-MMS allocation of goods exists and can be computed in polynomial time, while we also present a polynomial-time algorithm for finding a $\left(\frac{3n-1}{2n}\right)$-MMS allocation of chores. Furthermore, we consider the generalized scenario where items are partitioned into multiple *categories*, each with its own lower and upper quotas. In this setting, our algorithm computes an $\left(\frac{n}{2n-1}\right)$-MMS allocation of goods or a $\left(\frac{2n-1}{n}\right)$-MMS allocation of chores in polynomial time. These results extend previous work on the *cardinality constraints*, i.e., the special case where only upper quotas are imposed.

KEYWORDS

Fair division, Maximin shares, Approximation, Constraints, Quotas

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1 INTRODUCTION

Fairly allocating indivisible items among agents with differing preferences is a fundamental problem in society, arising in various important scenarios ranging from commodity distribution to property settlement to computing resource management [28, 70, 71]. Long-standing research on the allocation of divisible resources, known as "cake-cutting", has introduced compelling fairness notions such as *envy-freeness (EF)* [40] and *proportionality (PROP)* [77]. However, both EF and PROP are scarcely attainable with indivisible items.[1] While various relaxations have been explored [8], the concept of *maximin shares (MMS)* [29] has garnered significant attention within the class of share-based notions.

Generalizing the idea of the *cut-and-choose* protocol [77], the maximin share (MMS) of an agent is defined as the maximum value that he can guarantee for himself if he were to freely partition all items into the same number of bundles as there are agents, and then choose his least desirable bundle. An allocation is considered fair if every agent receives at least their own MMS. However, such an *MMS allocation* may not exist [62, 38], and finding an MMS allocation is strongly NP-hard [27, 16]. Consequently, substantial research has sought to achieve (multiplicatively) *approximate* MMS allocations [62], leading to remarkable progress in recent years [20, 9, 45, 2, 1, 16, 53, 54, 51, 55]. Continuing in similar vein, this paper adopts approximate MMS guarantees as our fairness criterion.

In addition to fairness desiderata, real-world applications often involve various restrictions on resource allocation [78]. One of the simplest constraints, which has gained little attention in the literature, is to impose *lower quotas* on the number of items assigned to each agent. For instance, consider the following scenario in a company: the directors of upcoming projects are planning to allocate employees to their respective projects. Each director prefers employees who are well-suited and capable of performing well for their project, while each project requires a minimum number of employees to be carried out. Then, how can these directors fairly distribute the employees among their teams? Beyond this specific example, lower quotas on the sizes of allocated bundles can arise in various applications of fair division, such as allocating courses to college students [30], distributing computing resources to tasks [71], assigning conference papers to reviewers [46], and dividing food donations to individuals or communities in need [6, 69].

1.1 Our Contribution

Building on the preceding motivations, we initiate the study of MMS guarantees under lower quotas, where agents are assumed to have heterogeneous additive valuations. Specifically, an allocation is feasible if the number of items assigned to each agent is between lower and upper quotas. Under these constraints, we establish that a $\left(\frac{2n}{3n-1}\right)$-MMS allocation exists and can be computed in polynomial time for any instance of goods[2] with n agents. For the case of chores, we present a polynomial-time algorithm to obtain a $\left(\frac{3n-1}{2n}\right)$-MMS allocation. These results both generalize and improve (notably for small n) on the best-known results for the special case without lower quotas [57], while also implying the first MMS guarantees for another subcase studied as *balanced* allocations [78].[3]

Furthermore, we study a generalized setting where items are partitioned into categories, each with its own lower and upper quotas on the number of items assigned from the category to any single agent. In this setup, we show that an $\left(\frac{n}{2n-1}\right)$-MMS

*A full version of this paper is available on arXiv [60].
[1]Neither is feasible even in the simplest case where two agents must divide a single indivisible item.

[2]An item is called a *good* (resp. *chore*) if it bears a non-negative (resp. non-positive) value for each agent.
[3]An allocation is said to be *balanced* if the number of items assigned to each agent differs by at most one.

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Table 1: Our polynomial-time approximate MMS guarantees and the best-known inapproximability bounds (implied by the unconstrained problem) for n heterogeneous agents with additive valuations.

(a) Goods (non-negative item values).

Category	Lower bound	Upper bound
Single	$\frac{2n}{3n-1}$ [Thm. 1]	$\begin{cases} \frac{39}{40} & (n=3) \\ 1-n^{-4} & (n \geq 4) \end{cases}$ [38]
Multiple	$\frac{n}{2n-1}$ [Thm. 3]	

(b) Chores (non-positive item values).

Category	Upper bound	Lower bound
Single	$\frac{2n}{3n-1}$ [Thm. 1]	$\frac{44}{43}$ $(n=3)$ [38]
Multiple	$\frac{2n-1}{n}$ [Thm. 4]	

(resp. $\left(\frac{2n-1}{n}\right)$-MMS) allocation exists and can be found in polynomial time for any instance of goods (resp. chores) with n agents. These extend the previous results for the special case in which all categories have only upper quotas [25, 57].

In both the former (single-category) and latter (multi-category) settings, our approach builds on Hummel and Hetland [57]'s work for the special case where only upper quotas are imposed. While the main routine of our algorithms for the multi-category setting is a natural extension of its counterpart in [57], the technical novelty of this paper lies primarily in the single-category setting, to which Hummel and Hetland [57]'s solutions do not extend in a straightforward manner; see Section 4.1 for a further discussion.

1.2 Related Work

Related Constraints in Fair Division. The earliest related studies include Ferraioli et al. [39]'s work on the constraint requiring every agent to receive the same number of items. Mackin and Xia [67] consider the setting where items are partitioned into categories, and each agent must receive at least one item per category. A subsequent series of work has studied the so-called *cardinality constraints*, where items are categorized, and allocations are restricted by category-wise upper quotas. Biswas and Barman [25] reveal that, under cardinality constraints, both an EF1 allocation and a 1/3-MMS allocation exist and can be found in polynomial time. Hummel and Hetland [57] improve the MMS approximation to 1/2 and further establish 2/3-MMS allocations for single-category instances, both in polynomial time. They also study the case of chores, achieving a 2-approximation for general instances and a 3/2-approximation for single-category instances in polynomial time. In the case of two agents, Shoshan et al. [76] develop a polynomial-time algorithm that computes a feasible allocation satisfying *Pareto optimality (PO)* and EF1 for either goods or chores, which has been generalized by Igarashi and Meunier [58]. Cookson et al. [33] consider the same model as ours, i.e., an extension of cardinality constraints endowed with lower quotas, where they prove that the *maximum Nash welfare (MNW)* solutions achieve PO and approximate EF1. Aside from the lower quotas we focus on, other generalizations of cardinality constraints include the *budget constraints* [81, 18, 19, 42, 34, 35, 37, 79], where items of varying sizes are assigned to agents subject to their capacities, and the *matroid constraints* [25, 26, 36, 80, 33, 4], which require assigned bundles to be independent sets. These constraints are further subsumed under *independence systems*, in which MMS approximation has been studied [66, 56].

Unconstrained MMS Approximation and Inapproximability. Given the non-existence and computational intractability of exact MMS allocations even in the unconstrained setting [74, 16], extensive work has studied approximate MMS guarantees in the unconstrained setting. The best-known approximation ratio is $\frac{7}{9}$ for the case of goods [55] and $\frac{13}{11}$ for chores [54]. On the negative side, Feige et al. [38] establish inapproximability bounds of $\frac{39}{40}$ for goods and $\frac{44}{43}$ for chores, which remain the best-known results even in our constrained setup.

Lower Quotas in Two-Sided Matching. In the many-to-one matching framework, commonly referred to as the *hospitals/residents problem* or *college admissions problem* [41, 49], one side often imposes lower quotas on the number of acceptable matches from the other side [11, 14]. The primary objective in this model is typically to ensure *stability* or its suitable relaxations [24, 52, 50, 59, 72, 82].

2 PRELIMINARIES

Our problem instance is a tuple $\mathcal{I} = \left(N, M, (v_i)_{i\in N}, \mathcal{C}, (q^-_C, q^+_C)_{C\in\mathcal{C}}\right)$, where $N = \{1, 2, \ldots, |N|\}$ is a finite set of *agents*, M is a finite set of *items*, $(v_i : 2^M \to \mathbb{R})_{i\in N}$ are the agents' respective *valuations*, and $\mathcal{C}$ is a partition of the item set M into disjoint *categories*, each endowed with a *lower quota* $q^-_C \in \mathbb{Z}_{\geq 0}$ and an *upper quota* $q^+_C \in \mathbb{Z}_{\geq 0}$, such that

$$q^-_C|N| \leq |C| \leq q^+_C|N|. \tag{1}$$

Each valuation $v_i : 2^M \to \mathbb{R}$ is additive, i.e., $v_i(S) = \sum_{g\in S} v_i(g)$ for any $S \subseteq M$, where $g \in M$ is understood as $\{g\} \subseteq M$ and treated similarly throughout the paper. A subset of items, $S \subseteq M$, is called a *bundle* and said to be *feasible* if $q^-_C \leq |S \cap C| \leq q^+_C$ for any category $C \in \mathcal{C}$. An *allocation* for $\mathcal{I}$ is an ordered partition $A = (A_i)_{i\in N}$ of M, and said to be *feasible* if each bundle $A_i \subseteq M$ is feasible. $\mathcal{F}(\mathcal{I})$ denotes the set of all feasible allocations for $\mathcal{I}$, which is non-empty due to Eq. (1).

