On Minimal Achievable Quotas in Multiwinner Voting

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ABSTRACT

Justified representation (JR) and extended justified representation (EJR) are well-established proportionality axioms in approval-based multiwinner voting. Both axioms are always satisfiable, but they rely on a fixed quota (typically Hare or Droop), with the Droop quota being the smallest one that guarantees existence across all instances. With this in mind, we take a step beyond the fixed-quota paradigm by studying instance-dependent proportionality notions. More specifically, we minimize the quota requirements for JR and EJR using the parameter α. We demonstrate that all commonly studied voting rules can have an additive gap to the optimum of $\frac{k^2}{(k+1)^2}$. Moreover, we examine the computational aspects of our instance-dependent quota and prove that determining the optimal value of α for a given approval profile that allows some committee to satisfy α-JR is NP-complete. To address this, we introduce an integer linear programming (ILP) formulation for computing committees that satisfy α-JR, and we provide positive computational results in the voter interval (VI) and candidate interval (CI) domains.

KEYWORDS

Justified representation; Proportionality; Approval voting

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1 INTRODUCTION

In the context of multiwinner approval voting, fairness is typically understood in terms of proportionality: the elected committee should reflect, to an appropriate extent, the fraction of voters who share similar approval sets. This guarantees that sufficiently large groups of voters receive adequate representation, while preventing smaller groups from being disregarded. Proportionality axioms, like justified representation (JR) and extended justified representation (EJR), formalize this intuition and have become a central concept in the study of fairness in multiwinner elections. Given the number of voters n and committee size k, a group of voters S of size at least n/k that agrees on a common candidate is called *cohesive* and should be able to demand some form of representation in the committee. The Hare quota ($= n/k$) is deeply ingrained in the definitions of most proportionality axioms in multiwinner voting, including JR and EJR, as well as proportional justified representation (PJR) and full justified representation (FJR).

Nevertheless, there are various situations where these proportionality axioms fail to capture intuitive fairness. JR demands that every cohesive group S has at least one voter $v \in S$ who approves a candidate in the committee. Consider, for example, the election instance provided in Figure 1. For the group S to be cohesive, it must contain at least 5 voters. It should be easy to see that the committee $W = \{c_3, c_4\}$ satisfies JR. However, JR permits that the groups $\{1, 2, 3, 4\}$ and $\{7, 8, 9, 10\}$ remain unrepresented. These groups fall just short of meeting the quota. This instance, however, permits a more balanced representation of all voters by selecting the committee $W' = \{c_1, c_2\}$.

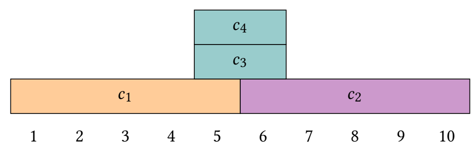

Figure 1: A voting instance with 10 voters, candidate set $\{c_1, c_2, c_3, c_4\}$ and $k = 2$ in which any committee containing c_3 satisfies JR. Voters are denoted by an integer and approve the candidates stacked above them.

This phenomenon extends to all fairness axioms defined in relation to these static quota requirements. Even the core is not immune to this issue.[1] Figure 2 illustrates an example in which the committee $\{c_1, c_2, c_3\}$ belongs to the core, even though voters 5–12 remain unrepresented. The three voters supporting c_4 once again fall short of the quota and can therefore not form a valid deviating coalition.

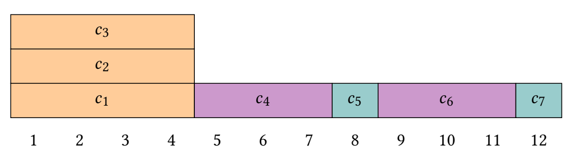

Figure 2: A voting instance with 12 voters, candidate set $\{c_1, \ldots, c_7\}$ and $k = 3$ in which the committee $W = \{c_1, c_2, c_3\}$ lies within the core even though a large group of voters remains unrepresented (example taken from [30]).

These observations regarding the limitations of proportionality are well established in the literature:
"Properties like the core (and also EJR, PJR, and JR) prevent specific pathological situations, but beyond their definitions, do not provide

[1] For a formal definition, consider the paper by Peters et al. [30].

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intuitive justifications for why a committee they allow should be selected" — Peters et al. [30].

For some proportionality axioms, representation cannot be guaranteed for groups smaller than the Droop quota (= $\frac{n}{k+1}$) [17]. Hence, fixed quotas may yield examples like that in Figure 1 or render the axiom unsatisfiable. This raises the question of whether, instead of relying on a uniform quota across all instances, one should adapt it with respect to the preference profile. Our experiments show that for a given instance, the quota for which JR or EJR remain satisfiable tend to be much smaller than $\frac{n}{k+1}$. A similar observation has been made for the ordinal case [2].

This paper analyzes variable quotas, contributing to and expanding upon the existing literature [2, 23]. The central idea is to reduce the quota n/k by a factor α, approaching the infimum value at which JR or EJR ceases to be feasible. In the case of JR, finding the optimal α-value corresponds to minimizing the size of the largest group of unrepresented voters who approve a common candidate.

To illustrate this concept more concretely, consider one more example: A company has four locations, each comprising five members. The task is to form a representative committee of three members. Within each location, all members approve a location representative who is aware of the problems at that location. Each location has a manager, a salesperson, and an engineer, with each role group approving a representative for their role. The remaining two members per location are working students, who only approve their location representative. Known approval-based multiwinner rules, such as PAV or CC, may elect three of the four location representatives because they receive the broadest approval support. However, this outcome fails to represent an entire location. We argue that a more proportionally fair outcome — and the optimal committee with respect to α-JR — would instead include the three group representatives, ensuring that no large and homogeneous segment of the electorate, such as an entire company location, remains unrepresented.

Our Contributions. In this paper, we study a class of proportionality axioms, denoted α-JR and α-EJR, which generalize the existing proportionality axioms JR and EJR. There are instances for which existing voting rules perform very badly with respect to these axioms. More specifically, we demonstrate that all commonly studied voting rules can produce committees with an additive gap to the optimal α-value of $\frac{k^2}{(k+1)^2}$. Additionally, we show that computing the optimal α-value for a voting instance is NP-hard, even for JR. To find the optimal value for JR, we formulate an ILP. Further, in the VI and CI domains, we can efficiently compute the optimal committee with respect to α-JR. For party-list profiles, we provide an algorithm that constructs an optimal α-EJR committee in polynomial time, corresponding to the d'Hondt method in the apportionment setting. Lastly, we provide empirical evaluation on the distribution of the optimal α-value for synthetically generated instances and show that existing voting rules can, to some extent, approximate the optimal α-value on average.

2 RELATED WORK

Our work builds on the growing literature on fairness and proportionality in approval-based multiwinner voting (see, e.g., [1, 28, 33, 35]). The central idea in this line of research is that voters who agree on a set of candidates should receive influence in the elected committee proportional to their size. The notions of JR and EJR, introduced by Aziz et al. [1], translate this into the setting of multiwinner approval voting, thereby marking the starting point of proportionality in approval-based multiwinner voting. Building on these axioms, subsequent work has extended proportionality axioms and analyzed their implications for prominent rules [7, 10]. These ideas have also been generalized beyond multiwinner elections to participatory budgeting [31], and general voting models with feasibility constraints [29].

One aspect that has received relatively little attention is the role of quotas, i.e., the requirement that a group of like-minded voters must reach a certain threshold to secure representation in the committee. The most common quota used in the literature is the Hare quota (= $\frac{n}{k}$). For this, it has been shown that JR and EJR are always satisfiable [1]. This also holds for some stronger versions of the axioms, like EJR+ and FJR. More recently, these representation axioms have been considered for the Droop quota (= $\frac{n}{k+1}$) [12]. In this paper, the authors demonstrate that proportionality axioms based on the Droop quota can be satisfied for any given voting instance. They show that many voting rules and concepts originally developed for the Hare quota can be effectively adapted to the Droop quota setting. Before this work, the Droop quota was primarily studied within the framework of apportionment [8, 32].

In this work, we go a step further and consider dynamic quotas. Droop [17] — after whom the quota is named — already observed that elections often result in situations where more voters than the quota support a common candidate, raising the question of how to allocate the "excess" votes that such candidates receive. That being said, we take a quantitative perspective of proportionality, aiming to measure the extent to which a given committee satisfies proportional representation. Similar quantitative approaches have been explored in prior work [2, 22, 23, 26].

The work closest to ours is that of Janson [23], who studied and surveyed thresholds of proportionality. More specifically, he asked for the smallest proportion of voters that can be guaranteed a good outcome, where a good outcome corresponds to committees satisfying various proportionality notions. While he mostly focused on adjusting the proportion of votes achievable by sequential Phragmén and Thiele rules, we consider additional voting rules and examine computational approaches and results related to minimizing the quota.

More recently, Bardal et al. [2] developed quantitative measures of proportionality in the ordinal setting, thereby strengthening the proportionality guarantees of so-called *solid coalitions*. Their work primarily focused on empirical analyses of real-world data, whereas our study centers on the theoretical aspects of optimizing the α-value in the approval setting and on exploring results in the voter- and candidate-interval domains. In parallel to us, they study the parameter α, which for values less than 1 makes *proportionality for solid coalitions* more demanding. Skowron et al. [36] formalized proportionality in the ranking setting when voters have approval preferences and introduced a measure of group satisfaction within this setting.

Alternative approaches to capturing fairness in multiwinner voting have been explored beyond JR–based axioms [11, 25, 27].

Setting	α-JR	α-EJR	α-EJR+
Computing $\alpha^*_\Phi(I)$	NP-complete (Theorem 5.2)	NP-hard (Theorem 5.2)	NP-complete (Theorem 5.2)
Computing $\alpha^W_\Phi(I)$	$O(nm)$ (Proposition 5.1)	coNP-hard (Theorem 5.4)	$O(mnk)$ (Proposition 5.5)
Restricted domains (computing $\alpha^*_\Phi(I)$)			
Party-list profiles	$O(\lvert P\rvert)$ (Theorem 6.2)	$O(k\log(\lvert P\rvert)+\lvert P\rvert))$ (Theorem 6.2)	$O(k\log(\lvert P\rvert)+\lvert P\rvert))$ (Theorem 6.2)
VI domain	$O(mn^2\log n)$ (Theorem 6.5)	?	?
CI domain	$O(m^2 n\log n)$ (Theorem 6.8)	?	?

Table 1: Overview of our computational results. Cells with a question mark indicate that the question is still open.

Brill et al. [9] introduced the axiom of *individual representation (IR)*, which strengthens the earlier concept of *semi-strong justified representation* proposed by Aziz et al. [1]. While *semi-strong JR* requires that every sufficiently large and cohesive group of voters is represented in the committee at least once, IR refines this principle by assigning each voter a guarantee relative to the largest ℓ-cohesive group she is able to form for a given profile. IR is not always satisfiable: there exist profiles for which no committee provides individual representation, and non-trivial approximation guarantees are unattainable without imposing restrictions on the domain of admissible profiles. Kehne et al. [25] shifted the focus on fairness to be more candidate-centric. Their desideratum of fairness states that candidates with similar sets of supporters should receive similar representation.

Lastly, we mention the *proportionality degree (PD)* as an alternative for quantifying proportionality in multiwinner approval voting. This approach strictly enforces the quota while allowing greater flexibility in determining each group's average satisfaction level. Skowron [34] used this to measure the proportionality guarantees that different voting rules can achieve. He showed that the sequential variant of the proportional approval voting rule has higher PD than the sequential Phragmén rule. Janeczko and Faliszewski [22] looked into computational aspects of the proportionality degree in committee elections. They show that deciding if a committee with a given PD exists is NP-hard and that verifying if a given committee provides a given PD is coNP-complete.