An instance $\mathcal{I}$ is said to be *of goods* (resp. *of chores*) if $v_i(g) \geq 0$ (resp. ≤ 0) for every pair $(i, g) \in N \times M$, where each item is called a *good* (resp. *chore*). An instance $\mathcal{I}$ is said to be *ordered* [57] when each category C has an ordering of its items $C = \left\{g^C_1, g^C_2, \ldots, g^C_{|C|}\right\}$ such that

$$v_i\left(g^C_1\right) \geq v_i\left(g^C_2\right) \geq \cdots \geq v_i\left(g^C_{|C|}\right) \qquad \forall i \in N. \tag{2}$$

Here, item $g^C_j \in C$ is said to be *more valuable* than item $g^C_{j'} \in C$ if $j < j'$. An instance $\mathcal{I}$ is said to be *single-category* if $\mathcal{C} = \{M\}$, where we let $(q^-, q^+) \coloneqq \left(q^-_M, q^+_M\right)$ and equate $\mathcal{I}$ with the tuple $(N, M, (v_i)_{i\in N}, (q^-, q^+))$; when $\mathcal{I}$ is also ordered, we let $g_j \coloneqq g^M_j$ for each $j \in \{1, 2, \ldots, |M|\}$.

Algorithm 1 Obtain a solution of the original instance from that of its corresponding ordered instance [57, Algorithm 2].

Input: An instance $\mathcal{I} = \left(N, M, (v_i)_{i \in N}, \mathcal{C}, (q_C^-, q_C^+)_{C \in \mathcal{C}}\right)$. An allocation $(\tilde{A}_i)_{i \in N}$ for the ordered instance $\tilde{\mathcal{I}} = \left(N, M, (\tilde{v}_i)_{i \in N}, \mathcal{C}, (q_C^-, q_C^+)_{C \in \mathcal{C}}\right)$ defined from $\mathcal{I}$ in the proof of Lemma 1
Output: An allocation $(A_i)_{i \in N}$ for the instance $\mathcal{I}$.

```
1: A_i ← ∅    for each i ∈ N.
2: for each C ∈ 𝒞 do
3:     for j = 1, 2, ..., |C| do
4:         Find i* ∈ N s.t. g_j^C ∈ Ã_{i*}.
5:         Find g* ∈ arg max {v_{i*}(g) | g ∈ C \ ⋃_{i∈N} A_i}.
6:         A_{i*} ← A_{i*} ∪ {g*}.
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Note that cardinality constraints are the special cases when $q_C^- = 0$ for every $C \in \mathcal{C}$, and that the unconstrained setting is represented by single-category instances with $(q^-, q^+) = (0, |M|)$.

Let us define the fairness criteria based on *maximin shares (MMS)*. We aim to design an algorithm that, given an arbitrary instance, computes an α-MMS allocation for some α as close to 1 as possible.

Definition 1. For an instance $\mathcal{I} = \left(N, M, (v_i)_{i \in N}, \mathcal{C}, (q_C^-, q_C^+)_{C \in \mathcal{C}}\right)$ and an agent $i \in N$, we define agent i's *maximin share (MMS)* as

$$\mu_i(\mathcal{I}) \coloneqq \max\left\{\min_{k \in N} v_i(P_k) \mid (P_k)_{k \in N} \in \mathcal{F}(\mathcal{I})\right\},$$

where any $(P_k)_{k \in N}$ achieving the maximum is called agent i's *MMS partition*. A feasible allocation $(A_i)_{i \in N}$ is said to be an *α-MMS allocation* for $\mathcal{I}$ and some $\alpha \in \mathbb{R}$ if $v_i(A_i) \geq \alpha\, \mu_i(\mathcal{I})$ for all $i \in N$. Particularly when these inequalities hold for $\alpha = 1$, $(A_i)_{i \in N}$ is said to be an *MMS allocation*.

It is shown that an arbitrary instance can be reduced to an ordered instance, which is well-known in the unconstrained setting [27] and also under cardinality constraints [57]. Because the known reduction preserves an instance except the valuation profile and does not alter the cardinality of solutions, we can focus solely on ordered instances as input even in the presence of lower quotas.

Lemma 1 (Corollary of [57, Theorem 3]). *For an arbitrary instance $\mathcal{I} = \left(N, M, (v_i)_{i \in N}, \mathcal{C}, (q_C^-, q_C^+)_{C \in \mathcal{C}}\right)$ and any $\alpha \in \mathbb{R}$, one can compute an ordered instance $\tilde{\mathcal{I}}$ in time $O(|N||M| \log |M|)$, such that an α-MMS allocation for $\mathcal{I}$ can be computed from any α-MMS allocation for $\tilde{\mathcal{I}}$ in time $O(|N||M| \log |M|)$.*

Proof. Let $\mathcal{I} = \left(N, M, (v_i)_{i \in N}, \mathcal{C}, (q_C^-, q_C^+)_{C \in \mathcal{C}}\right)$ be an arbitrary instance. Fix any ordering in each category $C = \left\{g_1^C, g_2^C, \ldots, g_{|C|}^C\right\}$. Following [57, Algorithm 1], we define an ordered instance $\tilde{\mathcal{I}} \coloneqq \left(N, M, (\tilde{v}_i)_{i \in N}, \mathcal{C}, (q_C^-, q_C^+)_{C \in \mathcal{C}}\right)$ by

$$\tilde{v}_i\left(g_j^C\right) \coloneqq (\text{the } j\text{-th largest value in } \{v_i(g) \mid g \in C\}) \quad \forall j \in \{1, 2, \ldots, |C|\}, \forall C \in \mathcal{C}, \forall i \in N,$$

where ties are broken arbitrarily. Clearly, $\mu_i(\tilde{\mathcal{I}}) = \mu_i(\mathcal{I})$ holds for every $i \in N$. Furthermore, given an allocation $\tilde{A} = (\tilde{A}_i)_{i \in N}$ for $\tilde{\mathcal{I}}$, Line 1 computes an allocation $A = (A_i)_{i \in N}$ for $\mathcal{I}$ such that

$$|A_i \cap C| = |\tilde{A}_i \cap C| \qquad \forall C \in \mathcal{C}, \forall i \in N,$$
$$v_i(A_i) \geq \tilde{v}_i(\tilde{A}_i) \qquad \forall i \in N.$$

Therefore, if $\tilde{A}$ is an α-MMS allocation for $\tilde{\mathcal{I}}$, then A is also an α-MMS allocation for $\mathcal{I}$. The ordered instance $\tilde{I}$ can be obtained in time $O(|N||M| \log |M|)$ by sorting the item values $(v_i(g))_{g \in M}$ for each agent $i \in N$. In addition, Algorithm 1 can be implemented to run in time $O(|N||M| \log |M|)$ by, for each $i \in N$, similarly computing the sorted array of item values in advance and using a one-directional pointer over it to avoid scanning the same item value more than once at Line 5. □

We also extend the concept of *valid reduction*, which has been leveraged largely in the unconstrained problem [27, 62, 9, 47, 43, 45, 2, 1, 51, 55] but also under constraints [48, 66, 57, 35]. Lemma 2 allows us to assign the feasible bundle B to any agent without decreasing other agents' MMS in the new instance $\mathcal{I}'$. This bundle B is designed not only to ensure that $\mathcal{I}'$ is well-defined, but also so that any agent's MMS partition contains an *item-wise more valuable* bundle than B by the pigeonhole principle, as formalized in the proof of Lemma 2. Corollary 1 then suggests that, as long as some agent i^* enjoys $v_{i^*}(B) \geq \alpha\, \mu_{i^*}(\mathcal{I})$, finding an α-MMS allocation for the original instance $\mathcal{I}$ can be reduced to finding one for $\mathcal{I}'$.

Lemma 2. *Let $\mathcal{I} = \left(N, M, (v_i)_{i \in N}, \mathcal{C}, (q_C^-, q_C^+)_{C \in \mathcal{C}}\right)$ be an ordered instance of goods, $i^* \in N$ be arbitrary, $C^* \in \mathcal{C}$ be non-empty, and $d \in \mathbb{Z}_{\geq 0}$ satisfy $|C^*| \geq d|N| + 1$. Let us also define the following:*

$$B \coloneqq \left\{g_j^{C^*} \mid d(|N| - 1) < j \leq d|N| + 1\right\} \cup \left\{g_{|C^*|-j}^{C^*} \mid 0 \leq j < \max\left\{q_{C^*}^-, |C^*| - q_{C^*}^+(|N| - 1)\right\} - d - 1\right\} \cup \bigcup_{C \in \mathcal{C} \setminus \{C^*\}} \left\{g_{|C|-j}^C \mid 0 \leq j < \max\left\{q_C^-, |C| - q_C^+(|N| - 1)\right\}\right\},$$

$$\mathcal{I}' \coloneqq \left(N \setminus \{i^*\}, M \setminus B, \left(v_i|_{2^{M \setminus B}}\right)_{i \in N \setminus \{i^*\}}, \{C \setminus B \mid C \in \mathcal{C}\}, \left(q_C^-, q_C^+\right)_{C \in \mathcal{C}}\right).$$

Then $\mathcal{I}'$ is well-defined as an ordered instance of goods and satisfies that

$$\mu_i(\mathcal{I}') \geq \mu_i(\mathcal{I}) \qquad \forall i \in N \setminus \{i^*\}.$$

Proof. By the definition of B, each category $C \in \mathcal{C}$ satisfies that

$$|B \cap C| = \begin{cases} \max\left\{d + 1, q_C^-, |C| - q_C^+(|N| - 1)\right\} & \text{if } C = C^*; \\ \max\left\{q_C^-, |C| - q_C^+(|N| - 1)\right\} & \text{otherwise.} \end{cases} \tag{3}$$

Eq. (1) and that $|C^*| \geq d|N| + 1$ together imply

$$d + 1 \leq \begin{cases} |C^*| - \left\lfloor \frac{|C^*|}{|N|} \right\rfloor (|N| - 1) & \text{if } d = \left\lfloor \frac{|C^*|}{|N|} \right\rfloor, \\ \left\lfloor \frac{|C^*|}{|N|} \right\rfloor & \text{otherwise} \end{cases} \tag{4}$$

$$\leq |C^*| - q_{C^*}^-(|N| - 1). \tag{5}$$

It follows from Eqs. (1), (3) and (5) that

$$|C| - q_C^+(|N| - 1) \leq |B \cap C| \leq |C| - q_C^-(|N| - 1) \quad \forall C \in \mathcal{C}, \tag{6}$$

which allows the instance $\mathcal{I}'$ to be well-defined as an ordered instance of goods.

Let $i \in N \setminus \{i^*\}$ be arbitrary. Let $(P_k)_{k=1}^{|N|}$ be agent i's MMS partition, which can be assumed, without loss of generality, to satisfy that

$$\left|P_1 \cap \left\{g_j^{C^*} \mid 1 \leq j \leq d|N|+1\right\}\right| \geq d+1 \tag{7}$$

by the pigeonhole principle and that $|C^*| \geq d|N|+1$. Given Eq. (7) and the definition of B, as well as that $(P_k)_{k=1}^{|N|} \in \mathcal{F}(\mathcal{I})$, there is an injection $f : B \setminus P_1 \to P_1 \setminus B$ that satisfies

$$f(g) \in C \text{ and } v_i(g) \leq v_i(f(g)) \quad \forall g \in C \cap (B \setminus P_1), \forall C \in \mathcal{C}, \tag{8}$$

Now, we define

$$\begin{aligned} \nu(M') := \mu_i \Big(&N \setminus \{i^*\}, M', (v_{i'}|_{2^{M'}})_{i' \in N \setminus \{i^*\}}, \\ &\{C \cap M' \mid C \in \mathcal{C}\}, (q_C^-, q_C^+)_{C \in \mathcal{C}}\Big) \end{aligned} \tag{9}$$

for any $M' \subseteq M$ that ensures the instance in the right-hand side is well-defined. Then from Eqs. (6), (8) and (9), as well as the definition of $(P_k)_{k=1}^{|N|}$, we finally obtain

$$\begin{aligned} \mu_i(\mathcal{I}') &= \nu(M \setminus B) \\ &\geq \nu(M \setminus ((B \cap P_1) \cup f(B \setminus P_1))) \\ &\geq \nu(M \setminus P_1) \\ &\geq \min\{v_i(P_k) \mid k \in \{2, 3, \ldots, |N|\}\} \\ &\geq \mu_i(\mathcal{I}), \end{aligned}$$

concluding the proof of Lemma 2. □

Corollary 1. *Let $\alpha \in \mathbb{R}$ be arbitrary. In Lemma 2, if we additionally have that $v_{i^*}(B) \geq \alpha\, \mu_{i^*}(\mathcal{I})$, and an α-MMS allocation $(A_i')_{i \in N \setminus \{i^*\}}$ for $\mathcal{I}'$ is given, the following $(A_i)_{i \in N}$ is an α-MMS allocation for $\mathcal{I}$:*

$$A_i := \begin{cases} B & \text{if } i = i^*, \\ A_i' & \text{otherwise} \end{cases} \qquad \forall i \in N.$$

3 MAIN RESULTS