3 PRELIMINARIES

We consider the standard approval-based multiwinner voting framework. For $x \in \mathbb{N}$, let $[x] = \{1, 2, \dots, x\}$. Let $N = \{v_1, \dots, v_n\}$ be the set of voters, $C = \{c_1, \dots, c_m\}$ be the set of candidates, and $k \in \mathbb{N}$ be the committee size with $k \leq m$. For every voter $v \in N$, we denote the set of candidates she approves of as $A_v \subseteq C$, and we write $N_c = \{v \in N : c \in A_v\}$ for the set of supporters of a candidate $c \in C$. A voting instance I is defined by an approval profile $A = (A_v)_{v\in N}$ and the committee size k, i.e., $I = (A, k)$. Finally, a committee W is a subset of the candidates with cardinality k. To determine which committee to select, various multiwinner voting rules have been proposed. In this paper, we consider commonly used approval-based multiwinner voting rules, including the method of equal shares (MES), the sequential Phragmén (seq-Phragmén), the Chamberlin-Courant (CC), proportional approval voting (PAV), and the greedy justified candidate rule (GJCR). We refer the reader to the full version of the paper for the formal definitions of these rules.

4 QUOTA-BASED STRENGTHENING OF PROPORTIONALITY

As stated in the beginning, most proportionality axioms rely on the principle that a group is entitled to representation only if it satisfies some cohesiveness condition. Cohesiveness in its general form demands that a voter group satisfies the Hare quota [1, 33].

Definition 4.1 (ℓ-Cohesiveness [1]). For $\ell \in [k]$, a group $S \subseteq N$ is ℓ-cohesive if (1) $|S| \geq \ell \cdot \frac{n}{k}$ and (2) $|\bigcap_{v\in S} A_v| \geq \ell$.

In this work, we study a more general version of this definition, which incorporates a parameter α to make the quota requirement dynamic. The parameter α acts as a scaling factor that adjusts the group size required for representation. Smaller α values correspond to a more demanding standard of proportionality, as even smaller groups can claim representation.

Definition 4.2 ((α, ℓ)-Cohesiveness). For $\ell \in [k]$ and $\alpha \in \mathbb{R}_{\geq 0}$, a group $S \subseteq N$ is (α, ℓ)-cohesive if (1) $|S| \geq \alpha \cdot \ell \cdot \frac{n}{k}$ and (2) $|\bigcap_{v\in S} A_v| \geq \ell$.

For $S = \emptyset$, it holds that $\bigcap_{v\in S} A_v = C$. Changing the quota size has, up until now, mostly been done for $\alpha \geq 1$, which relaxes the cohesiveness definition [14–16, 21, 24].

With this generalized cohesiveness definition, we modify existing proportionality axioms:

Definition 4.3 (α-JR). A rule satisfies α-JR if for every $(\alpha, 1)$-cohesive group S, there exists a voter $v \in S$ with $|A_v \cap W| \geq 1$.

Definition 4.4 (α-EJR). A rule satisfies α-EJR if for every (α, ℓ)-cohesive group S, there exists a voter $v \in S$ with $|A_v \cap W| \geq \ell$.

Definition 4.5 (α-EJR+). A committee W satisfies α-EJR+ if there does not exist a candidate $c \in C \setminus W$, a group of voters $S \subseteq N$, and $\ell \in [k]$ with $|S| \geq \alpha \cdot \ell \cdot \frac{n}{k}$ such that

$$c \in \bigcap_{v\in S} A_v \text{ and } |A_v \cap W| < \ell \text{ for all } v \in S.$$

Clearly, for the case $\alpha = 1$, the three proportionality axioms correspond to the original notions of JR, EJR, and EJR+. Therefore, several voting rules are known to guarantee satisfiability. More recently, it has been established that for any given instance, there exists an $\alpha < 1$ such that α-EJR+ can be satisfied.

Theorem 4.6 (Casey and Elkind [12]). *For every instance $I = (A, k)$, and any α with $\alpha > \frac{k}{k+1}$ there exists a committee W that satisfies α-EJR+ and is polynomial-time computable.*

In this paper, we are interested in the minimal achievable quota for a given instance I. Therefore, we would like to find the infimum α-value that still allows for an instance to satisfy α-Φ where $\Phi \in \{\text{JR}, \text{EJR}, \text{EJR+}\}$. That being said, let

$$\alpha_\Phi^W(I) := \inf\{\alpha\colon W \text{ satisfies } \alpha\text{-}\Phi\}$$

for the given instance I. We refer to $\alpha_\Phi^W(I)$ as the α-value of a committee W with respect to the proportionality axiom Φ. The minimum achievable α-value for an instance I is denoted by

$$\alpha_\Phi^*(I) = \min_{W \subseteq C : |W| = k} \alpha_\Phi^W(I).$$

We also refer to this as the optimal α-value for an instance I with respect to Φ. If the instance I is clear from the context, we omit it and just write α_Φ^* or α_Φ^W.

The notion of α_Φ^* is equivalent to finding the largest α-value for which all committees violate α-Φ. Observe that by definition, the value is greater than or equal to 0. [2]

If all voters with a non-empty ballot approve a candidate in the committee, we have $\alpha_{\text{JR}}^* = 0$, and if $|A_v \cap W| \geq \min(|A_v|, k)$ for all $v \in N$, then $\alpha_{\text{EJR}}^* = 0$ [3]. Note that $\alpha \geq 0$ since the proportionality axioms are always violated when $\alpha = 0$. For $\alpha > \frac{k}{k+1}$, all proportionality axioms can be satisfied, which was shown by [12].

All omitted proofs and further information can be found in the full version of the paper [3].