We present a polynomial-time algorithm to compute a $\left(\frac{2n}{3n-1}\right)$-MMS allocation for a single-category instance of goods with n agents, thereby establishing Theorem 1. This result extends and strictly improves upon Hummel and Hetland [57]'s 2/3-approximation for the special case without lower quotas. Section 4 is devoted to examining an existing approach and its issues, defining the proposed algorithm, discussing its key components, and establishing the desired guarantees through a careful scrutiny of the algorithm.

Theorem 1. *For an arbitrary single-category instance of goods $\mathcal{I} = (N, M, (v_i)_{i \in N}, (q^-, q^+))$, a $\left(\frac{2}{3-(\max\{|N|,1\})^{-1}}\right)$-MMS allocation for $\mathcal{I}$ can be computed in time $O(|N||M| \log |M|)$.*

A similar approach leads to the following result for the case of chores. As in the case of goods, Theorem 2 generalizes Hummel and Hetland [57]'s 3/2-approximation for their special case.

Theorem 2. *For an arbitrary single-category instance of chores $\mathcal{I} = (N, M, (v_i)_{i \in N}, (q^-, q^+))$, a $\left(\frac{3-(\max\{|N|,1\})^{-1}}{2}\right)$-MMS allocation for $\mathcal{I}$ can be computed in time $O(|N||M| \log |M|)$.*

For the setting with categorized items, we derive Theorems 3 and 4 for goods and chores, respectively, by leveraging Hummel and Hetland [57]'s results for the special case without lower quotas.

Theorem 3. *For an arbitrary (multi-category) instance of goods $\mathcal{I} = (N, M, (v_i)_{i \in N}, \mathcal{C}, (q_C^-, q_C^+)_{C \in \mathcal{C}})$, a $\left(\frac{1}{2-(\max\{|N|,1\})^{-1}}\right)$-MMS allocation for $\mathcal{I}$ can be computed in time $O(|N||M| \log |M|)$.*

Theorem 4. *For an arbitrary (multi-category) instance of chores $\mathcal{I} = (N, M, (v_i)_{i \in N}, \mathcal{C}, (q_C^-, q_C^+)_{C \in \mathcal{C}})$, a $\left(2 - \frac{1}{\max\{|N|,1\}}\right)$-MMS allocation for $\mathcal{I}$ can be computed in time $O(|N||M| \log |M|)$.*

4 A $\left(\frac{2n}{3n-1}\right)$-MMS ALLOCATION ALGORITHM FOR SINGLE-CATEGORY GOODS

4.1 An Existing Approach for Upper Quotas

Before discussing the proposed algorithm, we revisit an existing approach for approximate MMS guarantees only with upper quotas [57]. Whether constrained or unconstrained, one of the most common techniques is a simple greedy algorithm known as *bag-filling* [47, 43, 45, 2, 1, 64, 65, 35, 51]. In each round of this algorithm, a new bag is created as an empty set, and remaining items are added one by one until an unassigned agent finds it sufficiently valuable; then the round ends with the bag assigned to this agent. By adopting an additional type of update, Hummel and Hetland [57] develops an algorithm that computes a 2/3-MMS allocation under an upper quota. In each round of their algorithm, a new bag is created with the least valuable remaining items, many enough to ensure that the other remaining items can be divided into bundles under the upper quota. Then the other items are added one by one, as in the standard bag-filling, but not to violate the upper quota. If the bag reaches the quota and is still not valuable enough for any unassigned agent, the algorithm repeatedly exchanges the least valuable item in the bag with a higher-valued remaining item. This way of updates allows both the remaining items added to the bag and those not added to be sufficiently valuable and not too many (relative to the upper quota). However, additionally imposing a lower quota would also need both of them not to become too few, thereby raising a non-trivial challenge. Indeed, the correctness of their algorithm crucially relies on invariants concerning the values of certain subsets of remaining items, which would no longer hold consistently with lower quotas.

4.2 Overview of Our Algorithm

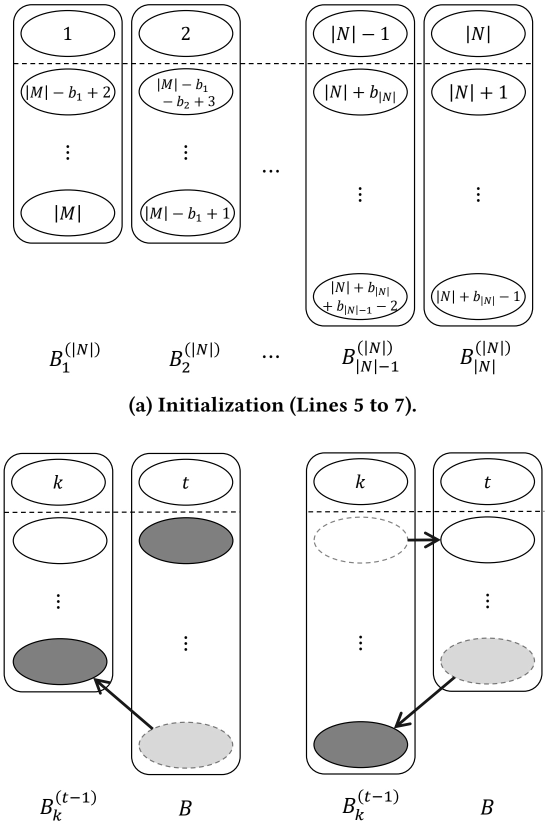
We show that ApproxGoods defined in Algorithm 2 certifies Theorem 1 together with the reduction to ordered instances. To address the above issue, we maintain multiple bags so that at every step, remaining items are left both sufficiently valuable and neither too many nor too few for unassigned agents. In its main routine (Lines 14 to 26), the algorithm incrementally updates a collection of bags to maintain certain invariant conditions, assigning one of them to an unassigned agent in each of the $|N|$ rounds. Unlike the standard bag-filling method, the bag B assigned in each round starts with items of sufficiently large size and value for any remaining agent and gradually becomes of less size and value through two types of updates: moving an item from B to another bag (Fig. 1(b)) and swapping an item in B with one in another bag (Fig. 1(c)). These updates are stopped when the value of this bag falls below a threshold for every agent. As formally shown in Section 4.3, valid reduction also helps preserve the invariants, while the polynomial runtime is attained by employing upper bounds on the agents' MMS values.

Algorithm 2 Compute a $\left(\frac{2}{3-(\max\{1,n\})^{-1}}\right)$-MMS allocation for a single-category ordered instance of goods with n agents.

1: **function** ApproxGoods($\mathcal{I} = (N, M, (v_i)_{i\in N}, (q^-, q^+)), \alpha$)
2: **if** $|M| \le |N|$ **then** ▷ *A trivial case.*
3: $A_i \leftarrow \begin{cases}\{g_i\} & \text{if } i \le |M|,\\ \emptyset & \text{otherwise}\end{cases}$ for each $i \in N = \{1, \dots, |N|\}$.
4: **return** $(A_i)_{i\in N}$
5: **for** $k \leftarrow |N|, |N|-1, \dots, 1$ **do** ▷ *Initialize the bags; see Fig. 1(a).*
6: $b_k \leftarrow \min\left\{q^+, |M| - \sum_{k'=k+1}^{|N|} b_{k'} - (k-1)\max\{q^-, 1\}\right\}$.
7: $B_k^{(|N|)} \leftarrow \{g_k\} \cup \left\{g_{|N|+j} \mid \sum_{k'=k+1}^{|N|}(b_{k'}-1) < j \le \sum_{k'=k}^{|N|}(b_{k'}-1)\right\}$.
8: $\hat{\mu}_i \leftarrow \min\left\{\frac{1}{|N|-r+1} v_i\left(\bigcup_{k=r}^{|N|} B_k^{(|N|)}\right) \mid 1 \le r \le |N|\right\}$ for each $i \in N$.
9: **if** $\exists i^* \in N$ s.t. $v_{i^*}(\{g_{|N|}, g_{|N|+1}\}) \ge \alpha\,\hat{\mu}_{i^*}$ **then** ▷ *Valid reduction.*
10: $A_{i^*} \leftarrow \{g_{|N|}, g_{|N|+1}\} \cup \{g_{|M|-j} \mid 0 \le j < \max\{q^-, |M| - q^+(|N|-1)\} - 2\}$.
11: $\mathcal{I}' \leftarrow \left(N \setminus \{i^*\}, M \setminus A_{i^*}, (v_i|_{2^{M\setminus A_{i^*}}})_{i\in N\setminus\{i^*\}}, (q^-, q^+)\right)$.
12: $(A_i)_{i\in N\setminus\{i^*\}} \leftarrow$ ApproxGoods($\mathcal{I}', \alpha$).
13: **return** $(A_i)_{i\in N}$
14: $N^{(|N|)} \leftarrow N$.
15: **for** $t \leftarrow |N|, |N|-1, \dots, 1$ **do**
16: $(B_1^{(t-1)}, B_2^{(t-1)}, \dots, B_{t-1}^{(t-1)}, B) \leftarrow (B_1^{(t)}, B_2^{(t)}, \dots, B_t^{(t)})$.
17: **for** $k \leftarrow t-1, t-2, \dots, 1$ **do**
18: **while** $\left(\exists i \in N^{(t)} \text{ s.t. } v_i(B) \ge \frac{3}{2}\alpha\,\hat{\mu}_i\right)$ and $B \setminus \{g_t\} \ne B_k^{(t)} \setminus \{g_k\}$ **do** ▷ *The latter implies that some items in $B \setminus \{g_t\}$ have not moved in the current while-loop.*
19: $g \leftarrow$ the least valuable item in $B \setminus \{g_t\}$.
20: **if** $|B| > |B_k^{(t)}|$ **then**
21: $B \leftarrow B \setminus \{g\}$, $B_k^{(t-1)} \leftarrow B_k^{(t-1)} \cup \{g\}$. ▷ *Move the item g; see Fig. 1(b).*
22: **else** ▷ *B and $B_k^{(t)}$ have become of equal size.*
23: $h \leftarrow$ the most valuable item in $B_k^{(t-1)} \setminus \{g_k\}$.
24: $B \leftarrow (B \setminus \{g\}) \cup \{h\}$, $B_k^{(t-1)} \leftarrow (B_k^{(t-1)} \setminus \{h\}) \cup \{g\}$. ▷ *Swap the items g and h; see Fig. 1(c).*
25: Find $i^{(t)} \in N^{(t)}$ s.t. $v_{i^{(t)}}(B) \ge \alpha\,\hat{\mu}_{i^{(t)}}$.
26: $A_{i^{(t)}} \leftarrow B$, $N^{(t-1)} \leftarrow N^{(t)} \setminus \{i^{(t)}\}$.
27: **return** $(A_i)_{i\in N}$.

Initialization of the Bags. All items in M are partitioned by Lines 5 to 7 into the initial bags $B_1^{(|N|)}, B_2^{(|N|)}, \dots, B_{|N|}^{(|N|)}$.[4] Their respective bag sizes $b_1, b_2, \dots, b_{|N|} \in \{\max\{1, q^-\}, \dots, q^+-1, q^+\}$ add up to $|M|$ and are lexicographically minimized among all such integer partitions; this is feasible since $|M| > |N|$ and $q^-|N| \le |M| \le q^+|N|$.

[4]Throughout Algorithm 2, the superscript indicates the number of agents yet to be assigned a bag, decreasing from $|N|$ to 0.



(a) Initialization (Lines 5 to 7).

(b) "Move" (Line 21).

(c) "Swap" (Line 24).

Figure 1: An illustration of key steps of ApproxGoods in Algorithm 2, where each item $g_j \in M$ is denoted by its index j. (a) At Lines 5 to 7, the item set M is split into the initial bags $B_1^{(|N|)}, B_2^{(|N|)}, \dots, B_{|N|}^{(|N|)}$ in a greedy manner. (b) At Line 21, an item is moved from the bag B to $B_k^{(t-1)}$; this continues until either the size of these bags are completely exchanged, or the bag value $v_i(B)$ falls below a threshold, $\frac{3}{2}\alpha\hat{\mu}_i$, for every remaining agent $i \in N^{(t)}$. (c) At Line 24, where $|B| = |B_k^{(t)}|$, an item from B is exchanged with another from $B_k^{(t-1)}$; this continues until either all items in these bags, except g_k and g_t, are completely exchanged from their initial state, or $v_i(B)$ falls below $\frac{3}{2}\alpha\hat{\mu}_i$ for every $i \in N^{(t)}$.

After the most valuable $|N|$ items, $g_1, g_2, \dots, g_{|N|}$, are placed into distinct bags, the other items are greedily packed into the bags from $B_{|N|}^{(|N|)}, B_{|N|-1}^{(|N|)}, \dots,$ to $B_1^{(|N|)}$ in descending order of value, subject to their fixed sizes; see also Fig. 1(a) and note that this order is identical across all agents as $\mathcal{I}$ is ordered. This configuration satisfies the properties stated in Lemma 4, which serve as the starting point for the invariant conditions formalized in Definition 2.

Valid Reduction. After initializing the bags, a valid reduction may occur in Lines 9 to 13 if the bundle $\{g_{|N|}, g_{|N|+1}\}$ is valued at $\alpha\hat{\mu}_{i^*}$ or more by some agent i^*; recall Corollary 1 and also notice Corollary 2 below. By repeatedly applying this reduction, we eventually reach an irreducible instance in which every item $g_{|N|+1}, g_{|N|+2}, \dots, g_{|M|}$ has a value less than $\frac{1}{2}\alpha\hat{\mu}_i$ for each agent i (Lemma 5), thereby providing a crucial setup for the main routine that follows.

Iterative Updates and Assignment of the Bags. Once the instance becomes irreducible, the bags are iteratively updated and assigned to agents through the outer for-loop (Lines 15 to 26). For each $t \in \{|N|, |N|-1, \dots, 1\}$, the inner for-loop (Lines 17 to 24) transforms the remaining t bags $B_1^{(t)}, B_2^{(t)}, \dots, B_t^{(t)}$ as a whole into the new $t-1$ bags $B_1^{(t-1)}, B_2^{(t-1)}, \dots, B_{t-1}^{(t-1)}$ plus the bundle B that is finally assigned to a remaining agent $i^{(t)}$. Let's take a look at its first iteration (with $k \leftarrow t-1$), where only B and $B_{t-1}^{(t-1)}$ are updated in two phases. We first move items from B to $B_{t-1}^{(t-1)}$ until their sizes match each other's initial sizes; see Fig. 1(b). Then we switch to swapping items between them until $B = (B_{t-1}^{(t)} \setminus \{g_{t-1}\}) \cup \{g_t\}$ (and equivalently until $B_{t-1}^{(t-1)} = (B_t^{(t)} \setminus \{g_t\}) \cup \{g_{t-1}\}$); see Fig. 1(c). Similar operations are performed with $k \leftarrow t-2, t-3, \dots, 1$, gradually sliding the remaining items (except $g_1, g_2 \dots, g_t$) between the bags, and monotonically reducing both the value and the size of B, as long as $v_i(B) \geq \frac{3}{2}\alpha\hat{\mu}_i$ for some remaining agent i. The validity of this entire routine relies on the invariant conditions and carefully examined in the proof of Lemma 7.

4.3 Proof of Theorem 1

In what follows, we show by induction on $|N|$ that ApproxGoods returns an α-MMS allocation for any single-category ordered instance of goods $\mathcal{I} = (N, M, (v_i)_{i\in N}, (q^-, q^+))$ and any real constant $\alpha \in \left[\frac{2}{3}, \frac{2}{3-(\max\{|N|,1\})^{-1}}\right]$, in time $O(|N||M|)$. This induction hypothesis is used to obtain Lemma 5, which concerns the recursion in Line 12. We eventually derive Theorem 1 by combining Lemmas 1, 3, 5 and 8 and Corollary 3; refer to Lemma 1 for the complexity of the reduction to an ordered instance, which proves dominant.

First, a trivial MMS allocation can be obtained when the number of items is at most that of agents, i.e., when $|M| \leq |N|$.

Lemma 3. *If Line 3 is reached, an MMS allocation $(A_i)_{i\in N}$ for $\mathcal{I}$ is returned at Line 4.*

Proof. The allocation $(A_i)_{i\in N}$ at Line 4 is feasible for $\mathcal{I}$ due to Eq. (1). If $|M| < |N|$, $\mu_i(\mathcal{I}) = 0$ holds for each $i \in N$; otherwise, $\mu_i(\mathcal{I}) = v_i(g_{|N|})$ holds for each $i \in N$. Therefore, it is clear in either case that $(A_i)_{i\in N}$ is an MMS allocation for $\mathcal{I}$. □

As described before, the bags are initialized by Lines 5 to 7 so that we can ensure the properties in Lemma 4, setting the ground for the invariant conditions defined later in Definition 2.

Lemma 4. *The bundles $B_1^{(|N|)}, B_2^{(|N|)}, \dots, B_{|N|}^{(|N|)}$ defined by Lines 5 to 7 are mutually disjoint and satisfy the following:*

$$B_1^{(|N|)} \cup B_2^{(|N|)} \cup \cdots \cup B_{|N|}^{(|N|)} = M, \tag{10}$$

$$q^- \leq |B_1^{(|N|)}| \leq |B_2^{(|N|)}| \leq \cdots \leq |B_{|N|}^{(|N|)}| \leq q^+, \tag{11}$$

$$v_i\left(B_r^{(|N|)} \cup B_{r+1}^{(|N|)} \cup \cdots \cup B_{|N|}^{(|N|)}\right) \geq (|N|-r+1)\,\mu_i(\mathcal{I}) \quad \forall r \in \{1,2,\dots,|N|\}, \forall i \in N. \tag{12}$$

Proof. Given Eq. (1) and $|M| > |N|$, these bundles are well-defined by Lines 5 to 7, mutually disjoint, and satisfying Eqs. (10) and (11). We fix any pair $i \in N$ and $r \in \{1,2,\dots,|N|\}$, for which the inequality in Eq. (12) is shown below. Let $(P_k)_{k=1}^{|N|}$ be agent i's MMS partition, which can be assumed, without loss of generality, to satisfy that

$$P_1 \cup P_2 \cup \cdots \cup P_{r-1} \supseteq \{g_1, g_2, \dots, g_{r-1}\} \quad \forall r \in \{1,2,\dots,|N|\}.$$

Combining this with $(P_k)_{k=1}^{|N|} \in \mathcal{F}(\mathcal{I})$ and the construction of $B_1^{(|N|)}, B_2^{(|N|)}, \dots, B_{|N|}^{(|N|)}$ yields that

$$\begin{aligned} v_i\left(B_r^{(|N|)} \cup B_{r+1}^{(|N|)} \cup \cdots \cup B_{|N|}^{(|N|)}\right) &\geq v_i\left(P_r \cup P_{r+1} \cup \cdots \cup P_{|N|}\right) \\ &\geq (|N|-r+1)\,\mu_i(\mathcal{I}), \end{aligned}$$

which concludes the proof. □

Eq. (12) of Lemma 4 also provides useful upper bounds on MMS values, $(\hat{\mu}_i)_{i\in N}$, defined by Line 8.

Corollary 2. *At Line 8, it holds for every $i \in N$ that $\hat{\mu}_i \geq \mu_i(\mathcal{I})$.*

Proof. This follows from Line 8 and Eq. (12) of Lemma 4. □

As $\frac{2}{3-\max\{|N|,1\}^{-1}}$ is non-increasing in $|N|$, Corollary 1 can be applied together with the induction hypothesis and Corollary 2.

Lemma 5. *If Line 10 is reached, an α-MMS allocation $(A_i)_{i\in N}$ for $\mathcal{I}$ is returned at Line 13. Otherwise, it follows at Line 14 that $v_i(g) < \frac{1}{2}\,\alpha\,\hat{\mu}_i$ for every $g \in M \setminus \{g_1, g_2, \dots, g_{|N|}\}$ and $i \in N$.*

Proof. By applying Lemma 2 for $C^* = M$ and $d = 1$, we see that the instance $\mathcal{I}'$ is well-defined by Line 11 as a single-category ordered instance of goods. Given the induction hypothesis and that

$$\frac{2}{3} \leq \alpha \leq \frac{2}{3-\max\{|N|,1\}^{-1}} \leq \frac{2}{3-\max\{|N \setminus \{i^*\}|,1\}^{-1}},$$

an α-MMS allocation $(A_i)_{i\in N\setminus\{i^*\}}$ for $\mathcal{I}'$ is obtained at Line 12. Because Line 9, 10, and Corollary 2 together ensure that

$$v_{i^*}(A_{i^*}) \geq \alpha\,\hat{\mu}_{i^*} \geq \alpha\,\mu_{i^*}(\mathcal{I}),$$

Corollary 1 allows $(A_i)_{i\in N}$ returned at Line 13 to be an α-MMS allocation for $\mathcal{I}$. As the instance $\mathcal{I}$ is ordered, Line 9 immediately gives the second statement of the Lemma 5. □

We formally define the invariant conditions maintained throughout the iterations over Lines 15 to 26.

Definition 2. Along with the outer for-loop over Lines 15 to 26, we consider the following conditions for each $t \in \{|N|, |N|-1, \dots, 1, 0\}$:

(C1) $B_1^{(t)}, B_2^{(t)}, \dots, B_t^{(t)}, A_{i^{(t+1)}}, A_{i^{(t+2)}}, \dots, A_{i^{(|N|)}}$ are disjoint.

(C2) $B_1^{(t)} \cup B_2^{(t)} \cup \cdots \cup B_t^{(t)} \cup A_{i^{(t+1)}} \cup A_{i^{(t+2)}} \cup \cdots \cup A_{i^{(|N|)}} = M$.

(C3) $q^- \leq |B_1^{(t)}| \leq |B_2^{(t)}| \leq \cdots \leq |B_t^{(t)}| \leq q^+$.

(C4) $B_k^{(t)} \cap \{g_1, g_2, \dots, g_{|N|}\} = \{g_k\} \quad \forall k \in \{1,2,\dots,t\}$.

(C5) $v_i(h_1) \leq v_i(h_2) \leq \cdots \leq v_i(h_t) \quad \forall i \in N^{(t)}$, $\forall h_1 \in B_1^{(t)} \setminus \{g_1\}, \forall h_2 \in B_2^{(t)} \setminus \{g_2\}, \dots, \forall h_t \in B_t^{(t)} \setminus \{g_t\}$.

(C6) $v_i\left(\bigcup_{k=r}^{t} B_k^{(t)}\right) \geq \left(t - r + 1 - (|N|-t)\left(\frac{3}{2}\,\alpha - 1\right)\right)\hat{\mu}_i \quad \forall r \in \{1,2,\dots,t\}, \forall i \in N^{(t)}$.

Lemma 4 and Corollary 2 together guarantee that these conditions initially hold for $t = |N|$.

Lemma 6. *At Line 14, Conditions* (C1) *to* (C6) *hold for* $t = |N|$.

Proof. Conditions (C1) to (C5) for $t = |N|$ follow from Lemma 4. Condition (C6) for $t = |N|$ follows from Line 8. □

Key to the overall correctness is Lemma 7: each iteration assigns an agent a sufficiently valuable and feasible bundle while ensuring that all the invariants are properly inherited by the next iteration.

Lemma 7. *Let* $s \in \{|N|, |N|-1\dots, 1\}$ *be arbitrary. Suppose that the for-loop over Lines 15 to 26 has successfully iterated for* $t \in \{|N|, |N|-1, \dots, s+1\}$*, and that Conditions* (C1) *to* (C6) *now hold for* $t = s$*. Then the next iteration with* $t = s$ *succeeds, where the following hold at Line 25:*

$$q^- \le |B| \le q^+. \tag{13}$$

$$\exists i^{(s)} \in N^{(s)} \ \text{ s.t. } \ v_{i^{(s)}}(B) \ge \alpha\, \hat{\mu}_{i^{(s)}}. \tag{14}$$

Furthermore, Conditions (C1) *to* (C6) *hold for* $t = s-1$ *at Line 26 of the same iteration.*