4.1 General Properties

This approach generates a whole class of proportionality axioms. We, therefore, start by showing some general relational results: The known implications chain of EJR+, EJR, and JR only holds for α_1-EJR+, α_2-EJR, and α_3-JR when $\alpha_1 \leq \alpha_2 \leq \alpha_3$.

Proposition 4.7. *For $\alpha_1 \leq \alpha_2 \leq \alpha_3$, it holds that α_1-EJR+ $\implies$ α_2-EJR $\implies$ α_3-JR.*

Proof Sketch. Let $\alpha \leq \alpha'$ and $\Phi \in \{\text{JR}, \text{EJR}, \text{EJR+}\}$. First, α-Φ implies α'-Φ since as α increases, the set of cohesive groups shrinks. Showing the implications between α'-EJR+, α'-EJR, and α'-JR follows directly from the original definitions of the proportionality axioms. □

Proposition 4.8. *For $1 \geq \alpha_1 > \alpha_2 > \alpha_3$, it holds that α_1-EJR+, α_2-EJR, and α_3-JR are incomparable.*

We next demonstrate that there can be a substantial gap between the optimal α-values for JR and EJR. This highlights that our framework offers greater flexibility in how representation is understood. Under the classical definitions, it is natural to focus on EJR, as it always implies JR. However, introducing the parameter α allows us to adjust the strength of representation: we can choose whether to prioritize smaller, cohesive groups or to emphasize the representation of larger ones.

Proposition 4.9. *There is an instance I such that the additive difference of $\alpha_{JR}^*(I)$ and $\alpha_{EJR}^*(I)$ is $k/(k+1)$.*

[2] $S = \emptyset$ is $(0, \ell)$-cohesive and violates α-Φ for all possible committees W.

[3] For $\alpha > 0$, every voter group S with $|S| \geq \alpha \ell \frac{n}{k}$ has at least one "satisfied" voter.

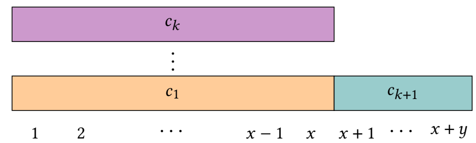

Figure 3: A voting instance $N = [x + y]$, $C = \{c_1, \dots c_{k+1}\}$ in which the optimal α-value for JR is 0, whereas the optimal α-value for EJR is $\min(\frac{x}{x+y}, \frac{k \cdot y}{x+y})$.

Proof Sketch. The committee $W = \{c_1, \dots, c_{k-1}, c_{k+1}\}$ achieves $\alpha_{\text{JR}}^W(I) = 0$, while for α_{EJR}^* we need to decide whether to select candidates $c_1, \dots, c_k$ or to replace one of them with c_{k+1}. Either we get a violation by a 1-cohesive group or by a k-cohesive group. When we set $x = ky$ in Figure 3, we can show that $\alpha_{\text{EJR}}^* = k/(k+1)$. □

This construction shows that even when a committee perfectly satisfies α-JR ($\alpha = 0$), its α-EJR value can be as large as $k/(k+1)$, demonstrating a maximal additive gap. Since α can be any value in $\mathbb{R}_{\geq 0}$, we prove that it is sufficient to only consider $k \cdot n$ many α-values to determine the optimum for JR and EJR. This technicality is important when designing efficient algorithms and is utilized in the restricted domain section.

Lemma 4.10. *For a given instance I, at most $\lceil \frac{n}{k} \rceil - 1 (\leq n)$ α-values need to be checked to determine $\alpha_{JR}^*(I)$. Further, it holds that $\alpha_{JR}^*(I) \in \mathbb{Q}$.*

Proof. For α-JR, it is only relevant what size the smallest 1-cohesive group has. The group size is between 1, meaning that a single voter forms a 1-cohesive group (together with a candidate she approves), and $\frac{n}{k}$, the size of the Hare quota. The relevant α-values come from the set $X = \{0, \frac{k}{n}, \frac{2k}{n}, \dots, \frac{rk}{n}\}$ with $r = \lceil \frac{n}{k} \rceil - 1$. Observe that if there does not exist a violation for $\alpha = \frac{k}{n}$ then α_{JR}^* is 0 by definition.

Now let α_{JR}^* be the optimal value. Thus, there exists some α_{JR}^*-JR violation witnessed by some (N', c). Therefore, it holds that $|N'| \geq \alpha_{\text{JR}}^* \cdot \frac{n}{k}$. It even has to be the case that this inequality is tight, since otherwise α_{JR}^* is clearly not optimal with (N', c) also witnessing a violation for a larger α. Thus, $\alpha_{\text{JR}}^* = |N'| \cdot \frac{k}{n} \in X$ which proves the claim. □

Lemma 4.11. *For a given instance I, at most $n \cdot k$ many α-values are needed to determine $\alpha_{EJR}^*(I)$. Further, it holds that $\alpha_{EJR}^*(I) \in \mathbb{Q}$.*

We can assume that $\alpha \in \mathbb{Q}$ since otherwise we can always consider the next larger $\alpha' \in \mathbb{Q}$ due to the two lemmas above.

4.2 Worst-Case Analysis of Existing Rules

Many known rules can always guarantee the Droop quota. Nevertheless, there are instances in which these voting rules perform very poorly with respect to the optimal α-value. Intuitively, this follows from the fact that many rules greedily select candidates approved by many voters and do not consider whether underrepresented voters approve of common candidates. Figure 4 visualizes such an instance for $k = 5$.

Example 4.12. Consider the following family of instances. Let $N = (k+1)(k+1)$. Furthermore, let $C = B \cup D$, with $B = \{b_1, \dots, b_{k+1}\}$ such that $N_{b_i} = \{(i-1) \cdot (k+1) + 1, \dots, i \cdot (k+1)\}$ for all $i \in [k+1]$ and $D = \{d_1, \dots, d_k\}$ such that $N_{d_i} = \{j \in N \colon (j \mod k+1) = i\}$ for all $i \in [k]$. Then each candidate in D and each candidate in B is approved by $k+1$ voters (see Figure 4 for a visualization of the case $k = 5$).