Proof. First, we verify that Line 25 is indeed reached, with all preceding operations done successfully; namely we show that the desired item exists at Line 19 and 23, as well as that the while-loop terminates. Given Condition (C5) for $t = s$, neither $|B|$ nor $v_i(B)$ for any $i \in N$ increases after Line 16 until 25. At Line 16, $B \ni g_s$ and $B_k^{(s-1)} \ni g_k$ for each $k \in \{1, 2, \dots, s-1\}$ hold due to Condition (C4) for $t = s$, remaining true until Line 26 by the definition of Lines 19 and 23. Let us go through a single iteration of the inner for-loop (Lines 17 to 24) with an arbitrary $k \in \{1, 2, \dots, s-1\}$. Clearly, nothing occurs if $k < s-1$ and the previous iteration has stopped with $v_i(B) < \frac{3}{2}\alpha\hat{\mu}_i$ for every $i \in N^{(s)}$. Otherwise, at the beginning of this iteration, we have that $B \setminus \{g_s\} = B_{k+1}^{(s)} \setminus \{g_{k+1}\}$, that $B_k^{(s-1)} = B_k^{(s)}$, and hence that $|B| = |B_{k+1}^{(s)}| \ge |B_k^{(s)}| = |B_k^{(s-1)}|$ due to Condition (C3) for $t = s$. Any single iteration of the while-loop (Lines 18 to 24) keeps $B \cup B_k^{(s-1)}$ unchanged and decreases $|B|$ by at most 1, while maintaining that $g_s \in B$, $g_k \in B_k^{(s)}$, and $|B| \ge |B_k^{(s)}|$. Therefore, $B \setminus \{g_s\}$ is never be empty at Line 19, until which it has been maintained that $B \setminus \{g_s\} \ne B_k^{(s)} \setminus \{g_k\}$. We then obtain that $B_k^{(s-1)} \setminus \{g_k\} \ne \emptyset$ at Line 23, where $|B_k^{(s-1)}| = |B_{k+1}^{(s)}| \ge |B_k^{(s)}| = |B| \ge 2$. Thanks to the choice of the items g and h at Lines 19 and 23, respectively, the while-loop successfully terminates.

Next, we prove that Eqs. (13) and (14) hold at Line 25, where we define the following:

$$k^* \coloneqq \begin{cases} 0 & \text{if } \exists i \in N^{(s)} \text{ s.t. } v_i(B) \ge \frac{3}{2}\,\alpha\,\hat{\mu}_i; \\ \min\left(\left\{k \in \{1, 2, \dots, s-1\} \mid B_k^{(s-1)} \ne B_k^{(s)}\right\} \cup \{s\}\right) & \text{otherwise.} \end{cases} \tag{15}$$

The value of k^* represents at which point the updates on B have stopped: $k^* = 0$ when all possible updates on B run out, i.e., when it holds at Line 25 that $B = (B_1^{(s)} \setminus \{g_1\}) \cup \{g_s\}$, $B_1^{(s-1)} = (B_2^{(s)} \setminus \{g_2\}) \cup \{g_1\}, \dots$, and $B_{s-1}^{(s-1)} = (B_s^{(s)} \setminus \{g_s\}) \cup \{g_{s-1}\}$; $k^* = s$ if B has not been updated at Line 25 since Line 16; otherwise (if $0 < k < s$), $B_{k^*}^{(s-1)}$ denotes the bag that has been last updated at either Line 21 or 24 (as $B_k^{(t-1)}$). Based on the observations in the previous paragraph and Conditions (C1) to (C4) for $t = s$, we see that the following hold at Line 25:

$$B_k^{(s-1)} = B_k^{(s)} \qquad \forall k \in \{1, 2, \dots, k^*-1\}. \tag{16}$$

$$B_k^{(s-1)} = (B_{k+1}^{(s)} \setminus \{g_{k+1}\}) \cup \{g_k\} \quad \forall k \in \{k^*+1, \dots, s-1\}. \tag{17}$$

$$\begin{cases} B = \left(B_{\max\{k^*,1\}}^{(s)} \setminus \left\{g_{\max\{k^*,1\}}\right\}\right) \cup \{g_s\} & \text{if } k^* \in \{0, s\}; \\ (B \setminus \{g_s\}) \cup B_{k^*}^{(s-1)} = B_{k^*}^{(s)} \cup \left(B_{k^*+1}^{(s)} \setminus \{g_{k^*+1}\}\right) & \text{otherwise.} \end{cases} \tag{18}$$

$$B \cap \left\{g_1, g_2, \dots, g_{|N|}\right\} = \{g_s\}. \tag{19}$$

$$q^- \le \left|B_{\max\{k^*,1\}}^{(s)}\right| \le |B| \le \left|B_{\min\{k^*+1,s\}}^{(s)}\right| \le q^+. \tag{20}$$

$$B_1^{(s-1)}, B_2^{(s-1)}, \dots, B_{s-1}^{(s-1)}, \text{ and } B \text{ are disjoint.} \tag{21}$$

In particular, Eq. (20) immediately gives Eq. (13). Moreover, applying the latter statement of Lemma 5 to the item $g \in B \setminus \{g_s\}$ at Line 19, along with the first condition of the while-loop, guarantees that

$$\exists i \in N^{(s)} \quad \text{s.t.} \quad v_i(B) > \frac{3}{2}\,\alpha\,\hat{\mu}_i - \frac{1}{2}\,\alpha\,\hat{\mu}_i = \alpha\,\hat{\mu}_i,$$

whenever B gets updated at either Line 21 or 24. Condition (C6) for $t = s$ also gives that at Line 16,

$$\begin{aligned} v_i(B) = v_i\left(B_s^{(s)}\right) &\ge \left(1 - (|N| - s)\left(\frac{3}{2}\,\alpha - 1\right)\right)\hat{\mu}_i \\ &\ge \left(1 - (|N| - 1)\left(\frac{3}{2}\,\alpha - 1\right)\right)\hat{\mu}_i \\ &\ge \frac{2}{3 - |N|^{-1}}\,\hat{\mu}_i \\ &\ge \alpha\,\hat{\mu}_i \qquad \forall i \in N^{(s)}, \end{aligned}$$

because $\alpha \in \left[\frac{2}{3}, \frac{2}{3-|N|^{-1}}\right]$. Therefore, we obtain that Eq. (14) always holds at Line 25.

Finally, we show that Conditions (C1) to (C6) hold for $t = s-1$ at Line 26. Conditions (C1) to (C5) for $t = s-1$ follow from those for $t = s$ and Eqs. (16) to (21). We fix any pair $r \in \{1, 2, \dots, s-1\}$ and $i \in N^{(s-1)} \subset N^{(s)}$, for which the inequality in Condition (C6) for $t = s-1$ is obtained in either of the following cases.

Case 1: Suppose $r > k^*$. Then given Eq. (17), Conditions (C4) and (C6) for $t = s$, and that $\alpha \ge \frac{2}{3}$, we obtain that

$$\begin{aligned} v_i\left(\bigcup_{k=r}^{s-1} B_k^{(s-1)}\right) &= v_i\left(\bigcup_{k=r}^{s-1}\left(\left(B_{k+1}^{(s)} \setminus \{g_{k+1}\}\right) \cup \{g_k\}\right)\right) \\ &\ge v_i\left(\bigcup_{k=r+1}^{s} B_k^{(s)}\right) \\ &\ge \left(s - r - (|N| - s)\left(\frac{3}{2}\,\alpha - 1\right)\right)\hat{\mu}_i \\ &\ge \left(s - r - (|N| - s + 1)\left(\frac{3}{2}\,\alpha - 1\right)\right)\hat{\mu}_i. \end{aligned}$$

Case 2: Suppose $1 \le r \le k^*$. Then given the definition of k^* in Eq. (15), we must have that $v_i(B) < \frac{3}{2}\alpha\hat{\mu}_i$ at Line 25. It also follows from that $r \le k^*$ and Eqs. (16) to (19) that

$$B \cup B_r^{(s-1)} \cup B_{r+1}^{(s-1)} \cup \cdots \cup B_{s-1}^{(s-1)} = B_r^{(s)} \cup B_{r+1}^{(s)} \cup \cdots \cup B_s^{(s)}.$$

These observations and Condition (C6) for $t = s$ together establish that

$$v_i\left(\bigcup_{k=r}^{s-1} B_k^{(s-1)}\right) \geq v_i\left(\bigcup_{k=r}^{s} B_k^{(s)}\right) - v_i(B)$$
$$> \left(s - r + 1 - (|N| - s)\left(\frac{3}{2}\,\alpha - 1\right)\right)\hat{\mu}_i - \frac{3}{2}\,\alpha\,\hat{\mu}_i$$
$$= \left(s - r - (|N| - s + 1)\left(\frac{3}{2}\,\alpha - 1\right)\right)\hat{\mu}_i. \qquad \square$$

Lemmas 6 and 7 and Corollary 2 together lead to Corollary 3.

Corollary 3. *If Line 14 is reached, an α-MMS allocation $(A_i)_{i\in N}$ for $\mathcal{I}$ is returned at Line 27.*

Proof. Lemmas 6 and 7 together ensure that Line 27 is successfully reached after Line 14, with Conditions (C1) to (C6) fulfilled for every $t \in \{|N|, |N|-1, \ldots, 1, 0\}$. Also given Conditions (C1) to (C2) for $t = 0$ and Eq. (13) of Lemma 7, it is implied that $(A_i)_{i\in N} = \left(A_{i(t)}\right)_{t\in\{|N|,|N|-1,\ldots,1\}}$ is a feasible allocation for $\mathcal{I}$. Corollary 2 and Eq. (14) of Lemma 7 also yield that

$$v_{i(t)}\left(A_{i(t)}\right) \geq \alpha\,\hat{\mu}_{i(t)} \geq \alpha\,\mu_{i(t)}(\mathcal{I}) \quad \forall t \in \{|N|, |N|-1, \ldots, 1\},$$

establishing that $(A_i)_{i\in N}$ is an α-MMS allocation for $\mathcal{I}$. $\square$

Lemma 8. *ApproxGoods runs in time $O(|N||M|)$.*

Proof. The recursion in Line 12 reduces $|N|$ by one, while Lines 5 to 11 run in time $O(|M|)$. In each of the $|N|$ iterations of the outer for-loop (Lines 15 to 26), the inner for-loop (Lines 17 to 24) iterates $t - 1 < |N| < |M|$ times, during which the while-loop (Lines 18 to 24) iterates at most $\sum_{k=2}^{t}(|B_k^{(t)}| - 1) \leq |M|$ times in total. $\square$

4.4 Tight Instances for the Algorithm

Our analysis above is tight for any number of agents, even when they have identical valuations.

Theorem 5. *For any $n \in \{1, 2, \ldots\}$, there is a single-category ordered instance $\mathcal{I}_n$ with n agents, $3n$ goods, and identical valuations, which satisfies the following: for any $\beta \in \left(\frac{2n}{3n-1}, \infty\right)$, no β-MMS allocation for $\mathcal{I}_n$ is obtained by ApproxGoods given $\mathcal{I}_n$ and any $\alpha \in \mathbb{R}$ as input.*

Proof. Let n be an arbitrary positive integer. We prove the claimed property of the single-category ordered instance of goods $\mathcal{I}_n = (N, M, (v_i)_{i\in N}, (q^-, q^+))$ defined as follows:

$$N := \{1, 2, \ldots, n\},$$
$$M := \{g_1, g_2, \ldots, g_{3n}\},$$
$$\left(q^-, q^+\right) := (3, 3);$$
$$v_i(g_j) := \begin{cases} \frac{2n-j}{3n-1} & \text{if } j \leq n, \\ \frac{1}{6n-2}\left\lceil \frac{5n+1-j}{2} \right\rceil & \text{if } n < j \leq 3n-2, \\ \frac{3n-j}{3n-1} & \text{otherwise} \end{cases} \quad \forall g_j \in M, \forall i \in N.$$

Given the partition $(P_k)_{k=1}^{n} \in \mathcal{F}(\mathcal{I}_n)$ of M defined as

$$P_k := \begin{cases} \{g_k, g_{n+k}, g_{3n+1-k}\} & \text{if } k \leq 2, \\ \{g_k, g_{3n-2k+3}, g_{3n-2k+4}\} & \text{otherwise} \end{cases} \quad \forall k \in \{1, 2, \ldots, n\},$$

we have that $\mu_i(\mathcal{I}_n) = v_i(P_k) = 1$ for any $k \in \{1, 2, \ldots, n\}$ and $i \in N$. Let us fix any $\beta \in \left(\frac{2n}{3n-1}, \infty\right)$, and suppose that ApproxGoods runs given $\mathcal{I}_n$ and some $\alpha \in \mathbb{R}$ as input. Because $|M| > |N|$ at Line 2, and it also follows at Line 8 that

$$v_i\left(B_k^{(|N|)}\right) = v_i(\{g_k, g_{3n-2k+1}, g_{3n-2k+2}\})$$
$$= \begin{cases} \frac{2n}{3n-1} & \text{if } k = 1, \\ \frac{3n}{3n-1} & \text{otherwise} \end{cases} \quad \forall k \in \{1, 2, \ldots, n\}, \forall i \in N,$$

Line 9 is reached with $\hat{\mu}_i = 1$ for every $i \in N$. The rest of the analysis depends on the regime of $\alpha \in \mathbb{R}$ and confirms that the algorithm never obtains a β-MMS allocation in any case.

Case 1: Suppose $\alpha \leq \frac{2n}{3n-1}$. Because Line 13 is reached with

$$v_{i^*}(A_{i^*}) = v_{i^*}(\{g_n, g_{n+1}, g_{3n}\}) = \frac{2n}{3n-1} < \beta = \beta\,\mu_{i^*}(\mathcal{I}_n),$$

$(A_i)_{i\in N}$ is then not a β-MMS allocation for $\mathcal{I}$.

Case 2: Suppose $\frac{2n}{3n-1} < \alpha \leq 1$. Then $B = B_t^{(|N|)}$ for each $t \in \{n, n-1\ldots, 2\}$ at Line 26. In the final iteration with $t = 1$, Line 16 is thus reached with

$$v_i(B) = v_i\left(B_1^{(1)}\right) = v_i\left(B_1^{(|N|)}\right) = \frac{2n}{3n-1} < \alpha = \alpha\,\hat{\mu}_i \quad \forall i \in N^{(1)},$$

which immediately makes Line 25 unsuccessful.

Case 3: Suppose $\alpha > 1$. Then ApproxGoods never successfully terminates; otherwise, Lines 25 and 26 would imply that

$$v_i(M) = \sum_{t=1}^{n} v_i\left(A_{i(t)}\right) = \sum_{t=1}^{n} v_{i(t)}\left(A_{i(t)}\right)$$
$$\geq \sum_{t=1}^{n} \alpha\,\hat{\mu}_{i(t)} = n\alpha > n \qquad \forall i \in N,$$

which contradicts the definition of $\mathcal{I}_n$. $\square$

5 DISCUSSION

A promising future work is to improve our MMS approximations, given their better counterparts in the unconstrained setting [55, 54]. A tantalizing question is whether imposing lower quotas strictly decreases the best MMS approximation achievable. Allowing for non-additive valuations would also be of interest.

Beyond MMS fairness, it may be even more desirable to obtain simultaneous guarantees across different fairness criteria [44], such as a pair of envy-based and share-based notions [10, 32, 5, 12]. Moreover, extending our results to randomized allocations [13, 15, 17, 3] and/or online arrivals [7, 83, 73, 75, 61] would be of both theoretical and practical significance.

In addition to fairness, allocation efficiency is another central issue. A large body of work has sought to simultaneously achieve fairness and efficiency, both in the unconstrained setting [31, 21, 23, 68, 22] and under constraints [76, 81, 33, 58], motivating us to explore the trade-off between MMS approximation and various efficiency measures both under quota constraints and beyond. It would also be relevant to quantify the *price* of (i.e., the welfare loss due to) lower quotas, as suggested by Lam et al. [63].

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REFERENCES

[1] Hannaneh Akrami and Jugal Garg. 2024. Breaking the 3/4 barrier for approximate maximin share. In *Proceedings of the 35th Annual ACM-SIAM Symposium on Discrete Algorithms*. 74–91.

[2] Hannaneh Akrami, Jugal Garg, Eklavya Sharma, and Setareh Taki. 2023. Simplification and improvement of MMS approximation. In *Proceedings of the 32nd International Joint Conference on Artificial Intelligence*. 2485–2493.

[3] Hannaneh Akrami, Kurt Mehlhorn, Masoud Seddighin, and Golnoosh Shahkarami. 2023. Randomized and deterministic maximin-share approximations for fractionally subadditive valuations. In *Proceedings of the 37th International Conference on Neural Information Processing Systems*. 58821–58832.

[4] Hannaneh Akrami, Roshan Raj, and László A. Végh. 2026. Matroids are equitable. In *Proceedings of the 37th Annual ACM-SIAM Symposium on Discrete Algorithms*. 5843–5860.

[5] Hannaneh Akrami and Nidhi Rathi. 2025. Achieving maximin share and EFX/EF1 guarantees simultaneously. In *Proceedings of the 39th AAAI Conference on Artificial Intelligence*. 13529–13537.

[6] Martin Aleksandrov, Haris Aziz, Serge Gaspers, and Toby Walsh. 2015. Online fair division: Analysing a food bank problem. In *Proceedings of the 24th International Joint Conference on Artificial Intelligence*. 2540–2546.

[7] Martin Aleksandrov and Toby Walsh. 2020. Online fair division: A survey. In *Proceedings of the 34th AAAI Conference on Artificial Intelligence*. 13557–13562.

[8] Georgios Amanatidis, Haris Aziz, Georgios Birmpas, Aris Filos-Ratsikas, Bo Li, Hervé Moulin, Alexandros A. Voudouris, and Xiaowei Wu. 2023. Fair division of indivisible goods: Recent progress and open questions. *Artificial Intelligence* 322 (2023), 103965.

[9] Georgios Amanatidis, Evangelos Markakis, Afshin Nikzad, and Amin Saberi. 2017. Approximation algorithms for computing maximin share allocations. *ACM Transactions on Algorithms* 13, 4 (2017), 1–28.

[10] Georgios Amanatidis, Evangelos Markakis, and Apostolos Ntokos. 2020. Multiple birds with one stone: Beating 1/2 for EFX and GMMS via envy cycle elimination. *Theoretical Computer Science* 841 (2020), 94–109.

[11] Ashwin Arulselvan, Ágnes Cseh, Martin Groβ, David F. Manlove, and Jannik Matuschke. 2018. Matchings with lower quotas: Algorithms and complexity. *Algorithmica* 80, 1 (2018), 185–208.

[12] Arash Ashuri and Vasilis Gkatzelis. 2025. Simultaneously satisfying MXS and EFL. In *Proceedings of the 26th ACM Conference on Economics and Computation*. 689–718.

[13] Haris Aziz. 2019. A probabilistic approach to voting, allocation, matching, and coalition formation. *The Future of Economic Design: The Continuing Development of a Field as Envisioned by Its Researchers* (2019), 45–50.

[14] Haris Aziz, Péter Biró, and Makoto Yokoo. 2022. Matching market design with constraints. In *Proceedings of the 36th AAAI Conference on Artificial Intelligence*. 12308–12316.

[15] Haris Aziz, Rupert Freeman, Nisarg Shah, and Rohit Vaish. 2024. Best of both worlds: Ex ante and ex post fairness in resource allocation. *Operations Research* 72, 4 (2024), 1674–1688.

[16] Haris Aziz, Gerhard Rauchecker, Guido Schryen, and Toby Walsh. 2017. Algorithms for max-min share fair allocation of indivisible chores. In *Proceedings of the 31st AAAI Conference on Artificial Intelligence*. 335–341.

[17] Moshe Babaioff, Tomer Ezra, and Uriel Feige. 2022. On best-of-both-worlds fair-share allocations. In *Proceedings of the 18th International Conference on Web and Internet Economics*. 237–255.

[18] Siddharth Barman, Arindam Khan, Sudarshan Shyam, and K. V. N. Sreenivas. 2023. Finding fair allocations under budget constraints. In *Proceedings of the 37th AAAI Conference on Artificial Intelligence*. 5481–5489.

[19] Siddharth Barman, Arindam Khan, Sudarshan Shyam, and K. V. N. Sreenivas. 2023. Guaranteeing envy-freeness under generalized assignment constraints. In *Proceedings of the 24th ACM Conference on Economics and Computation*. 242–269.

[20] Siddharth Barman and Sanath Kumar Krishnamurthy. 2020. Approximation algorithms for maximin fair division. *ACM Transactions on Economics and Computation* 8, 1 (2020), 1–28.

[21] Siddharth Barman, Sanath Kumar Krishnamurthy, and Rohit Vaish. 2018. Finding fair and efficient allocations. In *Proceedings of the 19th ACM Conference on Economics and Computation*. 557–574.

[22] Siddharth Barman and Paritosh Verma. 2025. Introspectively envy-free and efficient allocation of indivisible mixed manna. arXiv:2509.18673.

[23] Xiaohui Bei, Xinhang Lu, Pasin Manurangsi, and Warut Suksompong. 2021. The price of fairness for indivisible goods. *Theory of Computing Systems* 65 (2021), 1069–1093.

[24] Péter Biró, Tamás Fleiner, Robert W. Irving, and David F. Manlove. 2010. The college admissions problem with lower and common quotas. *Theoretical Computer Science* 411, 34-36 (2010), 3136–3153.

[25] Arpita Biswas and Siddharth Barman. 2018. Fair division under cardinality constraints. In *Proceedings of the 27th International Joint Conference on Artificial Intelligence*. 91–97.

[26] Arpita Biswas and Siddharth Barman. 2019. Matroid constrained fair allocation problem. In *Proceedings of the 33rd AAAI Conference on Artificial Intelligence*. 9921–9922.

[27] Sylvain Bouveret and Michel Lemaître. 2016. Characterizing conflicts in fair division of indivisible goods using a scale of criteria. *Autonomous Agents and Multi-Agent Systems* 30, 2 (2016), 259–290.

[28] Steven J. Brams and Alan D. Taylor. 1996. *Fair Division: From Cake-Cutting to Dispute Resolution*. Cambridge University Press.

[29] Eric Budish. 2011. The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. *Journal of Political Economy* 119, 6 (2011), 1061–1103.

[30] Eric Budish, Gérard P. Cachon, Judd B. Kessler, and Abraham Othman. 2017. Course match: A large-scale implementation of approximate competitive equilibrium from equal incomes for combinatorial allocation. *Operations Research* 65, 2 (2017), 314–336.

[31] Ioannis Caragiannis, David Kurokawa, Hervé Moulin, Ariel D. Procaccia, Nisarg Shah, and Junxing Wang. 2019. The unreasonable fairness of maximum Nash welfare. *ACM Transactions on Economics and Computation* 7, 3 (2019), 1–32.

[32] Bhaskar R. Chaudhury, Telikepalli Kavitha, Kurt Mehlhorn, and Alkmini Sgouritsa. 2021. A little charity guarantees almost envy-freeness. *SIAM J. Comput.* 50, 4 (2021), 1336–1358.