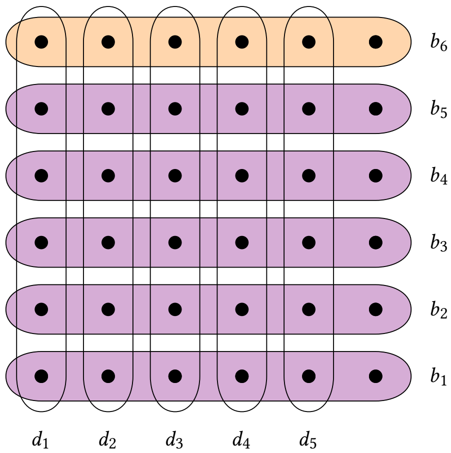

Figure 4: Visualization of Example 4.12 for $k = 5$. Voters are indicated by dots, and hyperedges correspond to the candidates. A committee W returned by most voting rules is indicated in violet, and the largest violation for W is shown in orange. The committee $D = \{d_1, \dots, d_5\}$ minimizes the α-value in this instance.

Many existing voting rules achieve an α-value of $\frac{k}{k+1}$ in this example, since they return the committee, visualized in violet, while leaving the large group of voters N_{b_6} unrepresented. Note that in this instance, rules like GJCR or MES would actually return no candidate, since the instance must contain a JR-violation for them to select any candidate. Therefore, we consider a natural way to adapt these rules to α by lowering the quota and giving more budget to the voters, respectively. We discuss these adaptations and define all the voting rules mentioned in the extended version. In the following, we show that all these rules have a comparably bad α-value on the instances defined in Example 4.12.

Theorem 4.13. *For CC, seq-CC, seq-Phragmén, PAV, α-MES, and α-GJCR, the additive difference to $\alpha^*_{JR}(I)$ can be of size $\frac{k^2}{(k+1)^2}$.*

Proof. Consider Example 4.12 [4]: All of the above rules might choose k from the $k+1$ candidates of the set B, i.e., the violet candidates from Figure 4. Without loss of generality, let $N_{b_{k+1}}$ be the group of voters that remain unrepresented. For them not to have a legitimate claim for a candidate in the committee, the quota $\alpha \cdot \frac{n}{k}$ must be greater than $|N_{b_{k+1}}|$. Consequently,

$$\alpha > |N_{b_{k+1}}| \cdot \frac{k}{n} = (k+1) \cdot \frac{k}{(k+1)(k+1)} = \frac{k}{k+1}.$$

On the other hand, note that there exists a committee $W = D$, such that for all pairs of non-represented voters v, v' with $v \neq v'$ it holds that $A_v \cap A_{v'} = \emptyset$. Thus, it holds that $\alpha^*_{\text{JR}}(I) \leq \frac{k}{n}$. Note that, in this instance, D is an optimal solution since no committee is approved by every voter. It follows that the α-value of these voting rules can be $\frac{k}{k+1} - \frac{k}{n} = \frac{k^2}{(k+1)^2}$. □

For seq-Phragmén, Janson [23] has already shown that the Droop quota is the smallest achievable quota in every instance. We include the result here for completeness.

Theorem 4.13 complements the results presented by Casey and Elkind [12]. They demonstrated that the proportionality axioms defined with the Droop quota are still attainable under current voting rules. We show that these algorithms cannot provide any axiomatic guarantee in the sense of satisfying the more demanding α-version of proportional representation notions when α is less than or equal to $\frac{k}{k+1}$ — corresponding to the Droop quota.

5 COMPUTING THE OPTIMAL α-VALUE

In this section, we analyze how to determine the optimal α-value for a given instance. We begin by showing that, for α-JR, potential violations can be detected efficiently for a given committee W.

Proposition 5.1. *For a given instance I and committee W, we can compute $\alpha^W_{JR}(I)$ in $O(nm)$.*

Proof. Given an instance I and a committee W, we want to compute the largest α for which there exists an α-JR violation. For this, we consider every candidate $c \notin W$ and compute the number of voters that support c but no candidate in W. We define the maximal size of such a group as

$$s_{\max} = \max_{c \notin W} |\{v \in N \colon A_v \cap W = \emptyset \text{ and } c \in A_v\}|.$$

We now claim that the largest α that causes an α-JR violation is equal to $s_{\max} \cdot \frac{k}{n}$. First, observe that by the definition of $s_{\max}$, there exists a group S of size $s_{\max}$ that commonly approves some candidate c but no candidate in W. For $\alpha = s_{\max} \cdot \frac{k}{n}$, S is $(\alpha, 1)$-cohesive since

$$\alpha \cdot \frac{n}{k} = s_{\max} \cdot \frac{k}{n} \cdot \frac{n}{k} = s_{\max},$$

and thus constitutes a violation of c. This implies that $\alpha^W_{\text{JR}} \geq s_{\max} \cdot \frac{k}{n}$.

On the other hand, by definition of α^W_{JR} there exists some α^W_{JR}-JR violation with witness (N', c). Thus, it holds that $|N'| \geq \alpha^W_{\text{JR}} \cdot \frac{n}{k}$. Further, since the voters in N' approve a common candidate c and do not approve any candidate in W, it holds that

$$|N'| \leq |\{v \in N \colon A_v \cap W = \emptyset \text{ and } c \in A_v\}| \leq s_{\max}.$$

Thus, $s_{\max} \geq \alpha^W_{\text{JR}} \cdot \frac{n}{k}$ which implies that $\alpha^W_{\text{JR}} \leq s_{\max} \cdot \frac{k}{n}$.
Since we need to compute $s_{\max}$, we must check, for every candidate $c \notin W$, how many voters are not represented and approve c. This can be done in $O(nm)$. □

Proposition 5.1 proves that, for JR, we can determine if there exists an α-JR violation for a given committee in polynomial time.