[33] Benjamin Cookson, Soroush Ebadian, and Nisarg Shah. 2025. Constrained fair and efficient allocations. In *Proceedings of the 39th AAAI Conference on Artificial Intelligence*. 13718–13726.

[34] Sijia Dai, Guichen Gao, Shengxin Liu, Boon Han Lim, Li Ning, Yicheng Xu, and Yong Zhang. 2023. Maximum Nash social welfare under budget-feasible EFX. *IEEE Transactions on Network Science and Engineering* 11, 2 (2023), 1810–1820.

[35] Bin Deng and Weidong Li. 2024. The budgeted maximin share allocation problem. *Optimization Letters* 19, 5 (2024), 955–968.

[36] Amitay Dror, Michal Feldman, and Erel Segal-Halevi. 2023. On fair division under heterogeneous matroid constraints. *Journal of Artificial Intelligence Research* 76 (2023), 567–611.

[37] Edith Elkind, Ayumi Igarashi, and Nicholas Teh. 2024. Fair division of chores with budget constraints. In *Proceedings of the 17th International Symposium on Algorithmic Game Theory*. 55–71.

[38] Uriel Feige, Ariel Sapir, and Laliv Tauber. 2021. A tight negative example for MMS fair allocations. In *Proceedings of the 17th International Conference on Web and Internet Economics*. 355–372.

[39] Diodato Ferraioli, Laurent Gourvès, and Jérôme Monnot. 2014. On regular and approximately fair allocations of indivisible goods. In *Proceedings of the 13th International Conference on Autonomous Agents and Multi-agent Systems*. 997–1004.

[40] Duncan K. Foley. 1967. Resource allocation and the public sector. *Yale Economic Essays* 7 (1967), 45–98.

[41] David Gale and Lloyd S. Shapley. 1962. College admissions and the stability of marriage. *The American Mathematical Monthly* 69, 1 (1962), 9–15.

[42] Marius Garbea, Vasilis Gkatzelis, and Xizhi Tan. 2023. EFx budget-feasible allocations with high Nash welfare. In *Proceedings of the 26th European Conference on Artificial Intelligence*. 795–802.

[43] Jugal Garg, Peter McGlaughlin, and Setareh Taki. 2019. Approximating maximin share allocations. In *Proceedings of the 2nd Symposium on Simplicity in Algorithms*. 20:1–20:11.

[44] Jugal Garg and Eklavya Sharma. 2026. Exploring relations among fairness notions in discrete fair division. In *Proceedings of the 25th International Conference on Autonomous Agents and Multi-Agent Systems*.

[45] Jugal Garg and Setareh Taki. 2021. An improved approximation algorithm for maximin shares. *Artificial Intelligence* 300, C (2021), 103547.

[46] Naveen Garg, Telikepalli Kavitha, Amit Kumar, Kurt Mehlhorn, and Julián Mestre. 2010. Assigning papers to referees. *Algorithmica* 58, 1 (2010), 119–136.

[47] Mohammad Ghodsi, Mohammad T. Hajiaghayi, Masoud Seddighin, Saeed Seddighin, and Hadi Yami. 2021. Fair allocation of indivisible goods: Improvement. *Mathematics of Operations Research* 46, 3 (2021), 1038–1053.

[48] Laurent Gourvès and Jérôme Monnot. 2019. On maximin share allocations in matroids. *Theoretical Computer Science* 754 (2019), 50–64.

[49] Dan Gusfield and Robert W. Irving. 1989. *The Stable Marriage Problem: Structure and Algorithms*. MIT Press.

[50] Koki Hamada, Kazuo Iwama, and Shuichi Miyazaki. 2016. The hospitals/residents problem with lower quotas. *Algorithmica* 74 (2016), 440–465.

[51] Ehsan Heidari, Alireza Kaviani, Masoud Seddighin, and AmirMohammad Shahrezaei. 2026. Improved maximin share guarantee for additive valuations. In *Proceedings of the 37th Annual ACM-SIAM Symposium on Discrete Algorithms*. 2239–2290.

[52] Chien-Chung Huang. 2010. Classified stable matching. In *Proceedings of the 21st Annual ACM-SIAM Symposium on Discrete Algorithms*. 1235–1253.

[53] Xin Huang and Pinyan Lu. 2021. An algorithmic framework for approximating maximin share allocation of chores. In *Proceedings of the 22nd ACM Conference on Economics and Computation*. 630–631.

[54] Xin Huang and Erel Segal-Halevi. 2023. A reduction from chores allocation to job scheduling. In *Proceedings of the 24th ACM Conference on Economics and Computation*. 908–908.

[55] Xin Huang and Shengwei Zhou. 2025. An FPTAS for 7/9-approximation to maximin share allocations. arXiv:2511.13056.

[56] Halvard Hummel. 2025. Maximin shares in hereditary set systems. *ACM Transactions on Economics and Computation* 13, 3 (2025), 1–33.

[57] Halvard Hummel and Magnus L. Hetland. 2022. Maximin shares under cardinality constraints. In *Proceedings of the 19th European Conference on Multi-Agent Systems*. 188–206. A full version is available at https://arxiv.org/pdf/2106.07300.

[58] Ayumi Igarashi and Frédéric Meunier. 2025. Fair and efficient allocation of indivisible items under category constraints. arXiv:2503.20260.

[59] Yuichiro Kamada and Fuhito Kojima. 2017. Stability concepts in matching under distributional constraints. *Journal of Economic Theory* 168 (2017), 107–142.

[60] Hirota Kinoshita and Ayumi Igarashi. 2026. Maximin shares with lower quotas. arXiv:2602.08966

[61] Pooja Kulkarni, Ruta Mehta, and Parnian Shahkar. 2025. Online fair division: Towards ex-post constant MMS guarantees. In *Proceedings of the 26th ACM Conference on Economics and Computation*. 638–638.

[62] David Kurokawa, Ariel D. Procaccia, and Junxing Wang. 2018. Fair enough: Guaranteeing approximate maximin shares. *J. ACM* 65, 2 (2018), 1–27.

[63] Alexander Lam, Bo Li, and Ankang Sun. 2025. The (exact) price of cardinality for indivisible goods: A parametric perspective. In *Proceedings of the 39th AAAI Conference on Artificial Intelligence*. 13985–13992.

[64] Bo Li, Minming Li, and Ruilong Zhang. 2021. Fair scheduling for time-dependent resources. In *Proceedings of the 35th International Conference on Neural Information Processing Systems*. 21744–21756.

[65] Bo Li, Fangxiao Wang, and Yu Zhou. 2023. Fair allocation of indivisible chores: Beyond additive costs. In *Proceedings of the 37th International Conference on Neural Information Processing Systems*. 54366–54385.

[66] Zhentao Li and Adrian Vetta. 2021. The fair division of hereditary set systems. *ACM Transactions on Economics and Computation* 9, 2 (2021), 1–19.

[67] Erika Mackin and Lirong Xia. 2016. Allocating indivisible items in categorized domains. In *Proceedings of the 25th International Joint Conference on Artificial Intelligence*. 359–365.

[68] Ryoga Mahara. 2026. Existence of fair and efficient allocation of indivisible chores. In *Proceedings of the 37th Annual ACM-SIAM Symposium on Discrete Algorithms*. 6742–6766.

[69] Marios Mertzanidis, Alexandros Psomas, and Paritosh Verma. 2024. Automating food drop: The power of two choices for dynamic and fair food allocation. In *Proceedings of the 25th ACM Conference on Economics and Computation*. 243–243.

[70] Hervé Moulin. 2004. *Fair Division and Collective Welfare*. MIT Press.

[71] Hervé Moulin. 2019. Fair division in the internet age. *Annual Review of Economics* 11, 1 (2019), 407–441.

[72] Meghana Nasre and Prajakta Nimbhorkar. 2017. Popular matchings with lower quotas. In *Proceedings of the 37th IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science*. 44:1–44:15.

[73] Ariel D. Procaccia, Ben Schiffer, and Shirley Zhang. 2024. Honor among bandits: No-regret learning for online fair division. In *Proceedings of the 38th International Conference on Neural Information Processing Systems*. 13183–13227.

[74] Ariel D. Procaccia and Junxing Wang. 2014. Fair enough: Guaranteeing approximate maximin shares. In *Proceedings of the 15th ACM Conference on Economics and Computation*. 675–692.

[75] Benjamin Schiffer and Shirley Zhang. 2025. Improved regret bounds for online fair division with bandit learning. In *Proceedings of the 39th AAAI Conference on Artificial Intelligence*. 14079–14086.

[76] Hila Shoshan, Noam Hazon, and Erel Segal-Halevi. 2023. Efficient nearly-fair division with capacity constraints. In *Proceedings of the 22nd International Conference on Autonomous Agents and Multiagent Systems*. 206–214.

[77] Hugo Steinhaus. 1948. The problem of fair division. *Econometrica* 16 (1948), 101–104.

[78] Warut Suksompong. 2021. Constraints in fair division. *ACM SIGecom Exchanges* 19, 2 (2021), 46–61.

[79] Yuanyuan Wang, Xin Chen, Qizhi Fang, Qingqin Nong, and Wenjing Liu. 2025. Guaranteeing fairness and efficiency under budget constraints. *Journal of Combinatorial Optimization* 49, 3 (2025), 1–21.

[80] Yuanyuan Wang, Xin Chen, and Qingqin Nong. 2024. The fairness of maximum Nash social welfare under matroid constraints and beyond. In *Proceedings of the 20th International Conference on Web and Internet Economics*. 172–189.

[81] Xiaowei Wu, Bo Li, and Jiarui Gan. 2025. Approximate envy-freeness in indivisible resource allocation with budget constraints. *Information and Computation* 303 (2025), 105264.

[82] Yu Yokoi. 2020. Envy-free matchings with lower quotas. *Algorithmica* 82, 2 (2020), 188–211.

[83] Shengwei Zhou, Rufan Bai, and Xiaowei Wu. 2023. Multi-agent online scheduling: MMS allocations for indivisible items. In *Proceedings of the 40th International Conference on Machine Learning*. 42506–42516.