[4] Example 4.12 can be adapted so that it does not rely on tie-breaking. The additive difference for this is then $\frac{k^2}{(k+2)(k+1)}$ (see full version of the paper).

Theorem 5.2. *For a constant $\alpha \in (0,1) \cap \mathbb{Q}$, deciding whether there exists a committee that satisfies α-Φ with $\Phi \in \{$JR, EJR, EJR+$\}$ is NP-hard.*

NP-completeness of α-JR and α-EJR+ then follows with Proposition 5.1 and Proposition 5.5, respectively. The intuition behind the reduction is as follows: We can ensure that certain candidates are selected if the support they receive is sufficiently large, and that every voter in the support only approves this specific candidate. Further, since $\alpha < 1$, we can choose the support of those candidates small enough that after picking those candidates, there remains a large number of voters unrepresented. Avoiding an α-JR violation for this group is then as hard as solving a vertex cover problem.

Since the problem of determining the optimal α-value for an instance is computationally hard, we provide a method to determine the value by repeatedly solving the following ILP:[5]

$$\sum_{c \in C} x_c = k, \quad (1)$$

$$y_v \leq \sum_{c \in A_v} x_c \quad \forall v \in N, \quad (2)$$

$$\sum_{v\colon c \in A_v} (1 - y_v) \;\leq\; \lceil \alpha \cdot \frac{n}{k} \rceil - 1 \quad \forall c \in C, \quad (3)$$

$$y_v, x_c \in \{0,1\}.$$

Given some α, the ILP finds a committee W satisfying α-JR if one exists. The idea of the ILP is as follows. The binary variables x_c indicate whether the candidate c is included in the committee, and the binary variables y_v can only be 1 if voter v is covered by the output committee, i.e., v has an approved candidate c in the committee (this means $x_c = 1$).

Condition (1) ensures that the committee contains precisely k candidates, and condition (2) guarantees that, if $y_v = 1$, the voter v approves at least one candidate in the committee. The last condition ensures that at most $\lceil \alpha \cdot n/k \rceil - 1$ voters approving some common candidate c are left uncovered, capturing the α-JR requirement. The constraint is equivalent to saying that at least $|N_c| - \lceil \alpha \cdot n/k \rceil + 1$ voters in N_c are covered. Note that the ILP does not need an objective function, as we are only interested in feasibility.

Theorem 5.3. *The ILP has a solution if and only if there exists a committee that satisfies α-JR.*

Having addressed the case of JR, we now turn our attention to EJR. In contrast to JR, verifying or optimizing α for EJR is computationally more demanding. In fact, even determining whether a given committee satisfies EJR for $\alpha = 1$ is coNP-hard [1]. This hardness directly extends to determining whether, for a fixed α, an α-EJR violation exists.

Theorem 5.4. *For a constant $\alpha \in (0,1] \cap \mathbb{Q}$, a given instance I and committee W, determining if W satisfies α-EJR is coNP-hard.*

For a given instance I and committee W, figuring out the optimal α-value with respect to EJR, i.e. finding $\alpha^W_{\text{EJR}}(I)$, can be solved by considering ℓ-cohesive groups individually with $1 \leq \ell \leq k$:

[5]Note that Lemma 4.10 limits the number of α-values that need to be considered for the ILP.

Namely, we can find for every ℓ and W the size of the largest group of voters, denoted $s_{\max}(W,\ell)$, such that

$$s_{\max}(W,\ell) = \max_{T \subseteq C, |T| = \ell} |\{v \in N\colon |A_v \cap W| < \ell \text{ and } T \subseteq A_v\}|.$$

Similar to JR, we can now directly determine the largest α_ℓ such that there exists an (α,ℓ)-cohesive group that causes an α-EJR violation. Now for every ℓ, we compute $\alpha_\ell = s_{\max}(W,\ell) \cdot \frac{k}{\ell \cdot n}$ and take the maximum over these values. This gives the largest α-value such that there exists an α-EJR violation. The reason that this does not give an efficient algorithm lies in the fact that determining $s_{\max}(W,\ell)$ is computationally hard for larger ℓ. This computational complexity does not carry over to $\alpha^W_{\text{EJR+}}(I)$.

Proposition 5.5. *For a given instance I and committee W, we can compute $\alpha^W_{EJR+}(I)$ in $O(mnk)$.*

6 RESTRICTED DOMAINS

Since it is computationally hard to compute $\alpha^*_\Phi(I)$ for general instances I and $\Phi \in \{$JR, EJR, EJR+$\}$, we look into some restricted domains in which we can compute the optimal α-value in polynomial time. These domains are well-studied restrictions to the general multiwinner voting setting. Preferences are aligned along a single dimension, such as voters and candidates positioned on a political spectrum, geographic proximity in local elections, or a preference for candidates differing along a skill or topic dimension.

6.1 Party-List Profiles

A commonly studied restricted domain is that of *party-list profiles*, where voters' preferences depend solely on the party affiliation of candidates.

Definition 6.1 (Lackner and Skowron [28]). We say that an approval profile A is a *party-list profile* if for every pair of voters $v_i, v_j \in N$ we have either $A_{v_i} = A_{v_j}$ or $A_{v_i} \cap A_{v_j} = \emptyset$. An election instance I is called a *party-list instance* if (1) A is a party-list profile, and (2) for each voter $v \in N$, it holds that $|A_v| \geq k$.

For α-JR, an optimal committee can be obtained by selecting one representative for each of the k most supported parties. If every party has received a candidate, the remaining seats may be filled arbitrarily. We can extend this positive result to α-EJR.

Theorem 6.2. *Computing $\alpha^*_{EJR+}(I)$ for party-list instances can be done in $O(k \log(|P|) + |P|)$ with P being the set of parties. Further, we can compute $\alpha^*_{JR}(I)$ in $O(|P|)$.*

In the proof, we use the d'Hondt method from the apportionment setting [13] and show that it constructs an optimal committee with regard to $\alpha^*_{\text{EJR+}}(I)$.

6.2 Voter-Interval Domain

Definition 6.3 (Elkind and Lackner [19]). Given an election instance I, we say I has *voter-interval (VI)* preferences if there exists a linear order $\sqsubset$ over N such that for all voters $v_1, v_2, v_3 \in N$ and for each candidate $c \in A_{v_1} \cap A_{v_3}$, we have that $v_1 \sqsubset v_2 \sqsubset v_3 \implies c \in A_{v_2}$.

Aziz et al. [1] showed that checking whether W provides EJR for an instance I is coNP-complete. However, to the best of our knowledge, there is no result on efficiently finding violations in ℓ-cohesive groups when we consider only restricted approval profiles. More specifically, we prove the following lemma, which allows us to efficiently check for an α-EJR violation given an instance I and a committee W.

Theorem 6.4. *Given an instance I in the VI domain, we can verify whether a committee W satisfies α-EJR in $O(n^3k)$.*

Furthermore, we show that we can compute the optimal value $\alpha^*_{\text{JR}}(I)$ for a given instance I in the VI domain. This shows that, by restricting voters' preferences, we can obtain a positive result in contrast to the hardness result shown in Theorem 5.2 for general instances.

Theorem 6.5. *Computing $\alpha^*_{JR}(I)$ in the VI domain can be done in $O(n^2m\log n)$.*

The greedy algorithm in Algorithm 1 constructs a subset of candidates that satisfies α-JR for a given I and α, which is cardinal-minimal.
By checking if the size of the returned committee is at most k, we can verify if, for this α-value, we obtain a witness (feasible committee) satisfying α-JR. Due to Lemma 4.10, the number of distinct α-values we need to consider is upper-bounded by n. We can go through the α-values using binary search to find the highest value such that Algorithm 1 returns a committee that has more than k candidates, as this shows that there does not exist a committee that can satisfy α-JR for the α-value given in this step.

In Algorithm 1, we use the observation that in the VI domain, every candidate c has a left-most and right-most supporting voter according to the VI order. We denote these voters by l_c and r_c, respectively. For $i \in [n]$, the algorithm considers the set $\{v_1, \dots v_i\}$ and fixes an α-JR violation, if existent, with a candidate approved by the right-most voter. Intuitively, this ensures that the violation to the left of v_i is resolved while covering as many voters as possible to the right, thereby postponing the next potential violation as long as possible.

Algorithm 1 Finding Optimal Committees in the VI Domain

Input: Election $I = (A, k)$ in the VI domain, $\alpha \in \mathbb{R}_{\geq 0}$
Output: Committee $W \subseteq C$
$W \leftarrow \emptyset$
for $i = 1, \dots, n$ **do**
  Check whether voters $\{v_1, \dots, v_i\}$ admit an α-JR violation under W
  if a violation is found **then**
    $W \leftarrow W \cup \{c\}$ with $c \in A_{v_i}$ and $r_{c'} \sqsubseteq r_c$ for all $c' \in A_{v_i}$.
return W

6.3 Candidate-Interval Domain

Definition 6.6 (Elkind and Lackner [19]). Given an election instance I, we say that I has *candidate-interval preferences (CI)* if there exists a linear order $\sqsubset$ over C such that for each voter $v \in N$ and all candidates $a, c \in A_v$, $b \in C$ we have that $a \sqsubset b \sqsubset c \implies b \in A_v$.

Intuitively, each voter approves a consistent interval of candidates. It turns out that the positive results for the VI domain also hold in the CI domain. More specifically, we show that we can find the optimal α-value for a given instance I and committee W with respect to EJR.

Theorem 6.7. *Given an instance I in the CI domain, we can verify whether a committee W satisfies α-EJR in $O(m^2kn)$.*

Again, as for the VI domain, it is possible to efficiently compute a committee that satisfies α-JR for the optimal value of α.

Theorem 6.8. *Computing $\alpha^*_{JR}(I)$ in the CI domain can be done in $O(m^2n\log n)$.*

We again have at most n α-values to check, and for each α-value, we use the greedy algorithm from Algorithm 2 to find a minimal committee satisfying α-JR. The size of the returned committee then indicates the satisfiability of a given α-value. Here, we iterate over the candidates in the linear order $\sqsubset$ given by the CI domain, removing candidates that can be omitted without α-JR violation. Intuitively, the algorithm has to keep a candidate if a set of voters with a relatively "early" interval has a violation, and later candidates cannot make up for this violation. This results in the algorithm constructing an optimal committee that pushes the candidates to the right regarding the linear order $\sqsubset$ over the candidates.

Algorithm 2 Finding Optimal Committees in the CI Domain

Input: Election $I = (A, k)$ in the CI domain, parameter $\alpha \in \mathbb{R}_{\geq 0}$
Output: Committee $W \subseteq C$
$W \leftarrow C$
for $i = 1, \dots, m$ **do**
  if $W \setminus \{c_i\}$ satisfies α-JR **then**
    $W \leftarrow W \setminus \{c_i\}$
return W

For both the CI and VI domains, we leave open the question of whether a polynomial-time algorithm exists to compute the optimal α-value for EJR, $\alpha^*_{\text{EJR}}(I)$, for a given instance I. The same holds for EJR+. The proposed left-to-right sweeping methodology does not work when $\ell > 1$.

7 EMPIRICAL EVALUATION

Finally, we complement our theoretical results with an empirical analysis. We generate synthetic voting instances to evaluate the resulting values of α and to assess the performance of existing multiwinner voting rules. Our primary focus is on α-EJR; additional plots for α-JR are available in the full version. Figure 5 reports the distribution of optimal values, α^*_{EJR}, under the Impartial Culture (IC) and the Euclidean Threshold (EUT) model, two commonly used probabilistic models for generating approval profiles. For each model, we consider instances of varying sizes—ranging from small to moderate—with up to 60 voters and 15 candidates. Both models are widely used in approval-based multiwinner voting [6, 18, 20]. Further details on the parameter configurations are provided in the full version.

The results demonstrate that in many instances α can be selected to be significantly smaller than $\frac{k}{k+1}$. In fact quite often α^*_{EJR} is closer

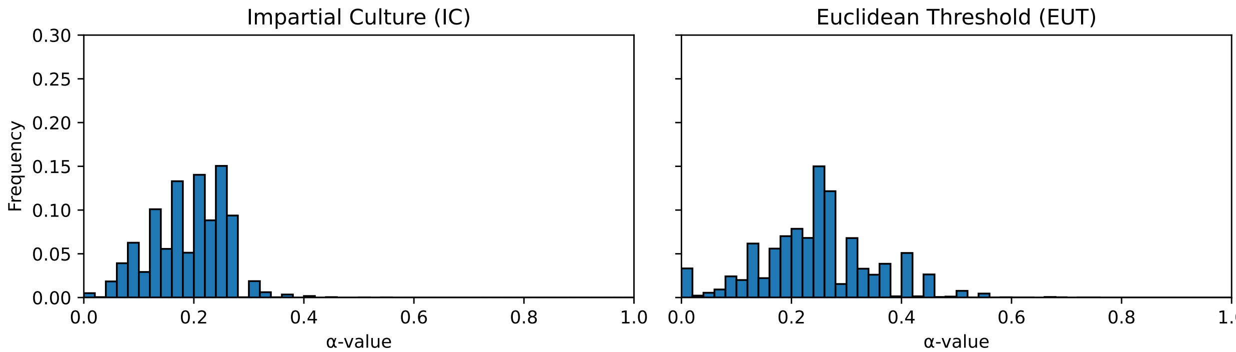

Figure 5: Distribution of optimal α-values, α^*_{EJR}, under the IC and the EUT model, based on 6400 generated instances each.

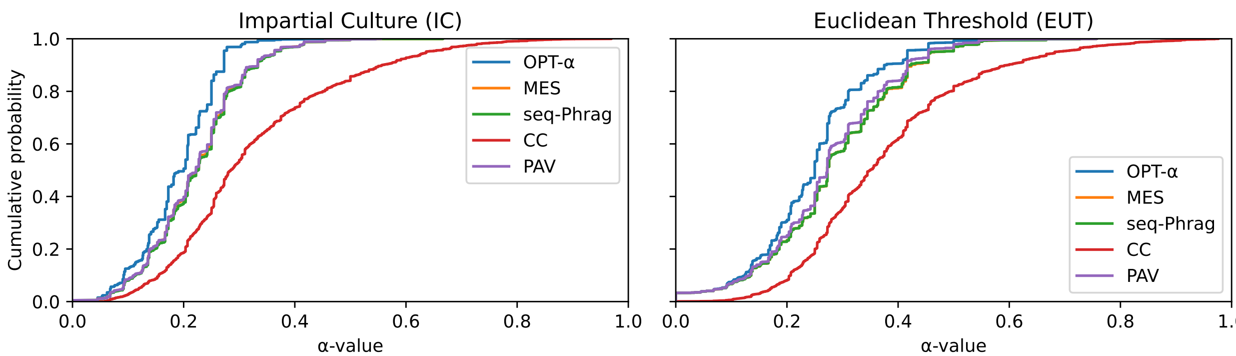

Figure 6: Cumulative distribution of the optimal α-value, α^*_{EJR}, and the α-value achieved by MES, seq-Phragmén, CC, and PAV under the IC and the EUT model, based on 6400 samples.

to 0 than it is to 1. Bardal et al. [2] show similar results when analyzing the α-value in the ranked setting for *proportionality of solid coalitions*. In addition to our theoretical analysis in Section 4.2, we evaluate the performance of existing voting rules on randomly generated instances. Figure 6 demonstrates that MES, seq-Phragmén, and PAV typically achieve committees with comparatively low α-values with respect to EJR, indicating that these rules perform well on average. Note that MES is completed with seq-Phragmén. Our experiments, however, also uncover instances in which all three rules yield committees whose α-values exceed the optimal value by more than a factor of four, highlighting substantial worst-case deviations even in average-case data. These cases are mostly seen in larger instances. In the full version, we provide more fine-grained information on how the voting rules perform under different instance sizes. CC does not perform very well, which is unsurprising because its only goal is to satisfy as many voters as possible and because it does not satisfy EJR for all instances. Further, we show that, on the contrary, CC performs best out of the considered rules for α^*_{JR}.

8 CONCLUSION

We demonstrate that our dynamic extensions of JR and EJR can lead to much fairer outcomes for various instances. Our axioms possess a clear fairness appeal, as they seek to minimize the underrepresentation of large aligned groups, which in real-world political settings may otherwise foster resentment. However, as with many other voting rules that aim to maximize an objective, such as CC or PAV, computing the optimal α-value is NP-hard, even for JR.

While existing voting rules can exhibit poor worst-case behavior theoretically, they perform reasonably well on our artificially generated datasets. A natural direction for future work is to evaluate these rules on real-world data, for instance, using the Polkadot dataset [5]. When restricting the set of voting instances, we obtain several positive results regarding the efficient computation and verification of α-values. Empirical results underscore the importance of considering adaptive quotas in multiwinner approval voting and, perhaps, in other settings. Further extensions could include generalizing our framework to weakly cohesive groups or developing broader notions of cohesiveness based on justified representation functions. Such generalizations may also prove useful in continuous settings, such as budget aggregation.

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